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WEIGHTED EXPONENTIAL INEQUALITIES

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Abstract. Necessary and sufficient conditions on weight pairs are found for the validity of a class of weighted exponential inequalities involving certain classical operators. Among the operators considered are the Hardy averaging operator and its variants in one and two dimensions, as well as the Laplace transform. Discrete analogues yield characterizations of weighted forms of Carleman's inequality.

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1. Introduction

It is well known ([10], [3]) that for nonnegative locally integrable functions u, v, and f defined on $\mathbb{R}^+$ and $1 < p < \infty$, the inequality

$$\int_0^\infty u(x)\left(\frac{1}{x}\int_0^x f(t)^{1/p}\,dt\right)^p dx \le B_p^p p(p')^{p-1}\int_0^\infty v(x)f(x)\,dx \tag{1.1}$$

holds, if and only if

$$B_p^p \equiv \sup_{s>0}\left(\int_s^\infty x^{-p}u(x)\,dx\right)\left(\int_0^s v(x)^{1-p'}\,dx\right)^{p-1} < \infty.$$

Here and throughout, p' denotes the conjugate index of p defined by $p' = p/(p-1)$. Moreover, the inequality (1.1), as well as those in the sequel, are interpreted in the sense that, if the right side is finite, so is the left side, and the inequality holds.

Now, if we assume f is positive a.e., then, since

$$\lim_{p\to\infty}\left(\frac{1}{x}\int_0^x f(t)^{1/p}\,dt\right)^p = \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right)$$

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(cf. [15], p. 344; 5c) it follows from Fatou's lemma and (1.1) that

$$\int_0^\infty u(x)\exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right)dx \le \liminf_{p\to\infty}\int_0^\infty u(x)\left(\frac{1}{x}\int_0^x f(t)^{1/p}\,dt\right)^p dx$$
$$\le C\int_0^\infty v(x)f(x)\,dx, \tag{1.2}$$

provided $\lim\limits_{p\to\infty} B_p^p p(p')^{p-1} = C$. This limit exists if $u(x) = v(x) = x^\lambda$, $\lambda \ge 0$, and also if $u(x) = v(x)$ is non-increasing. However, such a result for more general weights seems not to be known.

In this paper we consider the operator K defined by

$$(Kf)(x) = \int_0^\infty k(x,y)f(y)\,dy, \qquad f > 0,$$

where $k(x,y) \ge 0$ satisfies certain mild restrictions. Conditions on positive, locally integrable weights u, v are given for which a weighted exponential inequality of the form (1.4) below holds. Here, locally integrable refers to integrating on intervals (a,b), $0 < a < b < \infty$. For a number of operators, specifically, the Laplace transform and the general averaging operator

$$(P_\beta f)(x) = \beta x^{-\beta}\int_0^x t^{\beta-1}f(t)\,dt, \qquad \beta > 0, \tag{1.3}$$

we characterize the weight functions u and v for which the exponential inequality

$$\left(\int_0^\infty u(x)(\exp(K\log f)(x))^q\,dx\right)^{1/q} \le C\left(\int_0^\infty v(x)f(x)^p\,dx\right)^{1/p} \tag{1.4}$$

holds, where $0 < p \le q < \infty$ or $0 < q < p < \infty$, $p > 1$ (Theorem 2.2). The basic result of Heinig, Kerman and Krbec was given in [7], where some of the specific cases were announced and then generalized for the wide range of parameters p and q as above in [8]. The original result in [6] has in fact triggerred a number of papers, for example [11], [12]. In particular, in [11], the inequality (1.4) was obtained for $\beta = 1$ and $0 < p \le q < \infty$. A survey of earlier results can be found in [7]; we recall here various generalizations of Carleman's inequality in [9], [6], [5], [16].

Note that in the sequel all functions are assumed to be measurable, χ_E denotes the characteristic function of a set E and constants C may be different at different occurrences.

2. Exponential Integral Inequalities

Let K be defined by

$$(Kf)(x) = \int_0^\infty k(x,y) f(y)\, dy$$

where f is a positive function and $k(x,y) \geq 0$ satisfies

(i) $k(\lambda x \lambda y) = \lambda^{-1} k(x,y), \quad \lambda > 0$ (k is homogeneous of degree -1);

(ii) $\int_0^\infty k(1,t)\, dt = 1$;

(iii) $\exp\left(-\int_0^\infty k(1,t) \log t\, dt \right) = A, \qquad$ for some constant A.

Theorem 2.1. *Suppose K is the integral operator defined above with kernel K satisfying* (i), (ii) *and* (iii). *Let u and v be positive weight functions and set*

$$w(x) = u(x) \left(\exp[(K \log 1/v)(x)]\right)^{q/p}.$$

Given exponents $p, q > 0$, there holds

$$\left(\int_0^\infty u(x) \left(\exp[(K \log f)(x)]\right)^q\, dx \right)^{1/q} \leq C \left(\int_0^\infty v(x) f(x)^p\, dx \right)^{1/p}. \tag{2.1}$$

with $C > 0$ independent of $f > 0$ if

(a)

$$\sup_{y>0} \int_0^\infty \left[\frac{y k(x,y)}{x} \right]^{q/p} w(x)\, dx < \infty \tag{2.2}$$

when $0 < p \leq q$; and

(b)

$$\int_0^\infty \left[\int_0^\infty \frac{y k(x,y)}{x} w(x)\, dx \right]^r dy < \infty, \qquad \textit{where } 1/r = 1 - q/p, \tag{2.3}$$

when $0 < q < p$.

