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A FIXED POINT THEOREM OF LEGGETT–WILLIAMS TYPE WITH APPLICATIONS TO SINGLE- AND MULTIVALUED EQUATIONS

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Abstract. We establish a general fixed point theorem for multivalued maps defined on cones in Banach spaces. Applications to single and multivalued equations are presented.

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1. Introduction

In [7] we studied the existence of nonnegative solutions to the integral equation

$$y(t) = h(t) + \int_0^1 k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0,1]. \tag{1.1}$$

By imposing strong conditions on the kernel k and rather mild conditions on the nonlinearity f we were able to establish the existence via Krasnoselskii's fixed point theorem in a cone. In particular, the following condition was assumed on the kernel k:

$$\begin{cases} \exists\, M,\ 0 < M < 1,\ \kappa \in L^1[0,1] \ \text{ and an interval } \ [a,b] \subseteq [0,1],\ a < b, \\ \text{such that } \ k(t,s) \geq M\,\kappa(s) \geq 0 \ \text{ for } \ t \in [a,b] \ \text{ and a.e. } \ s \in [0,1], \\ \text{together with } \ k(t,s) \leq \kappa(s) \ \text{ for } \ t \in [0,1] \ \text{ and } \ \text{a.e. } \ s \in [0,1]. \end{cases} \tag{1.2}$$

Condition (1.2) was motivated from the theory of differential equations [3,4]. However for the general integral equation (1.1) condition (1.2) may be too restrictive. In this paper we strengthen the conditions on the nonlinearity f and weaken the conditions on the kernel k, and again we establish the existence of a nonnegative solution to (1.1). Also, we establish the existence to general discrete and multivalued equations. To establish the existence we use the fixed point theorem which was established by Leggett and Williams [6] in the single-valued case. By using a result of the authors [2] we are able to present a multivalued version of the Leggett–Williams theorem. This result will be needed to discuss multivalued equations.

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2. Fixed Point Theory

Let X be a retract of some Banach space $E = (E, \|\,.\,\|)$. Suppose for every open subset U of X and every upper semicontinuous map $A : \overline{U^X} \to 2^X$ (here 2^X denotes the family of nonempty subsets of X) which satisfies property (B) (to be specified later) with $x \notin A\,x$ for $x \in \partial_X U$ (here $\overline{U^X}$ and $\partial_X U$ denote the closure and boundary of U in X, respectively) there exists an integer, denoted by $i_X(A, U)$, satisfying the following properties:

(P1) if $x_0 \in U$ then $i_X(\hat{x}_0, U) = 1$ (here $\hat{x}_0$ denotes the map whose constant value is x_0);

(P2) for every pair of disjoint open subsets U_1, U_2 of U such that A has no fixed points on $\overline{U^X} \setminus (U_1 \cup U_2)$,

$$i_X(A, U) = i_X(A, U_1) + i_X(A, U_2);$$

(P3) for every upper semicontinuous map $H : [0,1] \times \overline{U^X} \to 2^X$ which satisfies property (B) and $x \notin H(t, x)$ for $(t, x) \in [0,1] \times \partial_X U$,

$$i_X(H(1,\,.\,), U) = i_X(H(0,\,.\,), U);$$

(P4) if Y is a retract of X and $A(\overline{U^X}) \subseteq Y$, then

$$i_X(A, U) = i_Y(A, U \cap Y).$$

Also assume the family

$$\begin{aligned}\{\, i_X(A, U) : X \text{ a retract of a Banach space } E,\ U \text{ open in } \\ X,\ A : \overline{U^X} \to 2^X \text{ is upper semicontinuous,} \\ \text{satisfies property (B) and } x \notin A\,x \text{ on } \partial_X U \,\}\end{aligned}$$

is uniquely determined by the properties (P1)–(P4).

We note that property (B) is any property on the map so that the fixed point index is well defined. In the applications in this paper property (B) will mean that the map is compact with convex compact values. Other examples of maps with a well defined index (e.g. property (B) could mean that the map is countably condensing with convex compact values) may be found in [11].

