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POLYMERSIONS OF A DISK WITH CRITICAL POINTS ON THE BOUNDARY

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ABSTRACT. The existence of an interior extension with critical points on the boundary is proved for a given closed normal curve and for nonnormal curves of specific kind.

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INTRODUCTION

The problem of the existence of an interior extension for a given closed normal curve was solved by Ch. Titus [1] in 1961. In 1967 S. Blanc [2] gave another solution of this problem. Subsequently, many authors investigated different aspects of this theory [3]–[9], but in al these papers the existence of interior extension was proved under the assumption that the images of critical points of this mapping do not lie on the given curve. On the other hand, M. Morse [10] introduced the concept of a partial branch point of the inverse function and extended his formula connecting the angular order of a curve with the sum of multiplicities of critical points for interior mappings possessing critical points on the boundary.

In this paper we investigate a problem of the existence of an interior extension with critical points on the boundary for a given closed normal curve (Section 3). In Section 4 an interior extension is constructed for a closed curve which is not normal.

1. PRELIMINARIES

Two continuous mappings (paths) $g : [a, b] \to \mathbb{R}^2$ and $h : [c, d] \to \mathbb{R}^2$ are called equivalent if there exists a sense preserving homeomorphism $\chi : [a, b] \to [c, d]$ such that $g = h \circ \chi$. The set of all equivalent paths is called a curve and each path belonging to the curve is called a representation of this curve. Let γ be a curve, and let $\gamma(x)$ be some representation of γ defined on $[a, b]$. The point $i(\gamma) = \gamma(a)$ is called the initial point of γ and $t(\gamma) = \gamma(b)$ the terminal point of γ. By $\langle\gamma\rangle$ is denoted the set of images of $\gamma(x)$. Let γ_1 and γ_2 be two curves such that $i(\gamma_2) = t(\gamma_1)$, and let $\gamma_1(x)$, $x \in [a, b]$ and $\gamma_2(x)$, $x \in [c, d]$

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be the corresponding representations. Take some $b' > b$ and a homeomorphism $\chi : [b, b'] \to [c, d]$, $\chi(b) = c$ and consider the mapping defined on $[a, b']$

$$(\gamma_1 \cdot \gamma_2)(x) = \begin{cases} \gamma_1(x), & \text{when} \quad x \in [a, b], \\ \gamma_2(\chi(x)), & \text{when} \quad x \in [c, d]. \end{cases}$$

The curve containing the mapping $(\gamma_1 \cdot \gamma_2)(x)$ will be denoted by $\gamma_1 \cdot \gamma_2$.

Let $\gamma(x) \in \gamma$, $x \in [a, b]$ and $[c, d] \subseteq [a, b]$. By $\gamma[c, d]$ we denote the curve generated by the restriction of $\gamma(x)$ to $[c, d]$ and by $\gamma[d, c]$ the curve generated by $\gamma(\omega(x))$, where $\omega(x)$ is a sense reversing homeomorphism of $[c, d]$ onto itself.

The point $v \in \langle\gamma\rangle$ is called a vertex (or node) if for some path $\psi \in \gamma$, $\operatorname{card} \psi^{-1}(v) > 1$ [4], [6]. It is clear that the definition of a vertex does not depend on the choice of representation. A vertex v is called a double point if $\psi^{-1} = \{x_1, x_2\}$ and $\langle\psi[x_1 - \varepsilon, x_1 + \varepsilon]\rangle$ separates $\langle\psi[x_2 - \varepsilon, x_2 + \varepsilon]\rangle \backslash \psi(x_2)$ for some $\varepsilon > 0$. The curve is called normal if it has a finite number of vertices and each of them is a double point [9].

Let f be an interior mapping [11], [12] of a domain D into $\mathbb{R}^2$. For every $z \in D$ there exists a neighborhood $U(z)$ such that the restriction of f to $U(z)\backslash\{z\}$ is a $(\mu(f, z) + 1)$-sheeted covering of $f(U(z))\backslash\{f(z)\}$. If $\mu(f, z) > 0$, we say that the covering $f|_{U(z)}$ is branched and the point z is called a critical point of the multiplicity $\mu(f, z)$. After G. Francis [6] we call an interior mapping having a finite number of critical points in D a polymersion.

A closed curve γ is called an interior boundary [1], [2] if there exists an interior mapping (polymersion) of the unit disk D^2, continuous in $\overline{D}^2$ such that the restriction of f to the counterclockwise oriented unit circle S^1 is a path belonging to γ. Another expression of this fact is: f is an interior extension of γ.

2. The existence of an interior extension for a normal closed curve

In this section a summary of the main result of the paper [5] is given for convenience.

Let $w = \gamma(\zeta)$, $\zeta \in S^1$, be a representation of a closed normal curve and let α_j, $j = \overline{0, N}$, be the collection (raying) of paths (rays) with the following properties:

1. $i(\alpha_j) = a_j$ lies in some bounded component of $\mathbb{R}^2 - \langle\gamma\rangle$, $t(\alpha_j) = b_j$ lies in the unbounded component of $\mathbb{R}^2\backslash\langle\gamma\rangle$. There is at least one a_j in each bounded component of γ.
2. Each α_j is transverse to γ.
3. α_j's are pairwise disjoint and do not meet nodes of γ.

The crossing point of γ and α_j is called positive if γ crosses α_j from right to left, and negative otherwise. Index crossing points of α_j and γ by $(j, k, \pm 1)$ (called hereafter letters), where the sign of the third coordinate coincides with the sign of the crossing and second coordinates k, are arranged in an increasing way along α_j. In what follows the expression "a point (a,b,c)" will mean the crossing point corresponding to the letter (a,b,c).

Choose $i(\gamma)$ and trace along γ. We make a sequence α (called the word of γ) of letters as they are encountered in the tracing of γ. Denote by $\mathcal{G}(\gamma)$ (the grouping) and $\mathcal{B}(\gamma)$ (the branching) collections of letters of α satisfying following conditions

(a) The sets of $\mathcal{A}(\gamma)=\mathcal{G}(\gamma)\cup\mathcal{B}(\gamma)$ are mutually disjoint.