Proof. Let $s > 0$ and $g = (v f^p)^{1/s}$. Then (2.1) is equivalent to

$$\left(\int_0^\infty w(x) \left(\exp[(K \log g)(x)]\right)^{qs/p}\, dx \right)^{p/(qs)} \leq C \left(\int_0^\infty g(x)^s\, dx \right)^{1/s}. \tag{2.4}$$

The change of variable $y = xt$, the conditions (i)–(iii) and finally Jensen's inequality yield for the left side of (2.4),

$$\left(\int_0^\infty w(x) \left(\exp\left[(K \log g)(x)\right]\right)^{qs/p}\, dx \right)^{p/(qs)}$$

$$= \left(\int_0^\infty w(x)\left(\exp\left[\int_0^\infty k(1,t)\log g(xt)\right]dt\right)^{qs/p} dx\right)^{p/(qs)}$$

$$= A\left(\int_0^\infty w(x)\left(\exp\left[\int_0^\infty k(1,t)\log tg(xt)\right]dt\right)^{qs/p} dx\right)^{p/(qs)}$$

$$\le A\left(\int_0^\infty w(x)\left(\int_0^\infty k(1,t)t\log g(xt)\,dt\right)^{qs/p} dx\right)^{p/(qs)}.$$

Setting $y = xt$ in the last inner integral and invoking (2.4) again, we have that (2.1) is a consequence of

$$\left(\int_0^\infty w(x)[(\widetilde{K}g)(x)]^{qs/p}\,dx\right)^{p/(qs)} \le C\left(\int_0^\infty g(x)^s\,dx\right)^{1/s}, \tag{2.5}$$

where $\widetilde{K}$ is the integral operator with the kernel $\widetilde{k}(x,y) = yk(x,y)/x$. According to [2], Lemma 2.3 (a), one has (2.5) with $s = 1$ when $0 < p \le q$ if and only if

$$\sup_{y>0}\int_0^\infty [\widetilde{k}(x,y)]^{q/p}w(x)\,dx = \sup_{y>0}\int_0^\infty \left[\frac{yk(x,y)}{x}\right]^{q/p} w(x)\,dx < \infty,$$

and (2.5) with $s = p/q$ when $0 < q < p$ if and only if

$$\int_0^\infty\left(\int_0^\infty \widetilde{k}(x,y)w(x)\,dx\right)^r dy = \int_0^\infty\left(\int_0^\infty \frac{yk(x,y)}{x}w(x)\,dx\right)^r dy < \infty,$$

where $1/r = 1 - q/p$ (by the duality counterpart to [2], Lemma 2.3 (b)). $\square$

The next result yields a complete characterization of the weights in (2.1) for the operator P_β with the kernel $k_\beta(x,y) = \beta x^{-\beta}y^{\beta-1}\chi_{(0,x)}$, $x, y > 0$, defined by (1.3).

Theorem 2.2. *Let $\beta > 0$ and suppose u, v are positive locally integrable functions on $(0,\infty)$ with $\int_0^a t^{\beta-1}v(t)^{-1}\,dt < \infty$ for all $a > 0$. Set $w(x) = u(x)\left(\exp[P_\beta\log(1/v)(x)]\right)^{q/p}$. Then, a necessary and sufficient condition that the inequality*

$$\left(\int_0^\infty u(x)\left(\exp[(P_\beta\log f)(x)]\right)^q\,dx\right)^{1/q} \le C\left(\int_0^\infty v(x)f(x)^p\,dx\right)^{1/p} \tag{2.6}$$

holds with $C > 0$ independent of $f > 0$ is

(a)

$$\sup_{y>0}\int_y^\infty \left(\frac{y^\beta}{x^{\beta+1}}\right)^{q/p} w(x)\,dx < \infty, \tag{2.7}$$

when $0 < p \le q < \infty$; and

(b)

$$\int_0^\infty \left(\int_y^\infty \frac{y^\beta w(x)}{x^{\beta+1}}\, dx \right)^r dy < \infty, \qquad 1/r = 1 - q/p, \tag{2.8}$$

when $0 < q < p < \infty$.

Proof. In view of the preceding theorem we need only prove the necessity. Suppose first that (2.5) – which is equivalent to (2.6) – holds with $0 < p \le q$, $\widetilde{K} = P_\beta$, and $s = 1$. Given fixed $y > 0$, define

$$h(t) = e^{1-1/\beta}\beta t^{\beta-1} y^{-\beta}\chi_{(0,y)}(t) + \beta t^{1-\beta} y^\beta \chi_{(y,\infty)}(t), \qquad t > 0.$$

Then, for $x > y$,

$$\begin{aligned}(P_\beta \log h)(x) &= \beta x^{-\beta} \int_0^y t^{\beta-1} \log\left(e^{1-1/\beta}\beta y^{-\beta} t^{\beta-1}\right) dt \\ &\quad + \beta x^{-\beta}\int_y^x t^{\beta-1}\log\left(y^\beta t^{1-\beta}\right) dt \\ &= -\frac{2}{\beta}\left(\frac{y}{x}\right)^\beta - (1-\beta)\log x + \beta\log y + 1 + \frac{1}{\beta} + \log\beta.\end{aligned}$$

Substituting this in (2.6) and restricting the integral on the left side to (y,∞), we obtain

$$\begin{aligned} C_1 &\int_y^\infty \left(\frac{y^\beta w(x)}{x^{\beta+1}}\right)^{q/p} dx \\ &\le C_1 \left(\int_y^\infty w(x)^{q/p} \exp\left[-\frac{2}{\beta}\frac{q}{p}(yx)^\beta\right] \exp\left[-\frac{q}{p}(1+\beta)\log x\right] \right. \\ &\qquad\qquad \left. \exp\left[\frac{q}{p}\beta\log y\right] dx \right)^{p/q} \\ &\le C\int_0^\infty h(x)\, dx = C\left(1 + e^{1-1/\beta}\right),\end{aligned}$$

where $C_1 = \beta\exp(1 + 1/\beta) > 0$, so that (2.7) holds.