In [2] we proved the following extension of the Petryshyn–Krasnoselskii theorem on compression and expansion of a cone. Let $C \subset E$ be a cone and $\eta > 0$ a constant. For notational purposes let

$$C_\eta = \{y \in C : \ \|y\| < \eta\}, \quad S_\eta = \{y \in C : \ \|y\| = \eta\} \quad \text{and} \quad \overline{C_\eta} = \{y \in C : \ \|y\| \le \eta\}.$$

Theorem 2.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space and X a retract of E. Suppose for every open subset U of X and every upper semicontinuous map $A : \overline{U^X} \to 2^X$ which satisfies property* (B) *with $x \notin A\,x$ for $x \in \partial_X U$*

there exists an integer $i_X(A,U)$ *satisfying* (P1)–(P4). *Furthermore, assume the family*

$$\{\, i_X(A,U) : X \ \text{ a retract of a Banach space } E,\ U \ \text{ open in } X,\ A : \overline{U^X} \to 2^X \text{ is upper semicontinuous, satisfies property (B) and } x \notin A\,x \ \text{ on } \partial_X U \,\}$$

is uniquely determined by the properties (P1)–(P4). *Let* $C \subset E$ *be a cone in* E, $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ *with* $R = \max\{r_1, r_2\}$ *and* $r = \min\{r_1, r_2\}$. *Let* $F : \overline{C_R} \to 2^C$ *be an upper semicontinuous map which satisfies property* (B) *and assume* $\mu = \sup\{\|F\,x\| : x \in \overline{C_{r_2}}\} < \infty$. *Furthermore, suppose* F *satisfies the following conditions:*

(H1) $x \notin \lambda\, F\, x$ *for* $\lambda \in [0,1]$ *and* $x \in S_{r_1}$;

(H2) $\exists v \in C \setminus \{0\}$ *with* $x \notin F\,x + \delta\, v$ *for* $\delta \geq 0$ *and* $x \in S_{r_2}$;

(H3) *the mapping* $H_3 : [0,1] \times \overline{C_{r_1}} \to 2^C$, *given by* $H_3(t,x) = t\,F\,x$, *satisfies property* (B);

(H4) $\exists \lambda_1 > \frac{(r_2+\mu)}{\|v\|}$ *such that the mapping* $H_4 : [0,1] \times \overline{C_{r_2}} \to 2^C$ *given by* $H_4(t,x) = F\,x + t\,\lambda_1\, v$ *satisfies property* (B).

Then F *has at least one fixed point* $y \in C$ *with* $r < \|y\| < R$.

We now use Theorem 2.1 to obtain a generalization of the Leggett–Williams fixed point theorem.

Theorem 2.2. *Let* $E = (E, \|\,.\,\|)$ *be a Banach space and* X *a retract of* E. *Suppose for every open subset* U *of* X *and every upper semicontinuous map* $A : \overline{U^X} \to 2^X$ *which satisfies property* (B) *with* $x \notin A\,x$ *for* $x \in \partial_X U$ *there exists an integer* $i_X(A,U)$ *satisfying* (P1)–(P4). *Furthermore, assume the family*

$$\{\, i_X(A,U) : X \ \text{ a retract of a Banach space } E,\ U \ \text{ open in } X,\ A : \overline{U^X} \to 2^X \text{ is upper semicontinuous, satisfies property (B) and } x \notin A\,x \ \text{ on } \partial_X U \,\}$$

is uniquely determined by the properties (P1)–(P4). Let $C \subset E$ *be a cone in* E, $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ *with* $R = \max\{r_1, r_2\}$ *and* $r = \min\{r_1, r_2\}$. *Let* $F : \overline{C_R} \to 2^C$ *be an upper semicontinuous map which satisfies property* (B) *and assume* $\mu = \sup\{\|F\,x\| : x \in \overline{C_{r_2}}\} < \infty$. *Furthermore, assume the following conditions are satisfied:*

(H5) $\exists\, u_0 \in C \setminus \{0\}$ *with* $y \not\leq x$ *for all* $y \in F\,x$ *and* $x \in S_{r_2} \cap C(u_0)$, *here* $C(u_0) = \{u \in C : \ \exists \lambda > 0 \ \text{ with } \ u \geq \lambda\, u_0\}$;

(H6) $\|y\| \leq \|x\|$ *for all* $y \in F\,x$ *and* $x \in S_{r_1}$.

Also suppose (H3) *and* (H4) *hold. Then* F *has at least one fixed point* $y \in C$ *with* $r \leq \|y\| \leq R$.