(b) If $A\in\mathcal{A}(\gamma)$ and $A'\in\mathcal{A}(\gamma)$, then $\gamma^{-1}(A)$ is contained in one component of $S^1\backslash\gamma^{-1}(A')$.

(c) Every set in $\mathcal{G}(\gamma)$ is of the form $\{(j,k,-1),(j,k',1)\}$ with $k<k'$.

(d) Every negative letter (i.e., the letter with a negative third coordinate) occurs in some set of $\mathcal{G}(\gamma)$.

(e) All letters belonging to some $B\in\mathcal{B}(\gamma)$ have the same first coordinate. In other words, each B consists of letters corresponding to the crossing points of one path of the raying.

Such a set $\mathcal{A}(\gamma)$ is called an assemblage for γ. The assemblage is called maximal if $\sum\limits_{B\in\mathcal{B}(\gamma)}(\operatorname{card}B-1)=\tau(\gamma)$, where $\tau(\gamma)$ is the tangent winding number of γ.

Theorem ([5]). *A normal closed curve has an interior extension iff it has a maximal assemblage. An assemblage determines multiplicities of critical points of corresponding interior extension f. Namely, for $B\in\mathcal{B}(\gamma)$ there exists $x\in D^2$ such that $f(x)=a_j$ and $\mu(f,x)=\operatorname{card}B-1$.*

Remark. The investigation of the problem of the existence of an interior extension was carried out in [5] under the condition that γ is smooth. But it is easy to show that the above-stated result remains true for a piece-wise smooth curve as well.

3. The Existence of an Interior Extension with Partial Critical Points

By $U(z)$ we denote a neighborhood of z in $\mathbb{R}^2$. For a point $\zeta\in S^1$ denote $U_+(\zeta)=U(\zeta)\cap\overline{D}^2$, $U_-(\zeta)=U(\zeta)\cap CD^2$, $U^+(\zeta)=U(\zeta)\cap D^2$, $U^-(\zeta)=U(\zeta)\cap\overline{D}^2$. Let $\gamma(\zeta)$ be a representation of a normal curve and let $\gamma[\zeta_1,\zeta_2]$ be a simple arc of this curve. Let $\zeta_0\in\gamma[\zeta_1,\zeta_2]$ and let $U(\gamma(\zeta_0))$ be a neighborhood of $\gamma(\zeta_0)$ such that $U(\gamma(\zeta_0))\backslash\langle[\zeta_1,\zeta_2]\rangle$ consists of two components. Denote by $U^+(\gamma(\zeta_0))$ the component of $U(\gamma(\zeta_0))\backslash\langle[\zeta_1,\zeta_2]\rangle$ lying to the left of $\gamma[\zeta_1,\zeta_2]$ and by $U^-(\gamma(\zeta_0))$ the other component. By $U_+(\gamma(\zeta_0))$ and $U_-(\gamma(\zeta_0))$ denote $U^+(\gamma(\zeta_0))\cup(\langle[\zeta_1,\zeta_2]\rangle\cap U(\gamma(\zeta_0)))$ and $U^-(\gamma(\zeta_0))\cup(\langle[\zeta_1,\zeta_2]\rangle\cap U(\gamma(\zeta_0)))$, respectively.

Let f be an interior extension of γ. A point $\widetilde{\zeta}\in S^1$ is called a partial critical point of f (or a partial branch point of f^{-1}) of multiplicity $\mu(f,\widetilde{\zeta})>0$ if there exists an extension F of f from $U_+(\widetilde{\zeta})$ to $U(\widetilde{\zeta})$ such that $F:U(\widetilde{\zeta})\backslash\{\widetilde{\zeta}\}\to$

$F(U(f(\widetilde{\zeta})))\backslash\{f(\widetilde{\zeta})\}$ is a $\mu(f,\widetilde{\zeta})+1$-sheeted covering and $F|_{U^-}(\widetilde{\zeta}) : U^-(\widetilde{\zeta}) \to F(U^-(f(\widetilde{\zeta}))$ is a homeomorphism.

The behavior of $f(z)$ in $U^(\widetilde{\zeta})$ can be described in the following way [12]: $\widetilde{\zeta} \in S^1$ is a partial critical point of f of multiplicity $\mu(f,\widetilde{\zeta})$ if there exist $2\mu(f,\widetilde{\zeta})$ simple arcs ℓ_j, $j = \overline{1, 2\mu(f,\widetilde{\zeta})}$, $\bigcap_{j=1}^{2\mu(f,\widetilde{\zeta})} \langle \ell_j \rangle = \{\widetilde{\zeta}\}$ passing through some $U^+(\widetilde{\zeta})$ such that the restriction of f to each component of $U^+(\widetilde{\zeta})\backslash \bigcap_{j=1}^{2\mu(f,\widetilde{\zeta})} \langle \ell_j \rangle$ is schlicht and maps this component onto $U^+(f(\widetilde{\zeta}))$ or $U^-(f(\widetilde{\zeta}))$. If we number these components in a consecutive way then the components with odd indices will be taken onto $U^+(f(\widetilde{\zeta}))$, while the components with even indices onto $U^-(f(\widetilde{\zeta}))$.

Theorem 1. *Let γ be a normal closed curve possessing a maximal assemblage $\mathcal{A}(\gamma)$ and let $B = \{(j,k_0,1),(j,k_1,1),\ldots,(j,k_\mu,1)\} \in \mathcal{B}(\gamma)$, $k_i < k_{j+1}$, $i = \overline{0, \mu-1}$, $\mu > 0$. Then there exists an interior extension $\widetilde{f}$ of γ such that $\widetilde{f}^{-1}$ has a partial branch point of multiplicity μ at the point $(j,k_0,1)$.*

Without loss of generality we can assume that the raying $\alpha_0, \alpha_1, \ldots, \alpha_N$ is numbered so that the set B consists of letters belonging to α_0.