Now, let $w_n(t) = w(t)\chi_{(1/n,n)}(t)$ and define h by

$$h_n(t) = t^{\beta q r/p}\left(\int_t^\infty \frac{w_n(s)}{s^{\beta+1}}\, ds\right)^{qr/p}.$$

Then

$$(P_\beta \log h_n)(y) = \beta y^{-\beta}\int_0^y x^{\beta-1}\log\left[x^{\beta q r/p}\left(\int_x^\infty \frac{w_n(t)}{t^{\beta+1}}\, dt\right)^{qr/p}\right] dx$$

$$= \frac{\beta^2 qr}{p} y^{-\beta} \int_0^y x^{\beta-1} \log x \, dx + \frac{\beta qr}{p} y^{-\beta} \int_0^y x^{\beta-1} \log\left[\int_x^\infty \frac{w_n(t)}{t^{\beta+1}} \, dt\right] dx$$

$$\geq \frac{\beta^2 qr}{p} y^{-\beta} \left(\frac{y^\beta}{\beta} \log y - \frac{y^\beta}{\beta^2}\right) + \frac{\beta qr}{p} \log\left[\int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right] y^{-\beta} \int_0^y x^{\beta-1} \, dx$$

$$\geq \log y^{\beta qr/p} - \frac{qr}{p} + \log\left[\left(\int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right)^{qr/p}\right].$$

Substituting in the equivalent formulation of (2.6) in which f is replaced by $v^{-1/p}h$ and v by $w \exp[(q/p)P_\beta \log v]$, we get

$$e^{-qr/p} \int_0^\infty w_n(y) y^{\beta qr/p} \left(\int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right)^{qr/p} \leq C \left(\int_0^\infty \left(y^\beta \int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right)^r dy\right)^{q/p}.$$

Now, $qr/p = r - 1$ and $\beta qr/p + \beta = \beta r$, so the left side of (2.5) equals

$$-\frac{e^{-qr/p}}{r} \int_0^\infty y^{\beta qr/p+\beta+1} \frac{d}{dy} \left(\int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right)^r dy$$

$$= \frac{e^{-qr/p}}{r} \left(\frac{\beta qr}{p} + \beta\right) \int_0^\infty \left(y^\beta \int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right)^r dy$$

Hence,

$$\int_0^\infty \left(y^\beta \int_y^\infty \frac{w_n(x)}{x^{\beta+1}} \, dx\right)^r dy \leq \left(\frac{r e^{qr/p}}{(\beta qr/p) + \beta} \cdot C\right)^{(q/p)-1},$$

and, letting $n \to \infty$, (2.8) follows by Fatou's lemma. □

Remark 2.3. a) For $q = p = 1$, Theorem 2.2 a) was given in [7]. The special case $\beta = 1$ in Theorems 2.2. a) and b) was proved in [11] and [12], respectively. The proof of the sufficiency of the respective conditions use arguments different from ours.

b) If $(u, v) = (x^\delta, x^\gamma)$, $\delta, \gamma \in \mathbb{R}$ where $q(1+\gamma) = p(1+\delta)$, $0 < p \leq q < \infty$, then (2.6) holds. This follows at once if one observes that $w(x) = e^{q\gamma/(\beta p)} x^{\delta + q\gamma/p}$ and (see (2.7))

$$\sup_{s>0} s^{a\beta/p} \left(\int_s^\infty t^{\delta - q\gamma/p - q(1+a\beta)/p} \, dt\right)^{1/q} < \infty.$$

But this supremum is finite if $q(1+\gamma) = p(1+\delta)$ and $p(\delta+1) < q(1+\gamma+a\beta)$, which is clearly the case.

A result similar to that of Theorem 2.2 holds also for the operator Q_β defined by

$$(Q_\beta f)(x) = \beta x^\beta \int_x^\infty y^{-\beta-1} f(y)\, dy, \qquad \beta > 0.$$

Corollary 2.4. *Let u, v and f be as in Theorem 2.2, and*

$$\widetilde{w}(x) = x^{-2} u(1/x) \left(\exp\left[P_\beta(t^2/v(1/t))\right]\right)^{q/p}$$

Then the inequality (2.6) with P_β replaced by Q_β is satisfied in the range $0 < p \le q < \infty$, if and only if (2.7), with w replaced by $\widetilde{w}$, holds.

In the case $0 < q < p < \infty$, (2.6) with P_β replaced by Q_β holds, if and only if, (2.8) with w replaced by $\widetilde{w}$ is satisfied.

Proof. Observe that the change of variable $y = 1/t$ in the definition of Q_β shows that $(Q_\beta f)(x) = (P_\beta \widetilde{f})(1/x)$, where $\widetilde{f}(x) = f(1/x)$. Hence Theorem 2.2 and some obvious changes of variables yields

$$\left(\int_0^\infty u(x) \left(\exp[Q_\beta \log f(x)]\right)^q \, dx\right)^{1/q} = \left(\int_0^\infty u(x) \left(\exp[P_\beta \log \widetilde{f}(1/x)]\right)^q \, dx\right)^{1/q}$$

and

$$\begin{aligned}\left(\int_0^\infty x^{-2} u(1/x) \left(\exp[P_\beta \log \widetilde{f}(x)]\right)^q \, dx\right)^{1/q} &\le C\left(\int_0^\infty x^{-2} v(1/x) \widetilde{f}(x)^p \, dx\right)^{1/p} \\ &= C\left(\int_0^\infty v(x) f(x)^p \, dx\right)^{1/p},\end{aligned}$$

and we are done. □

When $p = q = 1$ and $\beta = 1$, the following equivalent statements hold for $P = P_1$:

Proposition 2.5. *Let u, v and f be positive functions and set*

$$w(x) = u(x) \exp(P \log(1/v))(x), \qquad x > 0.$$

Then, the following statements are equivalent:

$$\int_0^\infty u(x) \exp(P \log f)(x)\, dx \le C_1 \int_0^\infty v(x) f(x)\, dx; \tag{2.9}$$

$$(Pw)(x) + \alpha^{-1}(Q_\alpha w)(x) \le C_2; \tag{2.10}$$

$$Pw \in L_\infty; \tag{2.11}$$

$$\int_0^\infty w(x)(Pf^{1/p})^p(x)\, dx \le C_3 \int_0^\infty f(x)\, dx, \qquad p > 1, \tag{2.12}$$

where C_3 remains bounded as $p \to \infty$.

For the proof see [7].