Proof. We assume $x \notin F\,x$ for $x \in S_r \cup S_R$ (otherwise we are finished). First we show condition $(H1)$ of Theorem 2.1 holds. Certainly $(H1)$ is true if $\lambda = 1$ since we are assuming $x \notin F\,x$ for $x \in S_r \cup S_R$. If $x \in \lambda\,F\,x$ for some $\lambda \in [0,1)$ and $x \in S_{r_1}$, then there exists $y \in F\,x$ with $x = \lambda\,y$ and so $(H6)$ implies that

$$r_1 \,=\, \|x\| \,=\, |\lambda|\,\|y\| \,<\, \|y\| \,\leq\, \|x\| \,=\, r_1,$$

which is a contradiction. Thus condition $(H1)$ of Theorem 2.1 holds.

Next we show condition $(H2)$ of Theorem 2.1 is true. Certainly, $(H2)$ is true if $\delta = 0$ since we assume $x \notin F\,x$ for $x \in S_r \cup S_R$. Now if $(H2)$ were false (for $\delta > 0$), then for each $v \in C \setminus \{0\}$ there would exist $\delta_v > 0$ and $x_v \in S_{r_2}$ with $x_v \in F\,x_v + \delta_v\,v$. In particular, since $u_0 \in C \setminus \{0\}$, there exists $\delta > 0$ and $x \in S_{r_2}$ with $x \in F\,x + \delta\,u_0$. Now there exists $y \in F\,x$ with $x = y + \delta\,u_0$. Since $x - \delta\,u_0 = y \in C$ (since $F : \overline{C_R} \to 2^C$) we have $x \geq \delta\,u_0$ i.e. $x \in C(u_0)$. Moreover, since $\delta\,u_0 \in C$, we have

$$x \,=\, y \,+\, \delta\,u_0 \,\geq\, y.$$

This contradicts $(H5)$. Thus condition $(H2)$ of Theorem 2.1 holds. Now apply Theorem 2.1. $\square$

From Theorem 2.2 we can deduce a result for upper semicontinuous, condensing maps.

Theorem 2.3. *Let $E = (E, \|\,.\,\|)$ be a Banach space, $C \subset E$ a cone in E, $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ with $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$. Let $F : \overline{C_R} \to K(C)$ be an upper semicontinuous, compact* (*or more generally a condensing* [10]) *map* (*here $K(C)$ denotes the family of nonempty convex compact subsets of C*). *Also suppose* (H5) *and* (H6) *hold. Then F has at least one fixed point $y \in C$ with $r \leq \|y\| \leq R$.*

Proof. In this theorem property (B) means the map is compact (or more generally condensing) with nonempty, convex, compact values. If X is a retract of E, U an open subset of X and $A : \overline{U^X} \to K(X)$ is an upper semicontinuous, compact (or, more generally, condensing) map, then there exists [10] an integer $i_X(A, U)$ satisfying (P1)–(P4) and, moreover,

$$\begin{aligned}\{\, i_X(A,U) : X \;&\text{ a retract of a Banach space } E,\ U \text{ open in}\\ &X,\ A : \overline{U^X} \to K(X) \text{ is an upper semicontinuous}\\ &\text{compact (or more generally condensing) map and}\\ &x \notin A\,x \;\text{ on }\; \partial_X\, U \,\}\end{aligned}$$

is uniquely determined by the properties (P1)–(P4). Trivially $(H3)$ and $(H4)$ are true here so the result follows from Theorem 2.2. $\square$

A special case of Theorem 2.3 is the Leggett–Williams fixed point theorem [6].

Theorem 2.4. *Let* $E = (E, \|\,.\,\|)$ *be a Banach space,* $C \subset E$ *a cone in* E, $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ *with* $R = \max\{r_1, r_2\}$ *and* $r = \min\{r_1, r_2\}$. *Let* $F : \overline{C_R} \to C$ *be a single-valued continuous, compact (or more generally condensing) map with*

(H7) $\exists\, u_0 \in C \setminus \{0\}$ *with* $F\,x \not\leq x$ *for* $x \in S_{r_2} \cap C(u_0)$, *here* $C(u_0) = \{u \in C :\ \exists\, \lambda > 0 \quad$ *with* $u \geq \lambda\, u_0\}$*;*

and

(H8) $\|F\,x\| \leq \|x\|$ *for* $x \in S_{r_1}$

holding. Then F *has at least one fixed point* $y \in C$ *with* $r \leq \|y\| \leq R$.