Let E be an open set such that it contains the ray α_0 and does not contain other rays of the raying and nodes of the curve γ. Moreover, we can choose E in such a way that every component of $E \cap \langle\gamma\rangle$ contains at most one crossing point belonging to α_0. Denote by $\langle\gamma_{k_i}\rangle$ the component of $E \cap \langle\gamma\rangle$ containing the point $(0,k_i,1) \in B$, $i = \overline{0,\mu-1}$.

Take a simple arc $\gamma[\zeta', \zeta'']$ such that $\langle\gamma[\zeta',\zeta'']\rangle \subset \langle\gamma_{k_0}\rangle$. Let $(0,k_0,1) = \gamma(\widetilde{\zeta}) = \gamma(e^{i\widetilde{\theta}})$, $\zeta' = e^{i\theta'}$, $\zeta'' = e^{i\theta''}$. Take $2\mu+2$ points $\zeta_k = e^{i\theta_k}$, $k = \overline{0, 2\mu+1}$, such that $\theta' < \theta_0 < \theta_2 < \cdots < \theta_{2\mu} < \widetilde{\theta} < \theta_{2\mu-1} < \theta_{2\mu-3} < \cdots \theta_1 < \theta_{2\mu+1} < \theta''$. Let $\widetilde{U}(\gamma(\widetilde{\zeta}))$ be a neighborhood such that $\langle\gamma[\zeta',\zeta'']\rangle \subset \widetilde{U}(\gamma(\widetilde{\zeta})) \subset E$.

Denote by g_1 a simple arc such that $i(g_1) = \gamma(\zeta_0)$, $t(g_1) = \gamma(\zeta_1)$, $\langle g_1\rangle\backslash\{\gamma(\zeta_0),\gamma(\zeta_1)\} \subset \widetilde{U}^+(\gamma(\widetilde{\zeta}))$, and by g_2 a simple arc such that $i(g_2) = \gamma(\zeta_1)$, $t(\gamma_2) = \gamma(\zeta_2)$, $\langle g_2\rangle\backslash\{\gamma(\zeta_1),\gamma(\zeta_2)\} \subset \widetilde{U}^-(\gamma(\widetilde{\zeta}))$ etc. Without loss of generality we can assume that all curves g_n, $n = \overline{1, 2\mu}$, have no common points except end points of the adjacent curves. Evidently, $g_{2\mu+1}$ is transversal to all curves g_n with odd indices. Denote $g = g_1 \cdot g_2 \cdots g_{2\mu+1}$ and parametrize this curve in some way by θ, $\theta_0 \le \theta \le \theta_{2\mu+1}$. Denote

$$\gamma^*(\theta) = \begin{cases} \gamma(e^{i\theta}), & 0 \le \theta \le \theta_0, \\ g(\theta), & \theta_0 \le \theta \le \theta_{2\mu+1}, \\ \gamma(e^{i\theta}), & \theta_{2\mu+1} \le \theta \le 2\pi. \end{cases}$$

Denote by D_{2k+1}, $k = \overline{0,\mu}$, the domain bounded by the closed Jordan curve $\gamma[\zeta_{2k},\zeta_{2k+1}] \cdot g_{2k+1}^{-1}$, and by D_{2k}, $k = \overline{1,\mu}$, the domain bounded by $g_{2k}^{-1} \cdot \gamma[\zeta_{2k},\zeta_{2k-1}]$. Take in each D_{2k}, $k = \overline{1,\mu}$, a point a_{N+k} and draw simple disjoint paths α_{N+k} through E on the left of α_0 such that $i(\alpha_j) = a_j$ and

$t(\alpha_j)$ lie in the unbounded component of $\mathbb{R}^2\backslash\langle\gamma\rangle$, $j = \overline{N+1, N+\mu}$. Moreover, we can choose paths α_j, $j = \overline{N+1, N+\mu}$, in such a way that the crossing point of α_j and $g_{2\ell}$ preceeds, on g, the crossing point of α_s and $g_{2\ell}$ when $N+1 \leq j < s \leq \ell+N$, $1 \leq \ell \leq \mu$. Number the crossing points along each α_j, $j = \overline{N+1, N+\mu}$, consecutively so that the second coordinate of $\langle\alpha_j\rangle \cap \langle g_2\rangle$ is equal to μ. It is evident that all crossings $\langle\alpha_j\rangle \cap \langle g_{2\ell}\rangle$, $j = \overline{N+1, N+\mu}$, $1 \leq \ell \leq \mu$, are negative. Choose two points p_1 and p_2 such that $p_1 \in \langle\alpha_0\rangle \cap \widetilde{\widetilde{U}}(\gamma(\widetilde{\widetilde{\zeta}}))$, $p_2 \in \langle\alpha_0\rangle \cap \widetilde{U}(\gamma(\widetilde{\zeta}))$, p_1 preceeds p_2 on α_0 and $(\langle\gamma^*\rangle\backslash\langle g\rangle) \cap \langle\alpha_0[p_1, p_2]\rangle = \varnothing$. Denote by q an arc such that $i(q) = p_1$, $t(q) = p_2$, $\langle q\rangle \cap (\langle\gamma\rangle\backslash\langle\gamma[\zeta', \zeta_0]\rangle) = \varnothing$ and $\langle q\rangle \cap \langle\gamma[\zeta', \zeta_0]\rangle$ consists of a unique point. Denote $\widetilde{\alpha_0} = \alpha_0[a_0, p_1] \cdot q \cdot \alpha_0[p_2, b_0]$.