In our next result we consider the Laplace transform L defined by

$$(Lf)(x) = \int_0^\infty e^{-xy} f(y)\, dy, \qquad x > 0.$$

A characterization of the weights for which (2.9) holds is given next. Our result generalizes Theorem 7 of [6].

Theorem 2.6. *Let u, v and f be positive functions. Then*

$$\int_0^\infty u(x) \exp\left[\frac{1}{x}(L \log f)(1/x)\right] dx \le C \int_0^\infty v(x) f(x)\, dx \tag{2.13}$$

if and only if

$$\sup_{t>0} t \int_0^\infty e^{-tx} w(1/x)\, dx < \infty, \tag{2.14}$$

where $w(x) = u(x) \exp\left[x^{-1} \int_0^x e^{-y/x} \log(1/v(y))\, dy\right]$.

Proof. Let $k(x,y) = x^{-1}e^{-y/x}$, then $x^{-1}(Lf)(x^{-1}) = (Kf)(x)$ satisfies the conditions of Theorem 2.1 (observe that $\int_0^\infty e^{-y} \log y\, dy$ exists, so (iii) is satisfied) whence the sufficiency part follows.

Conversely, if $g = vf$, then (2.13) is equivalent to

$$\int_0^\infty w(x) \exp(K \log g)(x)\, dx \le C \int_0^\infty g(x)\, dx \tag{2.15}$$

where $(Kh)(x) = (1/x)(Lh)(1/x)$. Now, let $g(x) = g_t(x) = t^{-1}e^{-x/t}$, $t > 0$ fixed. Then, $(K \log g_t)(x) = -(x/t) - \log t$ and substituting this into (2.15) we obtain that

$$(Kw)(t) = t^{-1} \int_0^\infty w(x) e^{-x/t}\, dx \le C.$$

Next, define K_1 by

$$(K_1 h)(x) = x \int_0^\infty e^{-x/y} y^{-2} h(y)\, dy.$$

Then,

$$(K_1(Kw))(x) \le (K_1 C)(x) = Cx \int_0^\infty e^{-x/y} y^{-2}\, dy = C.$$

The proof will be complete if we can show that there exists a constant $C_1 > 0$ such that $(K_1 w)(x) \le C_1(K_1(Kw))(x)$. But,

$$\begin{aligned}
(K_1(Kw))(x) &= x\int_0^\infty e^{-x/y}(Kw)(y)\,\frac{dy}{y^2}\\
&= x\int_0^\infty e^{-x/y}y^{-3}\left(\int_0^\infty e^{-t/y}w(t)\,dt\right)dy\\
&= x\int_0^\infty w(t)\left(\int_0^\infty e^{-(x+t)/y}y^{-3}\,dy\right)dt\\
&= x\int_0^\infty w(t)\left(\int_0^\infty e^{-(x+t)s}s\,ds\right)dt\\
&= x\int_0^\infty \frac{w(t)\,dt}{t^2(1+x/t)^2}\\
&\ge \frac{e}{4}x\int_0^\infty \frac{w(t)}{t^2}\,e^{-x/t}\,dt\\
&= \frac{e}{4}(K_1 w)(x).
\end{aligned}$$

Here the last inequality follows from $e^{-s} \le 4e^{-1}/(1+s)^2$, $s > 0$. $\square$

The characterizations of the weights for the averaging operator between weighted L^p–spaces in higher dimensions are known only in two dimensions (cf. [13]). By contrast, the corresponding characterization of weights for the exponential inequality of the averaging operator in higher dimensions carries over from the one dimensional case in a straightforward manner.

Below we give the two dimensional result only. However the higher dimensional results carries over in the same way.

We shall write $x = (x_1, x_2)$, $\mathbb{R}^2_+ = (0,\infty)\times(0,\infty)$, $dx = dx_1dx_2$, with the two dimensional averaging operator being given by

$$(P^2 f)(x) = \frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f(y)\,dy.$$

Theorem 2.7. *Let u, v and f be positive functions on $\mathbb{R}^2_+$. Then*

$$\int_{\mathbb{R}^2_+} u(x)\exp(P^2\log f)(x)\,dx \le C\int_{\mathbb{R}^2_+} v(x)f(x)\,dx \tag{2.16}$$

if and only if for any α_1, $\alpha_2 > 0$,

$$\sup_{y_1>0,y_2>0} y_1^{\alpha_1}y_2^{\alpha_2}\int_{y_1}^\infty\int_{y_2}^\infty x_1^{-(1+\alpha_1)}x_2^{-(1+\alpha_2)}w(x)\,dx = A < \infty \tag{2.17}$$

where $w(x) = u(x)\exp(P^2\log(1/v))(x)$.

Proof. The argument that (2.17) is sufficient for (2.16) follows well known lines. Thus, let $g = fv$. Then (2.16) is equivalent to

$$\int_{\mathbb{R}^2_+} w(x)\exp(P^2\log g)(x)\,dx \le C\int_{\mathbb{R}^2_+} g(x)\,dx \tag{2.18}$$

and an obvious substitution gives

$$\int_{\mathbb{R}^2_+} w(x)\exp\left(\int_0^1\!\!\int_0^1 \log g(x_1t_1, x_2t_2)\,dt\right)dx \le C\int_{\mathbb{R}^2_+} g(x)\,dx.$$

But for $\alpha_1 > 0$, $\alpha_2 > 0$,

$$e^{-(\alpha_1+\alpha_2)} = \exp\int_0^1\!\!\int_0^1 \log\left(t_1^{\alpha_1}t_2^{\alpha_2}\right)\,dt$$

and the left side of (2.18) becomes

$$\begin{aligned}
&e^{\alpha_1+\alpha_2}\int_{\mathbb{R}^2_+} w(x)\exp\left(\int_0^1\!\!\int_0^1 \log(t_1^{\alpha_1}t_2^{\alpha_2}g(x_1t_1, x_2t_2))\,dt\right)dx\\
&\qquad\le e^{\alpha_1+\alpha_2}\int_{\mathbb{R}^2_+} w(x)\int_0^1\!\!\int_0^1 t_1^{\alpha_1}t_2^{\alpha_2}g(x_1t_1, x_2t_2)\,dt\,dx,
\end{aligned}$$

by Jensen's inequality. Changing variables (twice) and interchanging the order of integration the latter is equal to

$$e^{\alpha_1+\alpha_2}\int_{\mathbb{R}^2_+} g(y)\left(y_1^{\alpha_1}y_2^{\alpha_2}\int_{y_1}^{\infty}\!\!\int_{y_2}^{\infty} w(x)x_1^{-(1+\alpha_1)}x_2^{-(1+\alpha_2)}\,dx\right)dy,$$

so that (2.17) yields (2.18) with $C = e^{\alpha_1+\alpha_2}A$.

