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ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY THE QUADRATIC FORM $x_1^2+\cdots+x_8^2+4x_9^2$

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Abstract. An explicit exact (non asymptotic) formula is derived for the number of representations of positive integers by the quadratic form $x_1^2+\cdots+x_8^2+4x_9^2$. The way by which this formula is derived, gives us a possibility to develop a method of finding the so-called Liouville type formulas for the number of representations of positive integers by positive diagonal quadratic forms in nine variables with integral coefficients.

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In [6], entire modular forms of weight $9/2$ on the congruence subgroup $\Gamma_0(4N)$ are constructed. The Fourier coefficients of these modular forms have a simple arithmetical sense. This allows one to get the Liouville type formulas for the number of representations of positive integers by positive diagonal quadratic forms in nine variables with integral coefficients.

Let $r(n;f)$ denote the number of representations of a positive integer n by the positive quadratic form $f=a_1x_1^2+a_2x_2^2+\cdots+a_9x_9^2$.

1. Preliminaries

1.1. In this paper N, a, d, n, q, r, s denote positive integers; u, v are odd positive integers; ω is a square free integer; p is a prime number; k, ℓ are non-negative integers; c, g, h, j, m, x, α, β, γ, δ are integers; z, τ are complex variables ($\operatorname{Im}\tau>0$); $e(z)=\exp(2\pi iz)$; $Q=e(\tau)$; $(\frac{h}{u})$ is the generalized Jacobi symbol; $\eta(\gamma)=1$ if $\gamma\geq 0$ and $\eta(\gamma)=-1$ if $\gamma<0$; a denotes the least common multiple of the coefficients of the quadratic form and Δ is its determinant. Further, $\sum\limits_{h \bmod q}$ and $\sum\limits_{h \bmod q}'$ denote respectively summation with respect to the complete and the reduced residue system modulo q; $\rho(n;f)$ is a sum of the singular series corresponding to $r(n;f)$.

Since

$$\vartheta_{gh}(z\mid\tau;c,N)$$
$$=\sum_{m\equiv c \pmod N}(-1)^{h(m-c)/N}e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big)e\Big(\Big(m+\frac{g}{2}\Big)z\Big) \qquad (1.1)$$

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(theta-function with characteristics g, h), we have

$$\frac{\partial}{\partial z}\,\vartheta_{gh}(z\mid\tau;c,N)=\pi i\sum_{m\equiv c \pmod N}(-1)^{h(m-c)/N}(2m+g)\times e\Big(\frac{1}{2N}\Big(m+\frac{g}{2}\Big)^2\tau\Big)e\Big(\Big(m+\frac{g}{2}\Big)z\Big). \tag{1.2}$$

Suppose

$$\begin{aligned}\vartheta_{gh}(\tau;c,N)&=\vartheta_{gh}(0\mid\tau;c,N),\\ \vartheta'_{gh}(\tau;c,N)&=\frac{\partial}{\partial z}\,\vartheta_{gh}(z\mid\tau;c,N)\Big|_{z=0}.\end{aligned} \tag{1.3}$$

It is known (see, e.g., [5], p.112) that

$$\begin{aligned}\vartheta_{g+2j,h}(\tau;c,N)&=\vartheta_{gh}(\tau;c+j,N),\\ \vartheta'_{g+2j,h}(\tau;c,N)&=\vartheta'_{gh}(\tau;c+j,N),\end{aligned} \tag{1.4}$$

$$\begin{aligned}\vartheta_{gh}(\tau;c+Nj,N)&=(-1)^{hj}\vartheta_{gh}(\tau;c,N),\\ \vartheta'_{gh}(\tau;c+Nj,N)&=(-1)^{hj}\vartheta'_{gh}(\tau;c,N).\end{aligned} \tag{1.5}$$

From (1.1) and (1.2), in particular according to (1.3), it follows that

$$\vartheta_{gh}(\tau;0,N)=\sum_{m=-\infty}^{\infty}(-1)^{hm}Q^{(2Nm+g)^2/8N}, \tag{1.6}$$

$$\vartheta'_{gh}(\tau;0,N)=\pi i\sum_{m=-\infty}^{\infty}(-1)^{hm}(2Nm+g)Q^{(2Nm+g)^2/8N}. \tag{1.7}$$

From (1.6) and (1.7) we obtain

$$\vartheta_{-g,h}(\tau;0,N)=\vartheta_{gh}(\tau;0,N),\quad \vartheta'_{-g,h}(\tau;0,N)=-\vartheta'_{gh}(\tau;0,N). \tag{1.8}$$

It is clear that

$$\prod_{k=1}^{9}\vartheta_{00}(\tau;0,2a_k)=1+\sum_{n=1}^{\infty}r(n;f)Q^n. \tag{1.9}$$

Further, put

$$\theta(\tau;f)=1+\sum_{n=1}^{\infty}\rho(n;f)Q^n. \tag{1.10}$$

It is known that

$$\rho(n;f)=\frac{\pi^{9/2}n^{7/2}}{\Gamma(9/2)\Delta^{1/2}}\sum_{q=1}^{\infty}A(q), \tag{1.11}$$

where $\Gamma(\cdot)$ is the gamma-function,

$$A(q)=q^{-9/2}\sideset{}{'}\sum_{h \bmod q}e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{9}S(a_kh,q) \tag{1.12}$$

and $S(a_k h, q)$ is the Gaussian sum.

1.2. For the convenience we quote some known results as the following lemmas.

Lemma 1 ([2], p. 811, the end of the p. 954). *The entire modular form $F(\tau)$ of weight r on the congruence subgroup $\Gamma_0(4N)$ is identically zero if in its expansion in the series*

$$F(\tau) = C_0 + \sum_{n=1}^{\infty} C_n Q^n,$$

$C_0 = C_n = 0$ *for all* $n \le \frac{r}{12} 4N \prod_{p|4N} (1 + \frac{1}{p})$.