Consider the system of rays $\widetilde{\alpha}_0, \alpha_1, \ldots, \alpha_N, \alpha_{N+1}, \ldots, \alpha_{N+\mu}$. As $g_{2\mu+1}$ is transversal to all g_{2k+1}, $k = \overline{0, \mu-1}$, we have at least one initial point of rays in each bounded component of $\mathbb{R}^2\backslash\langle\gamma^*\rangle$. Thus we obtain the raying for γ^*.

Now we are going to construct an assemblage for γ^*. Since the existence of an assemblage does not depend on a choice of the initial point of the curve, we can assume that $i(\gamma^*) = \gamma(\zeta_0)$. Then the word for γ^* has a following form:

$$\begin{aligned}
&(N+1, \mu, -1)(N+2, \mu, -1)(N+3, \mu, -1)\cdots(N+\mu, \mu, -1)\\
&(N+2, \mu-1, -1)(N+3, \mu-1, -1)(N+4, \mu-1, -1)\cdots(N+\mu, \mu-1, -1)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(N+\mu-1, 1, -1)(N+\mu, 2, -1)(N+\mu, 1, -1)S_0\\
&(N+\mu, \mu+r_1, 1)(N+\mu-1, \mu+r_1, 1)\cdots(N+1, \mu+r_1, 1)S_1\\
&(N+\mu, \mu+r_2, 1)(N+\mu-1, \mu+r_2, 1)\cdots(N+1, \mu+r_2, 1)S_2\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(N+\mu, \mu+r_n, 1)(N+\mu-1, \mu+r_n, 1)\cdots(N+1, \mu+r_n, 1)S_n,
\end{aligned}\tag{1}$$

where $n \geq \mu$ and $S_0, S_1, \ldots, S_n$ are the sets of letters belonging to rays $\widetilde{\alpha}_0, \alpha_1, \ldots, \alpha_N$. Select from list (1) all letters belonging to $\bigcup_{i=1}^{\mu}\langle\gamma_{k_i}\rangle \cap \langle\widetilde{\alpha}_0\rangle$. Let $r_{n_1}, r_{n_2}, \ldots, r_{n_\mu}$ be the second coordinates of these letters arranged in the order as they are encountered in word (1). Replace the numbering of arcs $\gamma_{k_1}, \ldots, \gamma_{k_\mu}$ in the following way. The arc containing the letter with the second coordinate r_{n_i} will have a new number $\mu+i$ and be denoted by $\gamma^*_{\mu+i}$. Then we replace the second coordinates of letters belonging to $\gamma^*_{\mu+i}$ by $\mu+i$ and mark the letters with changed second coordinates by the superscript $*$.

Consider the following list of sets:

$$\begin{gathered}
\Big\{(N+1, \mu, -1), (N+1, 2\mu, 1)^*\Big\}, \Big\{(N+2, \mu, -1), (N+2, 2\mu, 1)^*\Big\}, \ldots\\
\Big\{(N+\mu, \mu, -1), (N+\mu, 2\mu, 1)^*\Big\},\\
\Big\{(N+2, \mu-1, -1), (N+2, 2\mu-1, -1)^*\Big\}, \ldots\\
\Big\{(N+\mu, \mu-1, -1), (N+\mu, 2\mu-1, 1)^*\Big\}, \ldots,
\end{gathered}$$

$$\Big\{(N+k,\mu-(k-1),-1),(N+k,2\mu-k+1,1)^*\Big\}, \tag{2}$$
$$\Big\{(N+k+1,\mu-(k-1),-1),(N+k+1,2\mu-k+1,1)^*\Big\},\dots,$$
$$\Big\{(N+\mu,\mu-(k-1),-1),(N+\mu,2\mu-k+1,1)^*\Big\},\dots$$
$$\Big\{(N+\mu-1,2,-1),(N+\mu-1,\mu+2,1)^*\Big\},$$
$$\Big\{(N+\mu,2,-1),(N+\mu,\mu+2,1)^*\Big\},\Big\{(N+\mu,1,-1),(N+\mu,\mu+1,1)^*\Big\}.$$

Include all sets of list (2) as well as all sets of $\mathcal{G}(\gamma)$ into $\mathcal{G}(\gamma^*)$. $\mathcal{B}(\gamma^*)$ consists of all sets of $\mathcal{B}(\gamma)$ except B_0. We claim that $\mathcal{G}(\gamma^*)\cup\mathcal{B}(\gamma^*)$ is an assemblage for γ^*. It is evident that conditions (a), (c) (d), (e) are fulfilled.

Recall that $B=\{0,k_0,1),(0,\mu+1,1)^*,(0,\mu+2,1)^*,\dots,(0,2\mu,1)^*\}$ was included in $\mathcal{A}(\gamma)$ and hence the set $[\theta_0,\theta_{2\mu+1}]\cup(\gamma^*)^{-1}(\bigcup\limits_{j=1}^{\mu}\langle\gamma^*_{\mu+j}\rangle)$ does not separate, on S^1, any $(\gamma^*)^{-1}(A)$ for $A\in\mathcal{A}(\gamma)$, $A\neq B$. Thus we have to check that for any two sets A_1 and A_2 from the (2) $(\gamma^*)^{-1}(A_1)$ does not separate, on S^1, the set $(\gamma^*)^{-1}(A_2)$.