We now show that (2.16) implies (2.17).

For fixed $t_1 > 0$, $t_2 > 0$ substitute

$$\begin{aligned}
g(x) =& t_1^{-1}t_2^{-1}\chi_{(0,t_1)}\chi_{(0,t_2)}(x_2) + t_1^{-1}\chi_{(0,t_1)}(x_1)\frac{e^{-(1+\alpha_2)}t_2^{\alpha_2}}{x_2^{1+\alpha_2}}\chi_{(t_2,\infty)}(x_2)\\
&+ t_2^{-1}\chi_{(0,t_2)}(x_2)\frac{e^{-(1+\alpha_1)}t_1^{\alpha_1}}{x_1^{1+\alpha_1}}\chi_{(t_1,\infty)}(x_1)\\
&+ \frac{e^{-(2+\alpha_1+\alpha_2)}t_1^{\alpha_1}t_2^{\alpha_2}}{x_1^{1+\alpha_1}x_2^{1+\alpha_2}}\chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2)
\end{aligned}$$

in (2.16). The right side of (2.16) is

$$C\left[1 + e^{-1-\alpha_2}/\alpha_2 + e^{-1-\alpha_1}/\alpha_1 + e^{-2-\alpha_1-\alpha_2}/(\alpha_1\alpha_2)\right]$$

Let us write the left side in the form

$$\int_{\mathbb{R}_+^2} = \iint_{0\,0}^{t_1\,t_2} + \iint_{0\,t_2}^{t_1\,\infty} + \iint_{t_1\,0}^{\infty\,t_2} + \iint_{t_1\,t_2}^{\infty\,\infty} \equiv I_1 + I_2 + I_3 + I_4.$$

Each integral is positive, so, if we show that

$$I_4 = t_1^{\alpha_1} t_2^{\alpha_2} \iint_{t_1\,t_2}^{\infty\,\infty} w(x) x_1^{-(1+\alpha_1)} x_2^{-(1+\alpha_2)}\, dx,$$

the result will follow. We have

$$\begin{aligned}
I_4 &= \iint_{t_1\,t_2}^{\infty\,\infty} w(x) \exp\left[\frac{1}{x_1 x_2}\left(\iint_{0\,0}^{t_1\,t_2} + \iint_{0\,t_2}^{t_1\,x_2} + \iint_{t_1\,0}^{x_1\,t_2} + \iint_{t_1\,t_2}^{x_1 x_2}\right) \log g(y)\, dy\right] dx \\
&= \iint_{t_1\,t_2}^{\infty\,\infty} w(x) \exp\left[\frac{t_1 t_2}{x_1 x_2} \log(t_1^{-1} t_2^{-1})\right] \\
&\qquad \exp\left[\frac{t_1}{x_1 x_2} \int_{t_2}^{x_2} \log\left(\frac{t_1^{-1} e^{-(1+\alpha_2)} t_2^{\alpha_2}}{y_2^{1+\alpha_2}}\right) dy_2\right] \\
&\qquad \exp\left[\frac{t_2}{x_1 x_2} \int_{t_1}^{x_1} \log\left(\frac{t_2^{-1} e^{-(1+\alpha_1)} t_1^{\alpha_1}}{y_1^{1+\alpha_1}}\right) dy_1\right] \\
&\qquad \exp\left[\frac{1}{x_1 x_2} \iint_{t_1\,t_2}^{x_1 x_2} \log\left(\frac{e^{-(2+\alpha_1+\alpha_2)} t_1^{\alpha_1} t_2^{\alpha_2}}{y_1^{1+\alpha_1} y_2^{1+\alpha_2}}\right) dy\right] dx \\
&= \iint_{t_1\,t_2}^{\infty\,\infty} w(x) E_1(x) E_2(x) E_3(x) E_4(x)\, dx.
\end{aligned}$$

Clearly,

$$E_1(x) = \exp\left[-\frac{t_1 t_2}{x_1 x_2} \log t_1 - \frac{t_1 t_2}{x_1 x_2} \log t_2\right].$$

An elementary calculation shows that

$$\begin{aligned}
E_2(x) = \exp\Bigg[-\frac{t_1}{x_1} \log t_1 + \frac{t_1 t_2}{x_1 x_2} \log t_1 + \frac{\alpha_2 t_1}{x_1} \log t_2 \\
+ \frac{t_1 t_2}{x_1 x_2} \log t_2 - \frac{(1+\alpha_2) t_1}{x_1} \log x_2\Bigg]
\end{aligned}$$

so that

$$E_1(x) E_2(x) = \exp\left[-\frac{t_1}{x_1} \log t_1 + \frac{\alpha_2 t_1}{x_1} \log t_2 - \frac{(1+\alpha_2) t_1}{x_1} \log x_2\right].$$