3. Single-Valued Equations

To illustrate how the results of Section 2 can be applied, we first consider the integral equation

$$y(t) \;=\; h(t) + \int_0^1 k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0,1]. \tag{3.1}$$

The following conditions are assumed to be fulfilled:

$$f : [0,1] \times [0,\infty) \to [0,\infty) \quad \text{is continuous;} \tag{3.2}$$

$$\begin{gathered} k_t(s) = k(t,s) \in L^1[0,1] \quad \text{with} \quad k_t \geq 0 \\ \text{a.e. on} \quad [0,1], \quad \text{for each} \quad t \in [0,1]; \end{gathered} \tag{3.3}$$

$$\text{the map} \quad t \mapsto k_t \quad \text{is continuous from} \quad [0,1] \quad \text{to} \quad L^1[0,1]; \tag{3.4}$$

$$\inf_{t \in [0,1]} \int_0^1 k(t,s)\, ds \;>\; 0; \tag{3.5}$$

$$\left\{ \begin{array}{l} \text{there exists a nondecreasing continuous map} \\ \psi : [0,\infty) \to [0,\infty) \quad \text{and a continuous map} \quad \phi : [0,\infty) \to [0,\infty) \\ \ \text{with} \quad \phi(y) \leq f(t,y) \leq \psi(y) \ \text{for} \ \ (t,y) \in [0,1] \times [0,\infty); \end{array} \right. \tag{3.6}$$

$$h \in C[0,1] \quad \text{with} \quad h(t) \geq 0 \quad \text{for} \quad t \in [0,1]; \tag{3.7}$$

$$\exists\, r > 0 \quad \text{with} \quad r \;<\; \phi(r) \inf_{t \in [0,1]} \int_0^1 k(t,s)\, ds; \tag{3.8}$$

$$\frac{\phi(x)}{x} \quad \text{is nonincreasing on} \quad (0, r) \tag{3.9}$$

and

$$\exists\, R > r \quad \text{with} \quad R \;>\; \psi(R) \sup_{t \in [0,1]} \int_0^1 k(t,s)\, ds \;+\; |h|_0; \tag{3.10}$$

here $|h|_0 = \sup_{t \in [0,1]} |h(t)|$.

Theorem 3.1. *Suppose* (3.2)–(3.10) *hold. Then* (3.1) *has a nonnegative solution* $y \in C[0,1]$ *with* $r \le |y|_0 = \sup_{t\in[0,1]} |y(t)| < R$.

Remark 3.1. In (3.10) if $R \ge \psi(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds + |h|_0$, then the result of Theorem 3.1 is again true if $r \le |y|_0 < R$ is replaced by $r \le |y|_0 \le R$.

Proof of Theorem 3.1. Let

$$E = (C[0,1], |\,.\,|_0) \quad \text{and} \quad C = \{u \in C[0,1] :\ u(t) \ge 0 \ \text{ for } \ t \in [0,1]\}.$$

Also let $u_0 \equiv 1$ and note

$$C(u_0) = \{u \in C :\ \exists\, \lambda > 0 \ \text{ with } \ u(t) \ge \lambda \ \text{ for } \ t \in [0,1]\}.$$

Remark 3.2. Note

$$C(u_0) = \{u \in C :\ u(t) > 0 \ \text{ for } \ t \in [0,1]\}.$$

Since if $u(t) > 0$ for $t \in [0,1]$, then $u \in C[0,1]$ guarantees that there exists $t_0 \in [0,1]$ with $\min_{t\in[0,1]} u(t) = u(t_0)$, and consequently $u(t) \ge u(t_0) > 0$ for $t \in [0,1]$.

Now let $A : C \to C$ be defined by

$$A\,y(t) = h(t) + \int_0^1 k(t,s)\, f(s,y(s))\,ds \quad \text{for} \ \ t \in [0,1],$$

here $y \in C$. Notice (3.2)–(3.4), (3.7) and a standard argument [7] guarantees that

$$A : C \to C \quad \text{is continuous and completely continuous.}$$

We wish to apply Theorem 2.4 and so we first show

$$|A\,y|_0 \le |y|_0 \quad \text{for} \ \ y \in S_R. \tag{3.11}$$

To see this, notice if $y \in S_R$, then $|y|_0 = R$ and so (3.6) implies for $t \in [0,1]$ that

$$\begin{aligned} A\,y(t) = h(t) + \int_0^1 k(t,s)\, f(s,y(s))\,ds &\le |h|_0 + \psi(|y|_0) \int_0^1 k(t,s)\,ds \\ &\le |h|_0 + \psi(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds. \end{aligned}$$

This together with (3.10) gives

$$|A\,y|_0 \le |h|_0 + \psi(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds \; < \; R \; = \; |y|_0, \tag{3.12}$$

and so (3.11) is true.