Consider two sets $A_i=\{(N+k_i,m_i,-1),(N+k_i,\mu+m_i,1)\}$, $i=1,2$. Suppose that $(\gamma^*)^{-1}(N+k_1,m_1,-1)$ preceeds $(\gamma^*)^{-1}(N+k_2,m_2,-1)$. Then either $m_1>m_2$, or $m_1=m_2$ and $k_1<k_2$. In the former case $\mu+m_2<\mu+m_1$ and hence $(N+k_1,\mu+m_1,1)^*$ follows $(N+k_2,\mu+m_2,1)^*$. In the latter case the crossing points $(N+k_1,\mu+m_1,1)^*$ and $(N+k_2,\mu+m_1,1)^*$ lie on the same $\gamma^*_{\mu+m_1}$ and as $k_1<k_2$, $(N+k_2,\mu+m_1,1)^*$ preceeds $(N+k_1,\mu+m_1,1)^*$ on $\gamma^*_{\mu+m_1}$. In both cases $(\gamma^*)^{-1}(A_1)$ does not separate $(\gamma^*)^{-1}(A_2)$ on S^1. Thus $\mathcal{A}(\gamma^*)$ is an assemblage for γ^*. Moreover, $\mathcal{A}(\gamma^*)$ is a maximal assemblage since

$$\sum_{B\in\mathcal{B}(\gamma^*)}(\operatorname{card}B-1)=\sum_{B\in\mathcal{B}(\gamma)\setminus B_0}(\operatorname{card}B-1)=\tau(\gamma)-\mu=\tau(\gamma^*).$$

Denote by $f^*(\zeta)$, $\zeta\in\overline{D}^2$, the interior extension of γ^* and let ℓ be the arc of S^1 corresponding to g under the mapping f^*.

Now we are going to construct some bordered Riemann surface using the procedure of identification of boundary arcs of the domains D_k, $k=\overline{1,2\mu+1}$. Write boundaries of D_1 and D_2 in the form

$$\partial D_1=\gamma[\zeta_0,\widetilde{\zeta}]\cdot\gamma[\widetilde{\zeta},\zeta_1]g_1^{-1},$$
$$\partial D_2=g_2^{-1}\cdot\gamma[\zeta_1,\widetilde{\zeta}]\cdot\gamma[\widetilde{\zeta},\zeta_2]$$

and paste $\overline{D}_1$ and $\overline{D}_2$ along $\langle\gamma[\zeta_1,\widetilde{\zeta}]\rangle$. Denote by $\widetilde{D}_2$ the obtained domain. We have

$$\partial\widetilde{D}_2=\gamma[\zeta_0,\widetilde{\zeta}]\cdot\gamma[\widetilde{\zeta},\zeta_2]\cdot g_2^{-1}\cdot g_1^{-1}$$

and

$$\partial D_3=\gamma[\zeta_2,\widetilde{\zeta}]\cdot\gamma[\widetilde{\zeta},\zeta_3]\cdot g_3^{-1}$$

After pasting $\overline{\overline{D}}_2$ and $\overline{D}_3$ along $\langle\gamma[\zeta_2, \widetilde{\zeta}]\rangle$, we obtain the domain $\widetilde{D}_3$ which is no longer one-sheeted and

$$\partial\widetilde{D}_3 = \gamma[\zeta_0, \widetilde{\zeta}] \cdot \gamma[\widetilde{\zeta}, \zeta_3] \cdot g_3^{-1} \cdot g_2^{-1} \cdot g_1^{-1}.$$

After 2μ steps of pasting, we obtain the many-sheeted domain $D_{2\mu+1}$ and

$$\partial\widetilde{D}_{2\mu+1} = \gamma[\zeta_0, \widetilde{\zeta}] \cdot \gamma[\widetilde{\zeta}, \zeta_{2\mu+1}] \cdot g_{2\mu+1}^{-1} \cdot g_{2\mu}^{-1} \cdots g_1^{-1} = \gamma[\zeta_0, \zeta_{2\mu+1}] \cdot g^{-1}.$$

Denote by G the image of $D_{2\mu+1}$ under the mapping $h(w) = \sqrt[\mu+1]{w - \gamma(\widetilde{\zeta})}$. G is a simply-connected domain bounded by a simple closed curve.

Let D' be a simply connected domain such that $D' \subset C\overline{D}^2$ and $\overline{D}'\cap\overline{D}^2 = \langle\ell\rangle$. Consider the function $h(f^*(e^{i\theta}))$, $e^{i\theta} \in \langle\ell\rangle$, where we mean that the points $f^*(e^{i\theta})$ belonging to $\langle g\rangle$ are located on the boundary of the many-sheeted domain $\widetilde{D}_{2\mu+1}$. Then $e^{i\theta} \to h(f^*(e^{i\theta}))$ is a homeomorphism of $\langle\ell\rangle$ onto some arc of the boundary of the domain G. Let H be a homeomorphism of $\overline{D}'$ onto $\overline{G}$ such that

$$H\Big|_{\langle\ell\rangle} = h \circ f^*. \tag{3}$$

The existence of such a homeomorphism H is evident (it follows from Lemma 6.1 of [11]). Consider the function

$$\widetilde{f}(z) = \begin{cases} f^*(z), & \text{if} \quad z \in \overline{D}^2, \\ h^{-1} \circ H, & \text{if} \quad z \in \overline{D}'. \end{cases}$$

By (3) $\widetilde{f}(z)$ is continuous in $\overline{D}^2 \cup \overline{D}' \cup \langle\ell\rangle$. As $f^*(z)$ and $h^{-1} \circ H$ are interior mappings in D^2 and D', respectively, and H and h are homeomorphisms on $\langle\ell\rangle$ and $\langle g\rangle$, respectively, $\widetilde{f}(2)$ is an interior function in $D^2 \cup D' \cup \langle\ell\rangle$. The behavior of $\widetilde{f}$ in a vicinity of the point $m \in \langle\partial(D^2 \cup D' \cup \langle\ell\rangle\rangle$, $\widetilde{f}(m) = \gamma(\widetilde{\zeta})$ is defined by the behavior of h at the point $\gamma(\widetilde{\zeta})$ and hence m is a partial critical point of $\widetilde{f}$ of multiplicity μ. $\square$

Now we will prove the inverse statement.