Further calculation – elementary again, but rather tedious so we omit details – yields

$$\begin{aligned} E_3(x) &= \exp\Big[-\frac{t_2}{x_2}\log t_2 + \frac{t_1t_2}{x_1x_2}\log t_2 + \frac{\alpha_1 t_2}{x_2}\log t_1 \\ &\qquad + \frac{t_1t_2}{x_1x_2}\log t_1 - \frac{(1+\alpha_1)t_2}{x_2}\log x_1\Big], \\ E_4(x) &= \frac{t_1^{\alpha_1}t_2^{\alpha_2}}{x_1^{1+\alpha_1}x_2^{1+\alpha_2}}\exp\Big[-\frac{\alpha_1 t_2}{x_2}\log t_1 + \frac{t_1}{x_1}\log t_1 - \frac{t_1t_2}{x_1x_2}\log t_1 + \frac{t_2}{x_2}\log t_2 \\ &\qquad -\frac{\alpha_1 t_1}{x_1}\log t_2 - \frac{t_1t_2}{x_1x_2}\log t_2 + \frac{(1+\alpha_1)t_2}{x_2}\log x_1 + \frac{(1+\alpha_2)t_1}{x_1}\log x_2\Big] \end{aligned}$$

and

$$E_3(x)E_4(x) = \frac{t_1^{\alpha_1}t_2^{\alpha_2}}{x_1^{1+\alpha_1}x_2^{1+\alpha_2}}\exp\left[\frac{t_1}{x_1}\log t_1 - \frac{\alpha_2 t_1}{x_1}\log t_2 + \frac{(1+\alpha_2)t_1}{x_1}\log x_2\right],$$

thus

$$E_1(x)E_2(x)E_3(x)E_4(x) = \frac{t_1^{\alpha_1}t_2^{\alpha_2}}{x_1^{1+\alpha_1}x_2^{1+\alpha_2}}.$$

This proves the theorem. □

3. The Discrete case

It is not unexpected that integral estimates have corresponding discrete analogues. In this section we give discrete versions of Theorem 2.2 (with $\beta = 1$) and Theorem 2.7, which yield two–weight generalizations of a weighted Carleman inequality.

Theorem 3.1. *Let $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$, $\{a_n\}_{n=1}^\infty$ be sequences of positive numbers and set*

$$w_n = \left(u_n \exp\left[\frac{1}{n}\sum_{k=1}^{n}\log(1/v_k)\right]\right)^{q/p}, \qquad n = 1, 2, \dots.$$

(i) *If $0 < p \le q < \infty$, then there is a constant $C > 0$ such that the inequality*

$$\left(\sum_{n=1}^{\infty} u_n (a_1 a_2 \dots a_n)^{q/n}\right)^{1/q} \le C\left(\sum_{n=1}^{\infty} v_n a_n^p\right)^{1/p} \tag{3.1}$$

if and only if for any $a > 0$

$$\sup_{m\ge1} m^{a/p}\left(\sum_{n=m}^{\infty}\frac{w_n}{n^{(1+a)q/p}}\right)^{1/q} < \infty. \tag{3.2}$$

(ii) *If* $1 < q < p < \infty$, *then the inequality* (3.1) *holds, provided the condition* $\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} n^{-q(a+1)} w_n\right)^{r/q} m^{(a+1/q')r} < \infty$ *is satisfied, where* $a > 0$ *and* $1/r = 1/q - 1/p$.

Proof. (i) Let $b_n^p = v_n a_n^p$, $n = 1, 2 \ldots$. Then, (3.1) is equivalent to

$$\left(\sum_{n=1}^{\infty} w_n \left(\exp\left[\frac{1}{n}\sum_{k=1}^{n} \log b_k\right]\right)^q\right)^{1/q} \le C\left(\sum_{n=1}^{\infty} b_n^p\right)^{1/p}. \tag{3.3}$$

For any $p > 0$, if we write $\gamma = p/\alpha$ where $\alpha > 1$, then, $0 < p = \gamma\alpha \le q < \infty$, or $1 < \alpha \le q/\gamma$. With $c_k = b_k^\gamma$ the inequality (3.3) is equivalent to

$$\left(\sum_{n=1}^{\infty} w_n \left(\exp\left[\frac{1}{n}\sum_{k=1}^{n} \log c_k\right]\right)^{q/\gamma}\right)^{\gamma/q} \le C^\gamma\left(\sum_{n=1}^{\infty} c_n^\alpha\right)^{1/\alpha}. \tag{3.4}$$

To prove (3.4) define $f(t) = c_k$, if $k-1 < t \le k$, $k = 1, 2 \ldots$, and zero otherwise. Then, since $\int_0^1 \log y\, dy = -1$, Jensen's inequality shows that the left side of (3.4) is

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty} w_n \left(\exp\left[\frac{1}{n}\sum_{k=1}^{n}\int_{k-1}^{k} \log f(t)\, dt\right]\right)^{q/\gamma}\right)^{\gamma/q} \\
&\quad = \left(\sum_{n=1}^{\infty} w_n \left(\exp\left[\frac{1}{n}\int_0^n \log f(t)\, dt\right]\right)^{q/\gamma}\right)^{\gamma/q} \\
&\quad = \left(\sum_{n=1}^{\infty} w_n \left(\exp\left[\int_0^1 \log f(ny)\, dy\right]\right)^{q/\gamma}\right)^{\gamma/q} \qquad (t = ny) \\
&\quad = e\left(\sum_{n=1}^{\infty} w_n \left(\exp\left[\int_0^1 \log(yf(ny))\, dy\right]\right)^{q/\gamma}\right)^{\gamma/q} \\
&\quad \le e\left(\sum_{n=1}^{\infty} w_n \left(\int_0^1 yf(ny)\, dy\right)^{q/\gamma}\right)^{\gamma/q} \\
&\quad = e\left(\sum_{n=1}^{\infty} w_n n^{-2q/\gamma} \left(\int_0^n f(t)\, dt\right)^{q/\gamma}\right)^{\gamma/q} \\
&\quad = e\left(\sum_{n=1}^{\infty} \frac{w_n}{n^{2q/\gamma}} \left(\sum_{k=1}^{n} kc_k\right)^{q/\gamma}\right)^{\gamma/q}.
\end{aligned}$$

Applying the discrete version of the weighted Hardy inequality (cf. [1], [4]), the last expression is dominated by

$$\left(\sum_{n=1}^{\infty} k^{-\alpha}\left(k^\alpha c_k^\alpha\right)\right)^{1/\alpha} = \left(\sum_{n=1}^{\infty} c_k^\alpha\right)^{1/\alpha}$$

provided in case $0 < p \le q < \infty$

$$\begin{aligned}\sup_{m\ge 1}\left(\sum_{n=m}^{\infty}\frac{w_n}{n^{2q/\gamma}}\right)^{\gamma/q}&\left(\sum_{n=m}^{\infty}k^{-\alpha(1-\alpha')}\right)^{\gamma/q'}\\ &=\sup_{m\ge 1}m^{1+1/\alpha'}\left(\sum_{n=m}^{\infty}\frac{w_n}{n^{2q\alpha/p}}\right)^{p/(q\alpha)}\\ &=\left(\sup_{m\ge 1}m^{(2\alpha/p)-(1/p)}\left(\sum_{n=m}^{\infty}\frac{w_n}{n^{2q\alpha/p}}\right)^{1/q}\right)^{p/\alpha}\\ &<\infty.\end{aligned}$$

But with $a = 2\alpha - 1 > 0$ this becomes (3.2).