Lemma 2 ([6], Theorem on p. 64). *For given $N \ge a$, the function*

$$\psi(\tau; f) = \prod_{k=1}^{9} \vartheta_{00}(\tau; 0, 2a_k) - \theta(\tau; f) - \lambda \prod_{k=1}^{3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k),$$

where λ is an arbitrary constant, is an entire modular form of weight $9/2$, and the multiplier system

$$v(M) = i^{\eta(\gamma)(\operatorname{sgn}\delta - 1)/2} \cdot i^{(|\delta|-1)^2/4} \Big(\frac{\beta\Delta \operatorname{sgn}\delta}{|\delta|}\Big)$$

on $\Gamma_0(4N)$ ($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is a substitution matrix from* $\Gamma_0(4N)$) *if the following conditions hold:*

1) $2 \mid g_k$, $N_k \mid N$ $(k = 1, 2, 3)$, $a \mid N$,
2) $4 \mid N \sum_{k=1}^{3} \frac{h_k^2}{N_k}$, $4 \mid \sum_{k=1}^{3} \frac{g_k^2}{4N_k}$,
3) *for all α and δ with* $\alpha\delta \equiv 1 \pmod{4N}$

$$\operatorname{sgn}\delta \Big(\frac{N_1 N_2 N_3}{|\delta|}\Big) \prod_{k=1}^{3} \vartheta'_{\alpha g_k, h_k}(\tau; 0, 2N_k) = \Big(\frac{-\Delta}{|\delta|}\Big) \prod_{k=1}^{3} \vartheta'_{g_k h_k}(\tau; 0, 2N_k).$$

Lemma 3 (see, e.g., [4], p. 14, Lemma 10). *Let*

$$\chi_p = 1 + A(p) + A(p^2) + \cdots;$$

then for $s > 4$

$$\sum_{q=1}^{\infty} A(q) = \prod_{p} \chi_p.$$

Lemma 4 ([1], Theorem 2, p. 531). *Let n be a positive integer and f be a positive definite quadratic form in $s > 4$ variables with integral coefficients, Δ*

its determinant. If $2 \nmid s$, $2^k \,||\, n$, $2^{k'} \,||\, \Delta$, $\Delta n = 2^{k+k'}uv = r^2\omega$, $p^\ell \,||\, n$, $p^{\ell'} \,||\, \Delta$, $u = \prod\limits_{\substack{p|n\\ p\nmid 2\Delta}} p^\ell = r_1^2\omega_1$, $v = \prod\limits_{\substack{p|\Delta n,\, p|\Delta\\ p>2}} p^{\ell+\ell'} = r_2^2\omega_2$, *then*

$$\prod_p \chi_p = \frac{(s-1)!r_1^{2-s}}{2^{s-2}\pi^{s-1}|B_{s-1}|}\,\chi_2 \prod_{\substack{p|\Delta\\ p>2}} \chi_p \prod_{p|2\Delta}(1-p^{1-s})^{-1} L\Big(\frac{s-1}{2}, (-1)^{(s-1)/2}\omega\Big)$$
$$\times \prod_{\substack{p|r_2\\ p>2}}\Big(1-\Big(\frac{(-1)^{(s-1)/2}\omega}{p}\Big)p^{(1-s)/2}\Big)$$
$$\times \sum_{d|r_1} d^{s-2}\prod_{p|d}\Big(1-\Big(\frac{(-1)^{(s-1)/2}\omega}{p}\Big)p^{(1-s)/2}\Big),$$

where L *is the Dirichlet series and* B_{s-1} *is the Bernoulli number.*

If in Lemma 4 we put $s=9$, then by (1.11) and Lemma 3 we get

$$\rho(n;f) = \frac{\pi^{9/2}n^{7/2}}{\Gamma(9/2)\Delta^{1/2}}\,\frac{8!r_1^{-7}}{2^7\pi^8|B_8|}\,\chi_2\prod_{\substack{p|\Delta\\ p>2}}\chi_p\prod_{p|2\Delta}(1-p^{-8})^{-1}L(4,\omega)$$
$$\times\prod_{\substack{p|r_2\\ p>2}}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)$$
$$=\frac{n^{7/2}\cdot 48\cdot 30}{\pi^4\Delta^{1/2}r_1^7}\,\chi_2\prod_{\substack{p|\Delta\\ p>2}}\chi_p\prod_{p|2\Delta}(1-p^{-8})^{-1}L(4,\omega)$$
$$\times\prod_{\substack{p|r_2\\ p>2}}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)\sum_{d|r_1}d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big). \tag{1.13}$$

Lemma 5 ([3], p. 298). *The sum of Dirichlet series* $L(4,\omega)$ *is equal to*

$$L(4,1) = \frac{\pi^4}{2^5\cdot 3}\,, \quad L(4,2) = \frac{11\pi^4}{2^8\cdot 3\sqrt{2}}\,,$$
$$L(4,\omega) = -2\pi^4\omega^{-1/2}\sum_{0<h\le\omega/2}\Big(\frac{h}{\omega}\Big)\Big(\frac{h^2}{2^2\omega^2}-\frac{h^3}{3\omega^3}\Big)$$
$$\text{if}\quad \omega>1,\ \ \omega\equiv 1 \pmod 4,$$
$$L(4,\omega) = 2\pi^4\omega^{-1/2}\Big\{\sum_{0<h\le\omega/4}\Big(\frac{h}{\omega}\Big)\Big(\frac{h}{2^4\omega}-\frac{h^3}{3\omega^3}\Big)$$
$$-\sum_{\omega/4<h\le\omega/2}\Big(\frac{h}{\omega}\Big)\Big(\frac{1}{2^5\cdot 3}-\frac{3h}{2^4\omega}+\frac{h^2}{2\omega^2}-\frac{h^3}{3\omega^3}\Big)\Big\}$$
$$\text{if}\quad \omega\equiv 3 \pmod 4,$$