Theorem 2. *If a normal curve γ has an interior extension possessing partial critical points, then there exists an interior extension of γ without partial critical points.*

Let $\widetilde{f}$ be an interior extension of γ. Denote by ζ_j, $j = \overline{1, m}$, partial critical points of $\widetilde{f}$, and by μ_j their multiplicities. Let $U_0(\zeta_j)$ be a neighborhood of ζ_j such that $\widetilde{f}|_{\overline{U_0(\zeta_j)} \cap S^1}$ is a simple arc. Let $\ell_j = [\zeta_j', \zeta_j'']$ be an arc of S^1 such that $\zeta_j \in \langle\ell_j\rangle \subset U_0(\zeta_j)$ and let $U(\zeta_j)$ be a neighborhood of ζ_j such that $\partial U^-(\zeta_j) = \ell_j' \cdot \ell_j^{-1}$, where ℓ' is a simple arc $\langle\ell_j'\rangle \cap \overline{D}^2 = \{\zeta_j', \zeta_j''\}$. Denote $L_j = \widetilde{f} \circ \ell_j$, and consider a neighborhood $U(\gamma(\zeta_j))$ such that $\partial U(\gamma(\zeta_j))$ is a simple arc and $\partial U(\gamma(\zeta_j)) \cap \langle\gamma\rangle = \{\widetilde{f}(\zeta_j'), \widetilde{f}(\zeta_j'')\}$. Denote $\partial U^-(\gamma(\zeta_j)) = L_j' \cdot L_j^{-1}$ and let φ :

$\langle \ell'_j \rangle \to \langle L'_j \rangle$ be an arbitrary homeomorphism $\varphi(\zeta'_j) = \widetilde{f}(\zeta'_j)$, $\varphi(\zeta''_j) = \widetilde{f}(\zeta''_j)$. Denote by Φ the homeomorphism

$$\Phi : \overline{U}^{-}(\zeta_j) \to \overline{U}^{-}(\gamma(\zeta_j))$$

coinciding with φ on $\partial U^-(\zeta_j)$.

The mapping

$$f_j = \begin{cases} \widetilde{f}(z), & z \in \overline{D}^2, \\ \Phi(z), & z \in \overline{U}^{-}(\zeta_j) \end{cases}$$

is an interior mapping of the simply connected domain $D^2 \cup U^-(\zeta_j) \cup \langle \ell_j \rangle$ and from the definition of a partial critical point it follows that ζ_j is a (interior) critical point of f_j of multiplicity μ_j.

Repeat the above described procedure for every partial critical point of $\widetilde{f}$. Finally, we obtain a function f without partial critical points. The restriction of f to the boundary of the domain of definition is a representation of a normal curve Γ which differs from γ. Applying the theorem of Section 2 to Γ we conclude that there exists an assemblage $\mathcal{A}(\Gamma)$. As the partial critical points ζ_j, $j = \overline{1,m}$, of $\widetilde{f}$ have became the interior critical points of f, each ζ_j, $j = \overline{1,m}$, must be the initial point for a ray α_j, $j = \overline{1,m}$, included into the raying $R(\Gamma)$ generating $\mathcal{A}(\Gamma)$.

Take m points z_j, $z_j \in U^+(\gamma(\zeta_j))$, $j = \overline{1,m}$, and connect z_j with ζ_j by a simple arc λ_j in $U^+(\gamma(\zeta_j))$. Consider a new system R' of rays obtained from $R(\Gamma)$ by replacing of α_j by $\alpha'_j = \lambda_j \cdot \alpha_j$. It is clear that R' satisfies all conditions of raying for γ.

Let $B \in \mathcal{B}(\Gamma)$, and let B consist of points belonging to α'_j. If the crossing point of α_j with L'_j does not belong to B we include B into $\mathcal{B}(\gamma)$. If the crossing point of α_j with L'_j belongs to B, we replace this point by the crossing point of α'_j with ℓ and include obtained set B' into $\mathcal{B}(\gamma)$. All other sets of $\mathcal{B}(\Gamma)$ are included into $\mathcal{A}(\gamma)$ unchanged as well as all sets of $\mathcal{G}(\Gamma)$. It is evident that the set $\mathcal{B}(\gamma) \cup \mathcal{G}(\Gamma)$ is an assemblage for γ. As $\operatorname{card} B' = \operatorname{card} B$ the multiplicity of every z_j is equal to μ_j, $j = \overline{1,m}$. Hence by Theorem 24.2 of [11] $\mathcal{A}(\gamma)$ is a maximal assemblage for γ.

4. The existence of an interior exstension of a nonnormal curve

Let $\gamma(\zeta) \equiv \gamma(e^{i\theta})$ be a parametrization of a closed curve γ. The point $\widehat{\zeta} = e^{i\widehat{\theta}}$ is called a reversing point if there exist $\varepsilon' > 0$, $\varepsilon'' > 0$ and a sense reversing homeomorphism $\psi_{\widehat{\zeta}} : [e^{i(\theta-\varepsilon')}, e^{i\widehat{\theta}}] \to [e^{i\widehat{\theta}}, e^{i(\widehat{\theta}+\varepsilon'')}]$ such that

$$\gamma(\psi_{\widehat{\zeta}}(e^{i\theta})) = \gamma(e^{i\theta}), \quad \theta \in [\widehat{\theta} - \varepsilon', \theta].$$

Let f be the interior extension of γ and let $\widehat{\zeta}$ be a reversing point of γ. We say that $\widehat{\zeta}$ is a reversing critical point of $f(z)$ (or $f(\widehat{\zeta})$ is a reversing branch point of f^{-1}) of multiplicity $\mu(f, \widehat{\zeta}) \geq 0$ if there exist some $U(\widehat{\zeta})$, $U(f(\widehat{\zeta}))$ and $\mu(f, \widehat{\zeta})$ simple arcs ℓ_j passing through $U^+(\widehat{\zeta})$ and such that $\langle \ell_j \rangle$ have a unique common point at $\widehat{\zeta}$, $f(z)$ is schlicht in every component v_j, $j=\overline{1,\mu(f,\widehat{\zeta})+1}$, of $U(\widehat{\zeta})\setminus\cup\langle\ell_j\rangle$ and $f(z)$ takes every V_j onto $U(f(\widehat{\zeta}))\setminus\langle\gamma[e^{i(\theta-\varepsilon')},e^{i(\theta+\varepsilon'')}]\rangle$.