(ii) In case $1 < q < p < \infty$ we see that (just as in the previous case) the left side of (3.3) is not larger than

$$e\left(\sum_{n=1}^{\infty}w_n n^{-q(\alpha+1)}\left(\sum_{k=1}^{n}k^{\alpha}b_k\right)^{q}\right)^{1/q}, \qquad \alpha>0,$$

and again by the discrete form of Hardy's inequality ([15], [16]) this is dominated by

$$\left(\sum_{k=1}^{\infty}k^{-\alpha p}\left(k^{\alpha}b_k\right)^{p}\right)^{1/p}=\left(\sum_{k=1}^{\infty}b_k^p\right)^{1/p}$$

whenever

$$\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty}\frac{w_n}{n^{q(\alpha+1)}}\right)^{r/q}\left(\sum_{n=1}^{\infty}n^{\alpha p'}\right)^{r/q'}m^{\alpha p'}<\infty.$$

But since

$$\begin{aligned}(\alpha p'+1)\frac{r}{q'}+\alpha p'&=r\left(\frac{\alpha p'}{q'}+\frac{1}{q'}+\frac{\alpha p'}{q}-\frac{\alpha p'}{p}\right)\\ &=r\left(\alpha p'+\frac{1}{q'}-\alpha p'+\alpha\right)=r\left(\alpha+\frac{1}{q'}\right),\end{aligned}$$

this is the condition given in (ii) with $a = \alpha$.

To prove necessity of part (i) let $b_n^p = 1/k$ if $n \le k$ and $e^{-1-\alpha}n^{-\alpha-1}k^{\alpha}$ if $n > k$, $\alpha > 0$ and k fixed, $n = 1, \dots,$ in (3.3) Clearly the right side of (3.3) is finite in this case and we can write the left side in the form

$$\left(\sum_{n=1}^{k}+\sum_{n=k+1}^{\infty}\right)^{1/q}=(S_1+S_2)^{1/q}.$$

Now,

$$S_1=\sum_{n=1}^{k}w_n\left(\exp\left[\frac{1}{n}\sum_{j=1}^{n}\log(k-1/p)\right]\right)^{q}=k^{-q/p}\sum_{n=1}^{k}w_n$$

and

$$S_2 = \sum_{n=k+1}^{\infty} w_n \left(\exp\left[\frac{1}{n}\sum_{j=1}^{k}\log(1/k)\right]\right)^{q/p} \left(\exp\left[\frac{1}{n}\sum_{j=k+1}^{n}\log\left(\frac{e^{-(1+\alpha)}k^{\alpha}}{j^{\alpha+1}}\right]\right)\right)^{q/p}$$

$$= \sum_{n=k+1}^{\infty} w_n \left(\exp\left[\frac{-k}{n}\log k\right]\right)^{q/p} \left(\exp\left[\frac{n-k-1}{n}\log\left(e^{-(1+\alpha)}k^{\alpha}\right)\right]\right)^{q/p} \times$$

$$\times \left(\exp\left[-\frac{\alpha+1}{n}\sum_{j=k+1}^{n}\log j\right]\right)^{q/p}.$$

But the Mean Value Theorem shows that

$$\sum_{j=k+1}^{n}\log j = \sum_{j=k+2}^{n+1} j\log(j-1) - \sum_{j=k+1}^{n} j\log j$$

$$= \sum_{j=k+1}^{n} j[\log(j-1)-\log j] - (k+1)\log k + (n+1)\log n$$

$$= \sum_{j=k+1}^{n}\frac{-j}{c_j'} + (n+1)\log n - (k+1)\log k$$

m where $j-1 \le c_j' \le j$. Hence

$$\sum_{j=k+1}^{n}\log j \le \sum_{j=k+1}^{n}(-1) + (n+1)\log n - (k+1)\log k$$

so

$$S_2 \ge \sum_{n=k+1}^{\infty} w_n \left(\exp\left[-\frac{k}{n}\log k\right]\right)^{q/p} \left(e^{-(1+\alpha)}k^{\alpha}\right)^{q/p}$$

$$\times \left(\exp\left[\frac{k+1}{n}(1+\alpha)\right]\right)^{q/p} \left(\exp\left[-\frac{\alpha(k+1)}{n}\log k\right]\right)^{q/p}$$

$$\times \exp\left[-\frac{(\alpha+1)}{n}(-n+k+1+(n+1)\log n-(k+1)\log k)\right]$$

$$= e^{-(1+\alpha)q/p}k^{\alpha q/p}\sum_{n=k+1}^{\infty} w_n \left(\exp\left[-\frac{k}{n}\log k\right]\right)^{q/p} \left(\exp\left[-\frac{\alpha(k+1)}{n}\log k\right]\right)^{q/p}$$

$$\times \left(\exp\left[-(\alpha+1)\left(\frac{n+1}{n}\right)\log n\right]\right)^{q/p} \left(\exp\left[(\alpha+1)\frac{\alpha(k+1)}{n}\log k\right]\right)^{q/p}$$

$$\times \left(e^{\alpha+1}\right)^{q/p}\left(\exp\left[-(\alpha+1)\left(\frac{k+1}{n}\right)\right]\right)^{q/p}$$