Next we show

$$A\,y \not\le y \quad \text{for} \ \ y \in S_r \cap C(u_0). \tag{3.13}$$

To see this, let $y \in S_r \cap C(u_0)$ so that

$$|y|_0 = r \quad \text{and} \quad r \geq y(t) > 0 \;\; \text{for} \;\; t \in [0,1].$$

Now (3.6), (3.7) and (3.9) imply for $t \in [0,1]$ that

$$\begin{aligned} A\,y(t) &= h(t) + \int_0^1 k(t,s)\, f(s, y(s))\, ds \geq \int_0^1 k(t,s)\, \phi(y(s))\, ds \\ &= \int_0^1 k(t,s)\, \frac{\phi(y(s))}{y(s)}\, y(s)\, ds \geq \frac{\phi(r)}{r} \int_0^1 k(t,s)\, y(s)\, ds. \end{aligned}$$

Let $t_0 \in [0,1]$ be such that $\min_{t\in[0,1]} y(t) = y(t_0)$ and this together with the previous inequality yields for $t \in [0,1]$,

$$A\,y(t) \geq \frac{\phi(r)}{r}\, y(t_0) \int_0^1 k(t,s)\, ds \geq \left(\frac{\phi(r)}{r} \inf_{t\in[0,1]} \int_0^1 k(t,s)\, ds \right) y(t_0).$$

Use (3.8) to obtain

$$A\,y(t) \; > \; y(t_0) \quad \text{for} \;\; t \in [0,1].$$

In particular, $A\,y(t_0) > y(t_0)$ so that (3.13) is true. Now apply Theorem 2.4 to deduce that (3.1) has a nonnegative solution $y \in C[0,1]$ with $r \leq |y|_0 \leq R$. Note $|y|_0 \neq R$ since if $|y|_0 = R$, then from $y = A\,y$ we have (follow the ideas from (3.11) to (3.12)),

$$R = |y|_0 = |A\,y|_0 \leq |h|_0 + \psi(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds \; < \; R,$$

which is a contradiction. $\square$

Example 3.1. Consider the integral equation

$$y(t) = \int_0^1 k(t,s)\, [y(s)]^{\alpha}\, ds \quad \text{for} \;\; t \in [0,1], \tag{3.14}$$

with $0 < \alpha < 1$ and (3.3), (3.4) and (3.5) holding. Then (3.14) has a nonnegative solution $y \in C[0,1]$ with

$$|y|_0 \geq \left(\frac{1}{2} \inf_{t\in[0,1]} \int_0^1 k(t,s)\, ds \right)^{\frac{1}{1-\alpha}}.$$

Remark 3.3. Note $y \equiv 0$ is also a solution of (3.14).

To see this, let

$$f(t,y) = \psi(y) = \phi(y) = y^{\alpha}, \;\; h \equiv 0 \quad \text{and} \quad r = \left(\frac{1}{2} \inf_{t\in[0,1]} \int_0^1 k(t,s)\, ds \right)^{\frac{1}{1-\alpha}}.$$

Clearly, (3.2), (3.6) and (3.7) hold. To see (3.8), notice

$$\frac{r}{\phi(r)} = r^{1-\alpha} = \frac{1}{2} \inf_{t\in[0,1]} \int_0^1 k(t,s)\,ds \;<\; \inf_{t\in[0,1]} \int_0^1 k(t,s)\,ds.$$

Also (3.9) holds since $\frac{\phi(x)}{x} = \frac{1}{x^{1-\alpha}}$ and $0<\alpha<1$. To see (3.10), notice

$$\lim_{x\to\infty} \frac{x}{\psi(x)} = \lim_{x\to\infty} x^{1-\alpha} = \infty$$

so that there exists $R>r$ with (3.10) holding. The result follows from Theorem 3.1.

Example 3.2. Consider the integral equation

$$y(t) = \int_0^1 k(t,s)\,\left([y(s)]^{\alpha} + [y(s)]^{\beta}\right)\,ds \quad \text{for} \quad t\in[0,1], \tag{3.15}$$

with $0<\alpha<1$, $\beta\ge 1$ and (3.3), (3.4) and (3.5) holding. Furthermore, assume

$$\sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds \;<\; \frac{1}{2}. \tag{3.16}$$

Then (3.15) has a nonnegative solution $y\in C[0,1]$ with

$$\left(\frac{1}{2} \inf_{t\in[0,1]} \int_0^1 k(t,s)\,ds\right)^{\frac{1}{1-\alpha}} \;\le\; |y|_0 \;<\; 1.$$

To see this, let

$$\phi(y) = y^{\alpha},\;\; \psi(y) = y^{\alpha} + y^{\beta},\;\; h\equiv 0,$$

$$r = \left(\frac{1}{2} \inf_{t\in[0,1]} \int_0^1 k(t,s)\,ds\right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad R=1.$$

Clearly, (3.2), (3.6), (3.7)–(3.9) and (3.10) (with $R=1$, note (3.16)) hold. The result follows from Theorem 3.1.