$$L(4,\omega)=2\pi^4\omega^{-1/2}\Bigg\{\sum_{0<h\le\omega/16}\Big(\frac{h}{\omega/2}\Big)\Big(\frac{11}{2^8\cdot 3}-\frac{h^2}{\omega^2}\Big)$$
$$+\sum_{\omega/16<h\le 3\omega/16}\Big(\frac{h}{\omega/2}\Big)\Big(\frac{5}{2^7\cdot 3}+\frac{h}{2^4\omega}-\frac{2h^2}{\omega^2}+\frac{2^4}{3}\,\frac{h^3}{\omega^3}\Big)$$
$$+\sum_{3\omega/16<h\le\omega/4}\Big(\frac{h}{\omega/2}\Big)\Big(\frac{37}{2^8\cdot 3}-\frac{h}{2\omega}+\frac{h^2}{\omega^2}\Big)\Bigg\}$$
$$\textit{if}\quad \omega>2,\ \ \omega\equiv 2 \pmod 8,$$
$$L(4,\omega)=2\pi^4\omega^{-1/2}\Bigg\{\sum_{0<h\le\omega/16}\Big(\frac{h}{\omega/2}\Big)\Big(\frac{3h}{2^4\omega}-\frac{2^4h^3}{3\omega^3}\Big)$$
$$-\sum_{\omega/16<h\le 3\omega/16}\Big(\frac{h}{\omega/2}\Big)\Big(\frac{1}{2^8\cdot 3}+\frac{h}{2^2\omega}+\frac{h^2}{\omega^2}\Big)$$
$$-\sum_{3\omega/16<h\le\omega/4}\Big(\frac{h}{\omega/2}\Big)\Big(\frac{7}{2^6\cdot 3}-\frac{13h}{2^4\omega}+\frac{2^2h^2}{\omega^2}-\frac{2^4h^3}{3\omega^3}\Big)\Bigg\}$$
$$\textit{if}\quad \omega\equiv 6 \pmod 8).$$

In particular, using this lemma, by tedious calculations we get:

$$L(4,3)=\frac{23\pi^4}{2^4\cdot 3^4\sqrt{3}},\quad L(4,5)=\frac{17\pi^4}{2\cdot 3\cdot 5^3\sqrt{5}},\quad L(4,6)=\frac{29\pi^4}{2^7\cdot 3^2\sqrt{6}},$$
$$L(4,7)=\frac{113\pi^4}{3\cdot 4\cdot 7^3\sqrt{7}},\quad L(4,10)=\frac{1577\pi^4}{2^7\cdot 3\cdot 5^3\sqrt{10}},$$
$$L(4,11)=\frac{2153\pi^4}{2^4\cdot 3\cdot 11^3\sqrt{11}},\quad L(4,13)=\frac{493\pi^4}{2\cdot 3\cdot 13^3\sqrt{13}},$$
$$L(4,14)=\frac{2503\pi^4}{2^6\cdot 3\cdot 7^3\sqrt{14}},\quad L(4,15)=\frac{179\pi^4}{20\cdot 15^2\sqrt{15}},$$
$$L(4,17)=\frac{205\pi^4}{17^3\sqrt{17}},\quad L(4,19)=\frac{14933\pi^4}{48\cdot 19^3\sqrt{19}},$$
$$L(4,21)=\frac{187\pi^4}{9\cdot 21^2\sqrt{21}},\quad L(4,22)=\frac{24889\pi^4}{2^7\cdot 3\cdot 11^3\sqrt{22}},$$
$$L(4,23)=\frac{7093\pi^4}{2^2\cdot 3\cdot 23^3\sqrt{23}},\quad L(4,26)=\frac{43679\pi^4}{2^7\cdot 3\cdot 13^2\sqrt{26}},$$
$$L(4,29)=\frac{2669\pi^4}{2\cdot 29^3\sqrt{29}},\quad L(4,30)=\frac{36451\pi^4}{2^6\cdot 3^4\cdot 5^3\sqrt{30}},$$
$$L(4,31)=\frac{10357\pi^4}{6\cdot 31^3\sqrt{31}},\quad L(4,33)=\frac{2115\pi^4}{33^3\sqrt{33}},$$
$$L(4,34)=\frac{57241\pi^4}{2^6\cdot 3\cdot 17^3\sqrt{34}},\quad L(4,35)=\frac{61733\pi^4}{2^3\cdot 3\cdot 35^3\sqrt{35}}.$$

2. Formulas for $r(n;f)$ if $f=x_1^2+x_2^2+\cdots+x_8^2+4x_9^2$

Lemma 6. *The function*

$$\begin{aligned}\psi(\tau;f)&=\vartheta_{00}^8(\tau;0,2)\vartheta_{00}(\tau;0,8)-\theta(\tau;f)\\&-\frac{16\cdot 9}{17}\,\frac{1}{512(\pi i)^3}\,\vartheta_{80}'^3(\tau;0,24)\\&-\frac{64\cdot 27}{17}\,\frac{1}{2048(\pi i)^3}\,\vartheta_{16,0}'^2(\tau;0,24)\vartheta_{80}'(\tau;0,24)\\&-\frac{32\cdot 21}{17}\,\frac{1}{1024(\pi i)^3}\,\vartheta_{80}'^2(\tau;0,24)\vartheta_{16,0}'(\tau;0,24)\\&-\frac{896}{17}\,\frac{1}{4096(\pi i)^3}\,\vartheta_{16,0}'^3(\tau;0,24)\end{aligned}\tag{2.1}$$

is an entire modular form of weight $9/2$ *and the multiplier system*

$$v(M)=i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}\cdot i^{(|\delta|-1)^2/4}\Big(\frac{4\beta\operatorname{sgn}\delta}{|\delta|}\Big)$$

on $\Gamma_0(48)$ ($M=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$ *is a substitution matrix from* $\Gamma_0(48)$).