$$\ge e^{-(1+\alpha)q/p}k^{\alpha q/p}\sum_{n=k+1}^{\infty} w_n n^{-(\alpha+1)q/p}\left(\exp\left[\frac{1}{n}\log\left(\frac{k}{n^{\alpha+1}}\right)\right]\right)^{q/p}$$

since $\dfrac{k+1}{n} \leq 1$. But, since, for $k \geq 1$ $(n \geq 2)$,

$$\frac{1}{n}\log\left(\frac{k}{n^{\alpha+1}}\right) \geq -\frac{\alpha+1}{n}\log n \geq -\frac{\alpha+1}{n}\log 2,$$

it follows that

$$\begin{aligned}\left(\exp\left[\frac{1}{n}\log\frac{k}{n^{\alpha+1}}\right]\right)^{q/p} &= \exp\left[\frac{q}{p}\left(\frac{1}{n}\log\frac{k}{n^{\alpha+1}}\right)\right]\\ &\geq 2^{-q(\alpha+1)/(2p)}.\end{aligned}$$

Therefore

$$S_2 \geq e^{-(1+\alpha)q/p}2^{-q(\alpha+1)/(2p)}k^{\alpha q/p}\sum_{n=k+1}^{\infty} w_n n^{-(\alpha+1)q/p},$$

and so

$$\begin{aligned}(S_1+S_2)^{1/q} &\geq e^{-(1+\alpha)/p}2^{-(\alpha+1)/(2p)}\bigg(k^{-q/p}w_k + k^{\alpha q/p}\sum_{n=k}^{\infty} w_n n^{-(\alpha+1)q/p}\\ &\qquad -k^{\alpha q/p}w_k k^{-(\alpha+1)q/p}\bigg)^{1/q}\\ &= e^{-(1+\alpha)/p}2^{-(\alpha+1)/(2p)}\left(k^{\alpha q/p}\sum_{n=k}^{\infty}\frac{w_n}{n^{(\alpha+1)q/p}}\right)^{1/q}.\end{aligned}$$

This implies condition (3.2) with $a = \alpha$ and completes the proof of the theorem. $\square$

Remark 3.2. i) If $u_n = v_n = 1$, then $w_n = 1$, $n = 1, 2, \ldots$, and condition (3.2) is satisfied only if $p = q$, so in this case (3.1) reduces to Carleman's inequality (with $C = e$).

ii) If $p = q = 1$ and

$$\sup_{m\geq 1} m^a \sum_{n=m}^{\infty}\frac{w_n}{n^{(1+\alpha)}} = A,$$

then the constant C in (3.1) satisfies

$$w^{-(1+a)/2}A(1+3e^{-(1+a)}/(2a)) \leq C \leq e^a A.$$

The previous result with $p = q = 1$ has a higher dimensional analogue. We state here the two dimensional case and only sketch the proof. Details can be found in authors' preprint [8].

Theorem 3.3. *For $n, m \in N$ let $\{u_{n,m}\}$, $\{v_{n,m}\}$, $\{a_{n,m}\}$ be double sequences of positive numbers and*

$$w_{n,m} = u_{n,m}\exp\left(\frac{1}{nm}\sum_{\substack{1\leq k\leq n\\ 1\leq j\leq m}}\log 1/v_{k,j}\right).$$

Then,

$$\sum_{n,m\geq 1} u_{n,m}\Bigg(\prod_{\substack{1\leq k\leq n\\ 1\leq j\leq m}} a_{j,k}\Bigg)^{1/(nm)} \leq C \sum_{n,m\geq 1} v_{n,m}a_{n,m} \tag{3.5}$$

if and only if for some $\alpha_1 > 0$, $\alpha_2 > 0$,

$$\sup_{\substack{k\geq 1\\ j\geq 1}} k^{\alpha_1} j^{\alpha_2} \sum_{\substack{n\geq k\\ m\geq j}} n^{-\alpha_1-1}m^{-\alpha_2-1}w_{n,m} \equiv A < \infty. \tag{3.6}$$

For the proof observe that the substitution $b_{n,m} = a_{n,m}v_{n,m}$ allows us to write (3.5) in the equivalent form

$$\sum_{n,m\geq 1} w_{n,m} \exp\Bigg(\frac{1}{nm}\sum_{\substack{1\leq k\leq n\\ 1\leq j\leq m}} \log b_{j,k}\Bigg) \leq C \sum_{n,m\geq 1} b_{n,m}. \tag{3.7}$$

Now, let $f(s,t) = b_{j,k}$ if $k-1 < t \leq k$, $j-1 < s \leq j$, $k,j = 1,2\ldots,$ and zero otherwise. Since

$$e^{-\alpha_1-\alpha_2} = \exp \int_0^1\!\!\int_0^1 \log\left(t_1^{\alpha_1}t_2^{\alpha_2}\right)\, dt_1dt_2,$$

the left side of (3.7) takes the form

$$\sum_{n,m\geq 1} w_{n,m} \exp\Bigg(\frac{1}{nm}\sum_{\substack{1\leq k\leq n\\ 1\leq j\leq m}} \int_{k-1}^{k}\int_{j-1}^{j} \log f(s,t)\, dsdt\Bigg)$$

and this is the case dealt with in the previous theorem.

For the converse we substitute into (3.7) $b_{n,m}$ defined by

$$b_{n,m} = \begin{cases} k^{-1}j^{-1} & \text{if } n\leq k,\ m\leq j,\\ k^{-1}e^{-1-\alpha_2}j^{\alpha_2}m^{-\alpha_2-1} & \text{if } n\leq k,\ j>m,\\ j^{-1}e^{-1-\alpha_1}k^{\alpha_1}n^{-\alpha_1-1} & \text{if } n>k,\ j\leq m,\\ e^{-2-\alpha_1-\alpha_2}k^{\alpha_1}j^{\alpha_2}n^{-\alpha_1-1}m^{-\alpha_2-1} & \text{if } n>k,\ j\geq m, \end{cases}$$

where k, j are fixed. Then, after some calculation one obtains (3.6).

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