Next we discuss the discrete equation

$$y(i) = h(i) + \sum_{j=0}^{N} k(i,j)\, f(j,y(j)) \quad \text{for} \quad i\in\{0,1,\dots,T\}\equiv T^+; \tag{3.17}$$

here $N,\, T\in \mathbf{N} = \{1,2,\dots\}$ and $T\ge N$. Let $C(N^+,\mathbf{R})$ denote the class of maps w continuous on T^+ (discrete topology) with norm $|w|_0 = \sup_{i\in T^+} |w(i)|$. The following conditions will be assumed to be fulfilled:

$$f : N^+\times[0,\infty)\to[0,\infty) \quad \text{is continuous (here} \quad N^+ = \{0,1,\dots,N\}); \tag{3.18}$$

$$k(i,j)\ge 0 \quad \text{for} \quad (i,j)\in T^+\times N^+; \tag{3.19}$$

$$\inf_{i\in N^+}\sum_{j=0}^{N}k(i,j)\;>\;0; \tag{3.20}$$

$$\begin{cases}\text{there exists a nondecreasing continuous map}\\ \psi:[0,\infty)\to[0,\infty)\ \text{ and a continuous map }\ \phi:[0,\infty)\to[0,\infty)\\ \ \text{with }\ \phi(y)\le f(j,y)\le\psi(y)\ \text{for }\ (j,y)\in N^+\times[0,\infty);\end{cases} \tag{3.21}$$

$$h\in C(T^+,\mathbf{R})\ \text{ with }\ h(i)\ge 0\ \text{ for }\ i\in T^+; \tag{3.22}$$

$$\exists\ r>0\ \text{ with }\ r\;<\;\phi(r)\inf_{i\in N^+}\sum_{j=0}^{N}k(i,j); \tag{3.23}$$

$$\frac{\phi(x)}{x}\quad\text{is nonincreasing on }\ (0,r) \tag{3.24}$$

and

$$\exists\ R>r\ \text{ with }\ R\;>\;\psi(R)\sup_{i\in T^+}\sum_{j=0}^{N}k(i,j)\;+\;|h|_0. \tag{3.25}$$

Theorem 3.2. *Suppose* (3.18)–(3.25) *hold. Then* (3.17) *has a nonnegative solution* $y\in C(T^+,\mathbf{R})$ *with* $r\le|y|_0\;<\;R$.

Remark 3.4. In (3.25) if $R\;\ge\;\psi(R)\sup_{i\in T^+}\sum_{j=0}^{N}k(i,j)\;+\;|h|_0$, then the result of Theorem 3.2 is again true if $r\le|y|_0<R$ is replaced by $r\le|y|_0\le R$.

Proof of Theorem 3.2. Let

$$E=(C(T^+,\mathbf{R}),|\,.\,|_0)\quad\text{and}\quad C=\{u\in C(T^+,\mathbf{R}):\ u(i)\ge 0\ \text{ for }\ i\in T^+\}.$$

Also let $u_0\equiv 1$ and note

$$C(u_0)=\{u\in C:\quad u(i)\;>\;0\ \text{ for }\ i\in T^+\}.$$

Now let $A:C\to C$ be defined by

$$A\,y(i)=h(i)+\sum_{j=0}^{N}k(i,j)\,f(j,y(j))\quad\text{for }\ i\in T^+,$$

here $y\in C$. The standard argument [1, 4] guarantees that

$$A:C\to C\quad\text{is continuous and completely continuous.}$$

Now if $y\in S_R$, then

$$\begin{aligned}|A\,y|_0&\le|h|_0+\psi(|y|_0)\sup_{i\in T^+}\sum_{j=0}^{N}k(i,j)\\&=|h|_0+\psi(R)\sup_{i\in T^+}\sum_{j=0}^{N}k(i,j)\;<\;R\;=\;|y|_0\end{aligned}$$

so that

$$|A\,y|_0\le|y|_0\quad\text{for }\ y\in S_R. \tag{3.26}$$

Now let $y \in S_r \cap C(u_0)$ so that

$$|y|_0 = r \quad \text{and} \quad r \geq y(i) > 0 \ \text{ for } \ i \in T^+.$$

Then for $i \in T^+$ we have

$$A\,y(i) \geq \sum_{j=0}^{N} k(i,j)\,\frac{\phi(y(j))}{y(j)}\,y(j) \ \geq \ \frac{\phi(r)}{r} \sum_{j=0}^{N} k(i,j)\,y(j).$$