Proof. In Lemma 2 put $a=\Delta=4$; $g_1=g_2=g_3=8$; $g_1=g_2=16$, $g_3=8$; $g_1=g_2=8$, $g_3=16$; $g_1=g_2=g_3=16$; $h_1=h_2=h_3=0$; $N_1=N_2=N_3=N=12$. It is easy to verify that the last four summands in (2.1) satisfy the conditions 1) and 2) of this Lemma. Further,

$$\Big(\frac{N_1N_2N_3}{|\delta|}\Big)=\Big(\frac{12^3}{|\delta|}\Big)=\Big(\frac{3}{|\delta|}\Big),\quad\Big(\frac{-\Delta}{|\delta|}\Big)=\Big(\frac{-4}{|\delta|}\Big)=\Big(\frac{-1}{|\delta|}\Big).\tag{2.2}$$

If $\alpha\delta\equiv 1 \pmod{48}$, then $\alpha\delta\equiv 1\pmod{24}$, whence, in particular,

$$\alpha\equiv\pm1\pmod{24},\ \text{ then respectively }\ \delta\equiv\pm1\pmod{24}.\tag{2.3}$$

From (2.3) it follows that if $\alpha\equiv\pm1\pmod 4$ and $\alpha\equiv\pm1\pmod 3$, then respectively

$$\delta\equiv\pm1\pmod 4\quad\text{and}\quad\delta\equiv\pm1\pmod 3.\tag{2.4}$$

By (1.4), (1.5) and (1.8),

$$\begin{aligned}\vartheta_{8\alpha,0}'^3(\tau;0,24)&=\vartheta_{\pm8+8(\alpha\mp1),0}'^3(\tau;0,24)=\vartheta_{\pm8,0}'^3(\tau;4(\alpha\mp1),24)\\&=\vartheta_{\pm8,0}'^3(\tau;0,24)=\begin{cases}\vartheta_{80}'^3(\tau;0,24)&\text{for}\quad\alpha\equiv1\pmod{24},\\-\vartheta_{80}'^3(\tau;0,24)&\text{for}\quad\alpha\equiv-1\pmod{24}.\end{cases}\end{aligned}\tag{2.5}$$

Similarly,

$$\begin{aligned} &\vartheta'^2_{16\alpha,0}(\tau;0,24)\vartheta'_{8\alpha,0}(\tau;0,24) \\ &= \begin{cases} \vartheta'^2_{16,0}(\tau;0,24)\vartheta'_{80}(\tau;0,24) & \text{for} \quad \alpha \equiv 1 \pmod{24}, \\ -\vartheta'^2_{16,0}(\tau;0,24)\vartheta'_{80}(\tau;0,24) & \text{for} \quad \alpha \equiv -1 \pmod{24}, \end{cases} \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\vartheta'^2_{8\alpha,0}(\tau;0,24)\vartheta'_{16\alpha,0}(\tau;0,24) \\ &= \begin{cases} \vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24) & \text{for} \quad \alpha \equiv 1 \pmod{24}, \\ -\vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24) & \text{for} \quad \alpha \equiv -1 \pmod{24}, \end{cases} \end{aligned} \tag{2.7}$$

$$\vartheta'^3_{16\alpha,0}(\tau;0,24) = \begin{cases} \vartheta'^3_{16,0}(\tau;0,24) & \text{for} \quad \alpha \equiv 1 \pmod{24}, \\ -\vartheta'^3_{16,0}(\tau;0,24) & \text{for} \quad \alpha \equiv -1 \pmod{24}. \end{cases} \tag{2.8}$$

From (2.2) it follows that

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big) = \operatorname{sgn}\delta\Big(\frac{3}{|\delta|}\Big) = \begin{cases} \Big(\dfrac{3}{\delta}\Big) & \text{if} \quad \delta > 0, \\ -\Big(\dfrac{3}{-\delta}\Big) & \text{if} \quad \delta < 0. \end{cases} \tag{2.9}$$

If in (2.9) we have $\delta > 0$ and $\delta \equiv 1 \pmod{24}$, i.e., $\delta \equiv 1 \pmod 4$ and $\delta \equiv 1 \pmod 3$, then $(\frac{3}{\delta}) = (\frac{\delta}{3}) = (\frac{1}{3}) = 1$ and $(\frac{-1}{\delta}) = 1$. Thus if $\delta > 0$, then $\operatorname{sgn}\delta(\frac{3}{|\delta|}) = (\frac{-1}{\delta}) = (\frac{-1}{|\delta|})$. But if in (2.9) we have $\delta < 0$ and $\delta \equiv 1 \pmod{24}$, i.e., $\delta \equiv 1 \pmod 4$ and $\delta \equiv 1 \pmod 3$, then, as $-\delta \equiv -1 \pmod 4$, we get $(\frac{3}{-\delta}) = -(\frac{-\delta}{3}) = -(\frac{-1}{3}) = 1$ and $(\frac{-1}{-\delta}) = -1$. Thus if $\delta < 0$, then $\operatorname{sgn}\delta(\frac{3}{|\delta|}) = -(\frac{3}{-\delta}) = (\frac{-1}{-\delta}) = (\frac{-1}{|\delta|})$.

Now suppose that in (2.9) we have $\delta \equiv -1 \pmod{24}$; hence $\delta \equiv -1 \pmod 4$ and $\delta \equiv -1 \pmod 3$. Then if $\delta > 0$, we get $(\frac{3}{\delta}) = -(\frac{\delta}{3}) = 1$ and $(\frac{-1}{\delta}) = -1$, i.e., $\operatorname{sgn}\delta(\frac{3}{|\delta|}) = -(\frac{-1}{\delta}) = -(\frac{-1}{|\delta|})$. But if $\delta < 0$, then $(\frac{3}{-\delta}) = (\frac{-\delta}{3}) = 1$ and $(\frac{-1}{-\delta}) = 1$, i.e., $\operatorname{sgn}\delta(\frac{3}{-\delta}) = -(\frac{3}{-\delta}) = -(\frac{-1}{-\delta}) = -(\frac{-1}{|\delta|})$.