Let $Q=(q_1,q_2,\dots,q_m)$ be an ordered m-tuple of points of $\langle\gamma\rangle$ distinct from the chosen initial point of γ. It is allowed that there are coinciding points among q_j but we require that every pair of points with succesive indices consists of distinct points. Denote by γ_1 the arc of γ such that $i(\gamma_1)=i(\gamma)$ and $t(\gamma_1)=q_1$, by γ_2 the arc of γ or γ^{-1}, $i(\gamma_2)=q_1$, $t(\gamma_2)=q_2$ etc, $i(\gamma_{m+1})=q_m$, $t(\gamma_{m+1})=t(\gamma)$. Consider the curve $\gamma_Q=\gamma_1\cdot\gamma_2\cdots\gamma_{m+1}$.

The curve coincides with γ iff every q_{j-1} preceeds q_j on γ, $j=\overline{2,m}$. Denote by Q' the subset of Q consisting of points q_k not separating q_{k-1} and q_{k+1} on γ, $k=\overline{2,m-1}$. Denote $n=\operatorname{card}Q'$. It is clear that n is even. Let $\overline{\mu}=(\mu_1,\mu_2,\dots,\mu_m)$ be a vector where μ_j, $j=\overline{1,m}$, are positive integers. Denote $|\overline{\mu}|=\sum\limits_{j=1}^{m}\mu_j$ and $\nu=n/2$.

Theorem 3. *Let γ be a normal curve and let f be an interior extension of γ with a partial critical point $\widetilde{\zeta}$ of multiplicity $\widetilde{\mu}$. Then for any $Q\subset U(f(\widetilde{\zeta}))$ and $\overline{\mu}$ such that $|\overline{\mu}|+\nu=\widetilde{\mu}$ there exists an interior extension $\widehat{f}$ of γ_Q such that every $q_j\in Q\setminus Q'$ is a partial branch point and $q_j\in Q'$ is a reversing branch point of $\widehat{f}^{-1}$ of multiplicity μ_j, $j=\overline{1,m}$.*

Denote $n_k=\operatorname{card}(Q'\cap\{q_1,q_2,\dots,q_k\})$ and by $\widehat{\zeta}_j$, $j=\overline{1,m}$, the points corresponding to q_j, $j=\overline{1,m}$, under the mapping f. Let the set E, points ζ_k, $k=0,2\mu+1$, and curves g_k, $k=\overline{1,2\widetilde{\mu}+1}$, be chosen as in Section 3. We assume that $U(f(\widetilde{\zeta}))$ is chosen so that $\langle g_k\rangle\cap U(f(\widetilde{\zeta}))=\varnothing$, $k=\overline{1,2\widetilde{\mu}+1}$. Denote $f|_{S^1}=\gamma(\zeta)$. Then

$$\partial D_j=\gamma[\zeta_{j-1},\widehat{\zeta}_1]\cdot\gamma[\widehat{\zeta},\zeta_j]g_j^{-1},\quad j=1,2\mu_1+1+n_1.$$

Paste $\overline{D}_1$ and $\overline{D}_2$ along $\langle\gamma[\widehat{\zeta}_1,\zeta_1]\rangle$ and denote the obtained domain by $\overline{\widehat{D}}_2$. We have

$$\partial\widehat{D}_2=\gamma[\zeta_0,\widehat{\zeta}_1]\cdot\gamma[\widehat{\zeta}_1,\zeta_2]\cdot g_2^{-1}\cdot g_1^{-1}.$$

After pasting $\overline{\widehat{D}}_2$ and $\overline{D}_3$ along $\langle\gamma[\zeta_2,\widehat{\zeta}_1]\rangle$, we obtain $\overline{\widehat{D}}_3$ and

$$\partial\widehat{D}_3=\gamma[\zeta_0,\widehat{\zeta}_1]\cdot\gamma[\widehat{\zeta}_1,\zeta_3]\cdot g_3^{-1}\cdot g_2^{-1}\cdot g_1^{-1}.$$

After $2\mu_1+n_1$ steps of pasting we obtain the many-sheeted domain $\widehat{D}_{2\mu_1+1+n_1}$ with the boundary

$$\partial\widehat{D}_{2\mu_1+1+n_1}=\gamma[\zeta_0,\widehat{\zeta}_1]\cdot\gamma[\widehat{\zeta}_1,\zeta_{2\mu_1+1+n_1}]\cdot g_{2\mu_1+1+n_1}^{-1}\cdots g_1^{-1}.$$

The next domain $\overline{D}_{2\mu_1+n_1+2}$ is pasted with $\overline{\widehat{D}}_{2\mu_1+n_1+1}$ along $\langle\gamma[\zeta_{2\mu_1+n_1+1},\widehat{\zeta}_2]\rangle$. The procedure is continued by pasting $\overline{D}_{2\mu_1+n_1+k}$ with $\overline{\widehat{D}}_{2\mu_1+n_1+k-1}$ along $\langle\gamma[\zeta_{2\mu_1+n_1+k},\widehat{\zeta}_2]\rangle$ for $k\le 2\mu_1+n_2+1$. The domain $\overline{D}_{2\mu_1+2\mu_2+n_2+2}$ is pasted with $\overline{\widehat{D}}_{2\mu_1+2\mu_2+n_2+1}$ along $\langle\gamma[\zeta_{2\mu_1+2\mu_2+n_2+1},\widehat{\zeta}_3]\rangle$. The general rule of pasting is the following: the domain $\overline{D}_{2\sum\limits_{j=1}^{s-1}\mu_j+n_{s-1}+k}$ is pasted with $\overline{\overline{D}}_{2\sum\limits_{j=1}^{s-1}\mu_j+n_{s-1}+k-1}$ along $\langle\gamma[\zeta_{2\sum\limits_{j=1}^{s-1}\mu_j+k}, \widehat{\zeta}_s]\rangle$ for $2 \le k \le 2\mu_s + n_s + 1$. The domain $\overline{D}_{2\sum\limits_{j=1}^{s}\mu_j+n_{s-1}+2}$ is pasted with $\overline{\overline{D}}_{2\sum\limits_{j=1}^{s}\mu_j+n_{s-1}+1}$ along $\langle\gamma[\zeta_{2\sum\limits_{j=1}^{s}\mu_j+n_s+1}, \widehat{\zeta}_{s+1}]\rangle$, $1 \le s \le m-1$, $n_0 = 0$ and the domain $\overline{\overline{D}}_{2\sum\limits_{j=1}^{m} n_m} = \overline{\overline{D}}_{2\left(\sum\limits_{j=1}^{m}\mu_j+\nu\right)} = \overline{\overline{D}}_{2\widetilde{\mu}}$ is pasted with $\overline{D}_{2\widetilde{\mu}+1}$ along $\langle\gamma[\widehat{\zeta}_m, \zeta_{2\widetilde{\mu}+1}]\rangle$. We have