Let $i_0 \in N^+$ be such that $\inf_{i \in N^+} y(i) = y(i_0)$, so that for $i \in N^+$,

$$A\,y(i) \geq \frac{\phi(r)}{r}\,y(i_0) \sum_{j=0}^{N} k(i,j) \geq \left(\frac{\phi(r)}{r} \inf_{i \in N^+} \sum_{j=0}^{N} k(i,j) \right) y(i_0) \ > \ y(i_0).$$

In particular, $A\,y(i_0) > y(i_0)$ so

$$A\,y \not\leq y \quad \text{for} \quad y \in S_r \cap C(u_0). \tag{3.27}$$

Now apply Theorem 2.4. $\square$

4. Inclusions

In this section we use Theorem 2.3 to establish some new results for the integral inclusion

$$y(t) \ \in \ h(t) + \int_0^1 k(t,s)\,F(s,y(s))\,ds \quad \text{for} \quad t \in [0,1], \tag{4.1}$$

here $k : [0,1] \times [0,1] \to \mathbf{R}$, $h : [0,1] \to \mathbf{R}$ and $F : [0,1] \times \mathbf{R} \to K(\mathbf{R})$. The following conditions are assumed to hold:

$$F : [0,1] \times \mathbf{R} \to K([0,\infty)); \tag{4.2}$$

$$t \mapsto F(t,x) \ \text{ is measurable for every } \ x \in \mathbf{R}; \tag{4.3}$$

$$x \mapsto F(t,x) \ \text{ is upper semicontinuous for a.e. } \ t \in [0,1]; \tag{4.4}$$

$$\begin{cases} \text{for each } \ b > 0 \ \text{ there exists } \ h_b \in L^1[0,1] \ \text{ with } \ |F(t,x)| \leq h_b(t) \\ \text{for a.e. } \ t \in [0,1] \ \text{ and every } \ x \in \mathbf{R} \ \text{ with } \ |x| \leq b; \end{cases} \tag{4.5}$$

$$\begin{cases} \text{for each } \ t \in [0,1], \ k(t,s) \ \text{ is measurable on } \ [0,1] \ \text{ and} \\ k(t) = ess\ sup\,|k(t,s)|, \ 0 \leq s \leq 1, \ \text{ is bounded on } \ [0,1]; \end{cases} \tag{4.6}$$

$$\begin{gathered} \text{the map } \ t \mapsto k_t \ \text{ is continuous from } \ [0,1] \ \text{ to } \ L^\infty[0,1] \\ (\text{here } \ k_t(s) = k(t,s)); \end{gathered} \tag{4.7}$$

$$\text{for each } \ t \in [0,1], \ k(t,s) \geq 0 \ \text{ for a.e. } \ s \in [0,1]; \tag{4.8}$$

$$\begin{cases} \text{there exists a nondecreasing continuous map } \psi : [0,\infty) \to [0,\infty), \\ \text{a continuous map } \ \phi : [0,\infty) \to [0,\infty), \ \text{ and a } \ q \in L^1[0,1] \ \text{ with} \\ q(s)\,\phi(y) \leq u \leq q(s)\,\psi(y) \ \text{ for any } \ u \in F(s,y) \\ \text{ with } \ (s,y) \in [0,1] \times [0,\infty); \end{cases} \tag{4.9}$$

$$\inf_{t\in[0,1]} \int_0^1 k(t,s)\,q(s)\,ds \;>\; 0; \tag{4.10}$$

$$h \in C[0,1] \;\text{ with }\; h(t) \geq 0 \;\text{ for }\; t \in [0,1]; \tag{4.11}$$

$$\exists\, r > 0 \;\text{ with }\; r \;<\; \phi(r) \inf_{t\in[0,1]} \int_0^1 k(t,s)\,q(s)\,ds; \tag{4.12}$$

$$\frac{\phi(x)}{x} \;\text{ is nonincreasing on }\; (0,r) \tag{4.13}$$

and

$$\exists\, R > r \;\text{ with }\; R \;>\; \psi(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,q(s)\,ds \;+\; |h|_0, \tag{4.14}$$

here $|h|_0 = \sup_{t\in[0,1]} |h(t)|$.