Thus

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big) = \begin{cases} \Big(\dfrac{-1}{|\delta|}\Big) & \text{for} \quad \delta \equiv 1 \pmod{24}, \\ -\Big(\dfrac{-1}{|\delta|}\Big) & \text{for} \quad \delta \equiv -1 \pmod{24}. \end{cases} \tag{2.10}$$

Since in the investigated quadratic form $\Delta = 4$, we have $(\frac{-1}{|\delta|}) = (\frac{-4}{|\delta|})$, and according to (2.3), formula (2.10) can be rewritten as

$$\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big) = \begin{cases} \Big(\dfrac{-4}{|\delta|}\Big) & \text{for } \delta \equiv 1 \pmod{24}, \\ & \text{i.e., also for } \alpha \equiv 1 \pmod{24}, \\ -\Big(\dfrac{-4}{|\delta|}\Big) & \text{for } \delta \equiv -1 \pmod{24}, \\ & \text{i.e., also for } \alpha \equiv -1 \pmod{24}. \end{cases} \tag{2.11}$$

Hence by (2.11) and (2.5)–(2.8), for all α and δ with $\alpha\delta \equiv 1 \pmod{24}$ we have:

1) $\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^3_{8\alpha,0}(\tau;0,24) = \Big(\frac{-4}{|\delta|}\Big)\vartheta'^3_{80}(\tau;0,24)$,

2) $\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^2_{16\alpha,0}(\tau;0,24)\vartheta'_{8\alpha,0}(\tau;0,24) = \Big(\frac{-4}{|\delta|}\Big)\vartheta'^2_{16,0}(\tau;0,24)\vartheta'_{80}(\tau;0,24)$,

3) $\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^2_{8\alpha,0}(\tau;0,24)\vartheta'_{16\alpha,0}(\tau;0,24) = \Big(\frac{-4}{|\delta|}\Big)\vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24)$,

4) $\operatorname{sgn}\delta\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\vartheta'^3_{16\alpha,0}(\tau;0,24) = \Big(\frac{-4}{|\delta|}\Big)\vartheta'^3_{16,0}(\tau;0,24)$.

Thus the condition 3) of Lemma 2 also is satisfied. □

Theorem 1. *The following indentity takes place:*

$$\begin{aligned}\vartheta^8_{00}(\tau;0,2)\vartheta_{00}(\tau;0,8) = \theta(\tau;f) &+ \frac{16\cdot 9}{17}\,\frac{1}{512(\pi i)^3}\,\vartheta'^3_{80}(\tau;0,24)\\ &+ \frac{64\cdot 27}{17}\,\frac{1}{2048(\pi i)^3}\,\vartheta'^2_{16,0}(\tau;0,24)\vartheta'_{80}(\tau;0,24)\\ &+ \frac{32\cdot 21}{17}\,\frac{1}{1024(\pi i)^3}\,\vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24)\\ &+ \frac{896}{17}\,\frac{1}{4096(\pi i)^3}\,\vartheta'^3_{16,0}(\tau;0,24).\end{aligned}$$

Proof. According to Lemma 1, the function $\psi(\tau;f)$ will be identically zero if all coefficients of Q^n $(n \le 36)$ in its expansion in powers of Q are zero.

If in (1.13) we put $\Delta = 4$, i.e., $v = 1$, $r_2 = 1$, and $n = 2^k u$, then

$$\begin{aligned}\rho(n;f) &= \frac{2^{7k/2}u^{7/2}}{\pi^4\cdot 2r_1^7}\cdot 48\cdot 30\chi_2\,\frac{256}{255}\,L(4,\omega)\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)\\ &= \frac{2^{7k/2}u^{7/2}}{17\pi^4\cdot r_1^7}\cdot 12288\chi_2 L(4,\omega)\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big).\end{aligned}\tag{2.12}$$

A method of finding the values of χ_2 in the general case for an arbitrary quadratic form is developed in [7]. In the case of the quadratic form $f = x_1^2 + x_2^2 + \cdots + x_8^2 + 4x_9^2$, we get

$$\begin{aligned}\chi_2 &= 1 \quad \text{if} \quad k = 0,\\ &= \frac{7}{8} \quad \text{if} \quad k = 1,\\ &= \frac{143 - 255\cdot 2^{-7k/2-3}}{127} \quad \text{if} \quad k \ge 2,\ \ 2 \mid k,\ \ u \equiv 3 \pmod 4,\\ &= \frac{143 + (127\cdot(-1)^{(u^2-1)/8} - 8)2^{-7k/2-6}}{127}\\ &\qquad\qquad \text{if} \quad k \ge 2,\ \ 2 \mid k,\ \ u \equiv 1 \pmod 4,\\ &= \frac{143 - 255\cdot 2^{-(7k-1)/2}}{127} \quad \text{if} \quad k \ge 3,\ \ 2 \nmid k.\end{aligned}\tag{2.13}$$

In particular,

$$\begin{aligned}\chi_2 &= \frac{9224 + (-1)^{(u^2-1)/8}}{8192} \quad \text{if} \quad k = 2, \quad u \equiv 1 \pmod 4, \\ &= \frac{1151}{1024} \quad \text{if} \quad k = 2, \quad u \equiv 3 \pmod 4 \ \text{ and if } \ k = 3.\end{aligned} \tag{2.14}$$

Our aim now is to find the values of $\rho(n; f)$ for all $1 \leq n \leq 36$. In the table below the values of χ_2 are obtained by virtue of (2.13) and (2.14), the values of $L(4, \omega)$ are calculated by Lemma 5 and are placed after this lemma, and the values of $\rho(n; f)$ are calculated by (2.12).