$$\partial\widehat{D}_{2\sum\limits_{j=1}^{s-1}\mu_j+n_{s-1}+k} = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma\Big[\widehat{\zeta}_1, \zeta_{2\sum\limits_{j=1}^{s-1}\mu_j+n_{s-1}+k}\Big] \cdot g^{-1}_{2\sum\limits_{j=1}^{s-1}\mu_j+n_{s-1}+k} \cdots g_1^{-1}.$$

and

$$\partial\widehat{D}_{2\widetilde{\mu}+1} = \gamma[\zeta_0, \widehat{\zeta}_1] \cdot \gamma[\widehat{\zeta}_1, \widehat{\zeta}_2] \cdots \gamma[\widehat{\zeta}_m, \zeta_{2\widetilde{\mu}+1}] \cdot g^{-1}_{2\widetilde{\mu}+1} \cdot g^{-1}_{2\widetilde{\mu}} \cdot g_1^{-1}. \tag{4}$$

We claim that $\widehat{D}_{2\widetilde{\mu}+1}$ is a simply connected domain. Indeed, all domains involved in the pasting procedure are simply connected and each domain is pasted with only one other domain along only one boundary arc.

Let $\widehat{h} : \widehat{D}_{2\widetilde{\mu}+1} \to G$ be a homeomorphism onto a one-sheeted simply connected domain. Let the domain D' be the same as in the proof of Theorem 1, and let $\widehat{H}$ be a homeomorphism of D' onto G such that

$$\widehat{H}\Big|_{\langle\ell\rangle} = \widehat{h}(f(e^{i\theta})), \quad e^{i\theta} \in \langle\ell\rangle.$$

From (4) it follows that the function

$$\widehat{f} = \begin{cases} f(z), & z \in \overline{D}^2, \\ h^{-1} \circ \widehat{H}, & z \in \overline{D}' \end{cases}$$

is an interior extension of γ_Q.

To complete the proof, we have to show that the point q_j is a branch point (partial or reversing) of multiplicity μ_j, $j = \overline{1, m}$. It is clear that without loss of generality we can assume that $\widehat{f}$ is defined in $\overline{D}^2$, and the preimages of q_j under $\widehat{f}$ are the same points $\widehat{\zeta}_j$, $j = \overline{1, m}$.

Take some $\widehat{\zeta}_s$ and calculate the number of domains D_k such that q_s belongs to $\overline{D}_k$. From the construction of $\widehat{D}_k$ it follows that q_s belongs to $\overline{D}_{2\sum\limits_{j=1}^{s-1}\mu_j+n_{s-1}+k}$, $1 \le k \le 2\mu_s + n_s + 1$. Hence q_s belongs to $2\mu_s + 1$ domains D_k if $n_s = n_{s-1}$ and to $2\mu_s + 2$ domains D_k if $n_s = n_{s+1} + 1$. Thus $\widehat{\zeta}_s$ is a partial critical point of multiplicity μ_s if $n_s = n_{s-1}$ (q_s separates q_{s-1} and q_{s+1}) and $\widehat{\zeta}_s$ is a reversing critical point of multiplicity μ_s if $n_s = n_{s-1} + 1$ (q_s does not separate q_{s-1} and q_{s+1}).

References

1. C. J. Titus, The combinatorial topology of analytic functions on the boundary of a disk. *Acta Mathematica* **106**(1961), No. 1–2, 45–64.
2. S. J. Blank, Extending immersions of the circle. *Brandeis University, Waltham, Mass*, 1967.
3. M. L. Marx, Normal curves arising from light open mappings of the annulus. *Trans. Amer. Math. Soc.* **120**(1965), 46–56.
4. M. L. Marx, The branch point structure of extensions of interior boundaries. *Trans. Amer. Math. Soc.* **131**(1968), 79–97.
5. M. L. Marx, Extensions of normal immersions of S into R. *Trans. Amer. Math. Soc.* **187**(1974), 309–325.
6. G. K. Francis, Assembling compact Riemann surfaces with given boundary curves and branch points on the sphere. *Illionis J. Math.* **20**(1976), 198–217.
7. R. F. Verhey, Diffeomorphic invariants of immersed circles. *Trans. Amer. Math. Soc.* **163**(1973), 47–63.
8. C. L. Ezell and M. L. Marx, Branched extensions of curves in orientable surfaces. *Trans. Amer. Math. Soc.* **259**(1980), 515–547.
9. H. Whitney, On regular closed curves in the plane. *Compositio Mathematica* **4**(1937) 276–284.
10. S. Stoïlow, Leçons sur les principes topologiques de la théorie des fonctions analytiques. *Gauthier–Villars, Paris*, 1956.
11. M. Morse, Topological methods in the theorie of functions of a complex variable. *Princeton University Press, Princeton*, 1947.

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