Theorem 4.1. *Suppose* (4.2)–(4.14) *hold. Then* (4.1) *has a nonnegative solution* $y \in C[0,1]$ *with* $r \leq |y|_0 = \sup_{t\in[0,1]} |y(t)| < R$.

Remark 4.1. In (4.14) if $R \geq \psi(R) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,q(s)\,ds + |h|_0$, then the result of Theorem 4.1 is again true if $r \leq |y|_0 < R$ is replaced by $r \leq |y|_0 \leq R$.

Proof of Theorem 4.1. Let E, C, u_0, and $C(u_0)$ be as in Theorem 3.1. Now let $A : C \to 2^C$ be defined by

$$A\,y = \Big\{ v \in C[0,1] :\; \exists\, w \in \mathcal{F}\,y \;\text{ such that }\; v(t) = \int_0^1 k(t,s)\,w(s)\,ds \;\text{ for all }\; t \in [0,1] \Big\}$$

with

$$\mathcal{F}\,y = \{u \in L^1[0,1] :\; u(t) \in F(t, y(t)) \;\text{ for a.e. }\; t \in [0,1]\},$$

here $y \in C$.

Remark 4.2. Note A is well defined since (4.2)–(4.4) and [5] imply $\mathcal{F}\,y \neq \emptyset$. Notice (4.2)–(4.8), (4.11) and a standard argument [8] guarantees that

$$A : C \to K(C) \;\text{ is upper semicontinuous and completely continuous.}$$

We wish to apply Theorem 2.3, so we first show

$$|y|_0 \leq |x|_0 \;\text{ for all }\; y \in A\,x \;\text{ and }\; x \in S_R. \tag{4.15}$$

Let $x \in S_R$ and $y \in A\,x$. Then there exists a $v \in \mathcal{F}\,x$ with

$$y(t) = h(t) + \int_0^1 k(t,s)\,v(s)\,ds \;\text{ for }\; t \in [0,1].$$

Moreover, (4.9) guarantees that $v(s) \leq q(s)\, \psi(x(s))$ for $s \in [0,1]$, and so for $t \in [0,1]$ we have

$$|y(t)| \leq |h|_0 + \psi(|x|_0) \int_0^1 k(t,s)\, q(s)\, ds \leq |h|_0 + \psi(R) \sup_{t \in [0,1]} \int_0^1 k(t,s)\, q(s)\, ds.$$

This together with (4.14) yields

$$|y|_0 \leq |h|_0 + \psi(R) \sup_{t \in [0,1]} \int_0^1 k(t,s)\, q(s)\, ds \; < \; R \; = \; |x|_0,$$

and so (4.15) is true.

Next we show

$$y \not\leq x \quad \text{for all} \quad y \in A\, x \quad \text{and} \quad x \in S_r \cap C(u_0). \tag{4.16}$$

Let $x \in S_r \cap C(u_0)$ with $y \in A\, x$. Then

$$|x|_0 = r \quad \text{and} \quad r \geq x(t) > 0 \quad \text{for} \quad t \in [0,1]$$

and there exists $v \in \mathcal{F}\, x$ with

$$y(t) = h(t) + \int_0^1 k(t,s)\, v(s)\, ds \quad \text{for} \quad t \in [0,1].$$

Also (4.9) guarantees that $v(s) \geq q(s)\, \phi(x(s))$ for $s \in [0,1]$, and so for $t \in [0,1]$ we have

$$\begin{aligned} y(t) &\geq \int_0^1 k(t,s)\, q(s)\, \phi(x(s))\, ds = \int_0^1 k(t,s)\, q(s)\, \frac{\phi(x(s))}{x(s)}\, x(s)\, ds \\ &\geq \frac{\phi(r)}{r} \int_0^1 k(t,s)\, q(s)\, x(s)\, ds. \end{aligned}$$

Let $t_0 \in [0,1]$ be such that $\min_{s \in [0,1]} y(t) = y(t_0)$ and this together with the previous inequality and (4.12) yields

$$y(t) \geq \left(\frac{\phi(r)}{r} \inf_{t \in [0,1]} \int_0^1 k(t,s)\, q(s)\, ds \right) x(t_0) \quad \text{for} \quad t \in [0,1].$$

In particular $y(t_0) > x(t_0)$ and so (4.16) is true. Now apply Theorem 2.3. $\square$

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