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
1	1	1	$4\cdot 1$	2	1	1	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{128}{17}$
3	3	1	$4\cdot 3$	2	3	1	$\frac{23\pi^4}{2^4\cdot 3^4\sqrt{3}}$	$\frac{256\cdot 23}{17}$
5	5	1	$4\cdot 5$	2	5	1	$\frac{17\pi^4}{2\cdot 3\cdot 5^3\sqrt{5}}$	2048
7	7	1	$4\cdot 7$	2	7	1	$\frac{113\pi^4}{2^2\cdot 3\cdot 7^3\sqrt{7}}$	$\frac{1024\cdot 113}{17}$
9	9	3	$4\cdot 9$	6	1	1	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{128\cdot 2161}{17}$
11	11	1	$4\cdot 11$	2	11	1	$\frac{2153\pi^4}{2^4\cdot 3\cdot 11^3\sqrt{11}}$	$\frac{256\cdot 2153}{17}$
13	13	1	$4\cdot 13$	2	13	1	$\frac{493\pi^4}{2\cdot 3\cdot 13^3\sqrt{13}}$	59392
15	15	1	$4\cdot 15$	2	15	1	$\frac{179\pi^4}{20\cdot 15^2\sqrt{15}}$	$\frac{1024\cdot 1611}{17}$
17	17	1	$4\cdot 17$	2	17	1	$\frac{205\pi^4}{17^3\sqrt{17}}$	$\frac{4096\cdot 615}{17}$
19	19	1	$4\cdot 19$	2	19	1	$\frac{14933\pi^4}{48\cdot 19^3\sqrt{19}}$	$\frac{256\cdot 14933}{17}$
21	21	1	$4\cdot 21$	2	21	1	$\frac{187\pi^4}{9\cdot 21^2\sqrt{21}}$	315392
23	23	1	$4\cdot 23$	2	23	1	$\frac{7093\pi^4}{2^2\cdot 3\cdot 23^3\sqrt{23}}$	$\frac{1024\cdot 7093}{17}$

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
25	25	5	$4\cdot 25$	10	1	1	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{128\cdot 78001}{17}$
27	27	3	$4\cdot 27$	6	3	1	$\frac{23\pi^4}{2^4\cdot 3^4\sqrt{3}}$	$\frac{1024\cdot 12581}{17}$
29	29	1	$4\cdot 29$	2	29	1	$\frac{2669\pi^4}{2\cdot 29^3\sqrt{29}}$	964608
31	31	1	$4\cdot 31$	2	31	1	$\frac{10357\pi^4}{6\cdot 31^3\sqrt{31}}$	$\frac{2048\cdot 10357}{17}$
33	33	1	$4\cdot 33$	2	33	1	$\frac{2115\pi^4}{33^3\sqrt{33}}$	$\frac{4096\cdot 6345}{17}$
35	35	1	$4\cdot 35$	2	35	1	$\frac{61733\pi^4}{8\cdot 3\cdot 35^3\sqrt{35}}$	$\frac{512\cdot 61733}{17}$
2	1	1	$8\cdot 1$	2	2	$\frac{7}{8}$	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	$\frac{16\cdot 77}{17}$
6	3	1	$8\cdot 3$	2	6	$\frac{7}{8}$	$\frac{29\pi^4}{2^7\cdot 3^2\sqrt{6}}$	$\frac{32\cdot 1827}{17}$
10	5	1	$8\cdot 5$	2	10	$\frac{7}{8}$	$\frac{1577\pi^4}{2^7\cdot 3\cdot 5^3\sqrt{10}}$	$\frac{32\cdot 11039}{17}$
14	7	1	$8\cdot 7$	2	14	$\frac{7}{8}$	$\frac{2503\pi^4}{2^6\cdot 3\cdot 7^3\sqrt{14}}$	$\frac{64\cdot 17521}{17}$
18	9	3	$8\cdot 9$	6	2	$\frac{7}{8}$	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	$\frac{16\cdot 170555}{17}$
22	11	1	$8\cdot 11$	2	22	$\frac{7}{8}$	$\frac{24889\pi^4}{2^7\cdot 3\cdot 11^3\sqrt{22}}$	$\frac{32\cdot 7\cdot 24889}{17}$
26	13	1	$8\cdot 13$	2	26	$\frac{7}{8}$	$\frac{43679\pi^4}{2^7\cdot 3\cdot 13^3\sqrt{26}}$	$\frac{32\cdot 305753}{17}$
30	15	1	$8\cdot 15$	2	30	$\frac{7}{8}$	$\frac{36451\pi^4}{2^6\cdot 3^4\cdot 5^3\sqrt{30}}$	$\frac{64\cdot 255157}{17}$
34	17	1	$8\cdot 17$	2	34	$\frac{7}{8}$	$\frac{57241\pi^4}{2^6\cdot 3\cdot 17^3\sqrt{34}}$	$\frac{64\cdot 400687}{17}$
4	1	1	$16\cdot 1$	4	1	$\frac{9225}{8192}$	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 9225}{17}$

n	u	r_1	Δn	r	ω	χ_2	$L(4,\omega)$	$\rho(n;f)$
8	1	1	$32\cdot 1$	4	2	$\frac{1151}{1024}$	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	$\frac{16\cdot 12661}{17}$
12	3	1	$16\cdot 3$	4	3	$\frac{1151}{1024}$	$\frac{23\pi^4}{2^4\cdot 3^4\sqrt{3}}$	$\frac{32\cdot 26473}{17}$
16	1	1	$64\cdot 1$	8	1	$\frac{1180681}{2^{20}}$	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 1180681}{17}$
20	5	1	$16\cdot 5$	4	5	$\frac{9223}{8192}$	$\frac{17\pi^4}{2\cdot 3\cdot 5^3\sqrt{5}}$	295136
24	3	1	$32\cdot 3$	4	6	$\frac{1151}{1024}$	$\frac{29\pi^4}{2^7\cdot 3^2\sqrt{6}}$	$\frac{32\cdot 300411}{17}$
28	7	1	$16\cdot 7$	4	7	$\frac{1151}{1024}$	$\frac{113\pi^4}{3\cdot 4\cdot 7^3\sqrt{7}}$	$\frac{128\cdot 130063}{17}$
32	1	1	$128\cdot 1$	8	2	$\frac{18743041}{2^{17}\cdot 127}$	$\frac{11\pi^4}{2^8\cdot 3\sqrt{2}}$	$\frac{16\cdot 11\cdot 147583}{17}$
36	9	3	$16\cdot 9$	12	1	$\frac{9225}{8192}$	$\frac{\pi^4}{2^5\cdot 3}$	$\frac{2\cdot 9225\cdot 2161}{17}$

Thus according to (1.10) we get

$$\begin{aligned}\theta(\tau;f) = 1 &+ \frac{128}{17}Q + \frac{16\cdot 77}{17}Q^2 + \frac{256\cdot 23}{17}Q^3 + \frac{2\cdot 9225}{17}Q^4 + 2048Q^5\\ &+ \frac{32\cdot 1827}{17}Q^6 + \frac{1024\cdot 113}{17}Q^7 + \frac{16\cdot 12661}{17}Q^8 + \frac{128\cdot 2161}{17}Q^9\\ &+ \frac{32\cdot 11039}{17}Q^{10} + \frac{256\cdot 2153}{17}Q^{11} + \frac{32\cdot 26473}{17}Q^{12} + 59392Q^{13}\\ &+ \frac{64\cdot 17521}{17}Q^{14} + \frac{1024\cdot 1611}{17}Q^{15} + \frac{2\cdot 1180681}{17}Q^{16}\\ &+ \frac{4096\cdot 615}{17}Q^{17} + \frac{16\cdot 170555}{17}Q^{18} + \frac{256\cdot 14933}{17}Q^{19}\\ &+ 295136Q^{20} + 315392Q^{21} + \frac{32\cdot 7\cdot 24889}{17}Q^{22}\\ &+ \frac{1024\cdot 7093}{17}Q^{23} + \frac{32\cdot 300411}{17}Q^{24} + \frac{128\cdot 78001}{17}Q^{25}\\ &+ \frac{32\cdot 305753}{17}Q^{26} + \frac{1024\cdot 12581}{17}Q^{27} + \frac{128\cdot 130063}{17}Q^{28}\\ &+ 964608Q^{29} + \frac{64\cdot 255157}{17}Q^{30} + \frac{2048\cdot 10357}{17}Q^{31}\end{aligned}$$

$$+\frac{16\cdot 11\cdot 147583}{17}Q^{32}+\frac{4096\cdot 6345}{17}Q^{33}+\frac{64\cdot 400687}{17}Q^{34}$$
$$+\frac{512\cdot 61733}{17}Q^{35}+\frac{2\cdot 9225\cdot 2161}{17}Q^{36}+\cdots. \tag{2.15}$$

Using (1.6), we have

$$\vartheta_{00}(\tau;0,8)=\sum_{m=-\infty}^{\infty}Q^{(16m)^2/64}=\sum_{m=-\infty}^{\infty}Q^{4m^2}$$
$$=1+2Q^4+2Q^{16}+2Q^{36}+\cdots \tag{2.16}$$

and

$$\vartheta_{00}(\tau;0,2)=\sum_{m=-\infty}^{\infty}Q^{(4m)^2/16}=\sum_{m=-\infty}^{\infty}Q^{m^2}$$
$$=1+2Q+2Q^4+2Q^9+2Q^{16}+2Q^{25}+2Q^{36}+\cdots. \tag{2.17}$$

From (2.17) it follows that

$$\vartheta_{00}^2(\tau;0,2)=1+4Q+4Q^2+4Q^4+8Q^5+4Q^8+4Q^9+8Q^{10}$$
$$+8Q^{13}+4Q^{16}+8Q^{17}+4Q^{18}+8Q^{20}+12Q^{25}+8Q^{26}$$
$$+8Q^{29}+4Q^{32}+8Q^{34}+4Q^{36}+\cdots.$$

It is not difficult to verify, but by very tedious calculations, that

$$\vartheta_{00}^4(\tau;0,2)=1+8Q+24Q^2+32Q^3+24Q^4+48Q^5+96Q^6+64Q^7$$
$$+24Q^8+104Q^9+144Q^{10}+96Q^{11}+96Q^{12}+112Q^{13}$$
$$+192Q^{14}+192Q^{15}+24Q^{16}+144Q^{17}+312Q^{18}+160Q^{19}$$
$$+144Q^{20}+256Q^{21}+288Q^{22}+192Q^{23}+96Q^{24}+248Q^{25}$$
$$+336Q^{26}+320Q^{27}+192Q^{28}+240Q^{29}+576Q^{30}+256Q^{31}$$
$$+24Q^{32}+384Q^{33}+432Q^{34}+384Q^{35}+312Q^{36}+\cdots$$

and

$$\vartheta_{00}^8(\tau;0,2)=1+16Q+112Q^2+448Q^3+1136Q^4+2016Q^5$$
$$+3136Q^6+5504Q^7+9328Q^8+12112Q^9+14112Q^{10}$$
$$+21312Q^{11}+31808Q^{12}+35168Q^{13}+38528Q^{14}$$
$$+56448Q^{15}+74864Q^{16}+78624Q^{17}+84784Q^{18}+109760Q^{19}$$
$$+143136Q^{20}+154112Q^{21}+149184Q^{22}+194688Q^{23}$$
$$+261184Q^{24}+252016Q^{25}+246176Q^{26}+327040Q^{27}$$
$$+390784Q^{28}+390240Q^{29}+395136Q^{30}+476672Q^{31}$$
$$+599152Q^{32}+596736Q^{33}+550368Q^{34}$$
$$+693504Q^{35}+859952Q^{36}+\cdots. \tag{2.18}$$

From (2.18) and (2.16), again by tedious calculations, we get

$$\begin{aligned}
&\vartheta_{00}^{8}(\tau;0,2)\vartheta_{00}(\tau;0,8)\\
&= 1+16Q+112Q^2+448Q^3+1138Q^4+2048Q^5+3360Q^6\\
&+6400Q^7+11600Q^8+16144Q^9+20384Q^{10}+32320Q^{11}\\
&+50464Q^{12}+59392Q^{13}+66752Q^{14}+99072Q^{15}\\
&+138482Q^{16}+148992Q^{17}+162064Q^{18}+223552Q^{19}\\
&+295136Q^{20}+315392Q^{21}+325024Q^{22}+425216Q^{23}\\
&+566112Q^{24}+584464Q^{25}+572768Q^{26}+759040Q^{27}\\
&+976768Q^{28}+964608Q^{29}+964544Q^{30}+1243648Q^{31}\\
&+1530448Q^{32}+1534464Q^{33}+1510208Q^{34}\\
&+1866368Q^{35}+2344530Q^{36}+\cdots . \qquad (2.19)
\end{aligned}$$

From (1.7) it follows that

1) $$\vartheta_{80}'^{3}(\tau;0,24)=\Big(8\pi i\sum_{m=-\infty}^{\infty}(6m+1)Q^{(6m+1)^2/3}\Big)^3=$$

$$=(\delta\pi i)^3Q\Big(1-15Q^8+96Q^{16}-335Q^{24}+672Q^{32}-\cdots\Big), \qquad (2.20)$$

i.e.,

$$\frac{1}{512(\pi i)^3}\,\vartheta_{80}'^{3}(\tau;0,24)=Q-15Q^9+96Q^{17}-335Q^{25}+672Q^{33}+\cdots . \qquad (2.21)$$

2) $$\begin{aligned}\vartheta_{16,0}'^{2}(\tau;0,24)\vartheta_{80}'(\tau;0,24)&=\Big(16\pi i\sum_{m=-\infty}^{\infty}(3m+1)Q^{4(3m+1)^2/3}\Big)^2\\
&\times\Big(8\pi i\sum_{m=-\infty}^{\infty}(6m+1)Q^{(6m+1)^2/3}\Big)\\
&=2048(\pi i)^3Q^3\Big(1-4Q^4-Q^8+20Q^{12}-13Q^{16}\\
&-20Q^{20}+12Q^{24}-40Q^{28}+70Q^{32}+76Q^{36}+\cdots\Big), \qquad (2.22)\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2048(\pi i)^3}\,\vartheta_{16,0}'^{2}(\tau;0,24)\vartheta_{80}'(\tau;0,24)&=Q^3-4Q^7-Q^{11}+20Q^{15}\\
&-13Q^{19}-20Q^{23}+12Q^{27}-40Q^{31}+70Q^{35}+\cdots . \qquad (2.23)\end{aligned}$$

3) $$\begin{aligned}\vartheta_{80}'^{2}(\tau;0,24)\vartheta_{16,0}'(\tau;0,24)&=\Big(8\pi i\sum_{m=-\infty}^{\infty}(6m+1)Q^{(6m+1)^2/3}\Big)^2\\
&\times\Big(16\pi i\sum_{m=-\infty}^{\infty}(3m+1)Q^{4(3m+1)^2/3}\Big)\end{aligned}$$

$$= 1024(\pi i)^3Q^2\Big(1-2Q^4-10Q^8+20Q^{12}+39Q^{16}-74Q^{20}$$
$$-70Q^{24}+100Q^{28}+44Q^{32}+58Q^{36}+\cdots\Big), \qquad (2.24)$$

i.e.,

$$\frac{1}{1024(\pi i)^3}\,\vartheta_{80}'^{2}(\tau;0,24)\vartheta_{16,0}'(\tau;0,24) = Q^2-2Q^6-10Q^{10}+20Q^{14}$$
$$+39Q^{18}-74Q^{22}-70Q^{26}+100Q^{30}+44Q^{34}+\cdots. \qquad (2.25)$$

4) $$\vartheta_{16,0}'^{3}(\tau;0,24) = \Big(16\pi i\sum_{m=-\infty}^{\infty}(3m+1)Q^{4(3m+1)^2/3}\Big)^2$$
$$= (16\pi i)^3Q^4\Big(1-6Q^4+12Q^8-8Q^{12}+12Q^{20}-48Q^{24}$$
$$+48Q^{28}-15Q^{32}+60Q^{36}+\cdots\Big), \qquad (2.26)$$

i.e.,

$$\frac{1}{4096(\pi i)^3}\,\vartheta_{16,0}'^{3}(\tau;0,24) = Q^4-6Q^8+12Q^{12}-8Q^{16}$$
$$+12Q^{24}-48Q^{28}+48Q^{32}-15Q^{36}+\cdots. \qquad (2.27)$$

By (2.1), (2.19), (2.15), (2.21), (2.23), (2.25) and (2.27), one can verify that all coefficients of Q^n ($n\le 36$) in the expansion of $\psi(\tau;f)$ in powers of Q are zero. Thus, according to Lemma 1, the theorem is proved. □

Theorem 2. *Let $n=2^ku$, $4n=r^2\omega$, $r_1^2\mid u$. Then*

$$r(n;f) = \frac{12288u^{7/2}}{17\pi^4r_1^7}\,L(4,\omega)\sum_{d\mid r_1}d^7\prod_{p\mid d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)$$
$$+\frac{16\cdot 9}{17}\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 1 \pmod 4,$$
$$= \frac{12288u^{7/2}}{17\pi^4r_1^7}\,L(4,\omega)\sum_{d\mid r_1}d^7\prod_{p\mid d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)$$
$$+\frac{64\cdot 27}{17}\sum_{\substack{4(x_1^2+x_2^2)+x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 3,\ x_3\equiv 1 \pmod 6}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 3 \pmod 4,$$
$$= \frac{12288\cdot 2^{7k/2}u^{7/2}}{17\pi^4r_1^7}\,\chi_2L(4,\omega)\sum_{d\mid r_1}d^7\prod_{p\mid d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)$$
$$+\frac{32\cdot 21}{17}\sum_{\substack{x_1^2+x_2^2+4x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 6,\ x_3\equiv 1 \pmod 3}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 2 \pmod 4,$$

$$r(n;f)=\frac{12288\cdot 2^{7k/2}u^{7/2}}{17\pi^4 r_1^7}\,\chi_2 L(4,\omega)\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)$$
$$+\frac{896}{17}\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3 \quad \textit{if} \quad n\equiv 0 \pmod 4.$$

The values of χ_2 *can be calculated by formulas* (2.13) *and* (2.14), *the values of* $L(4,\omega)$ *by Lemma* 5.

Remark. If $p^2\nmid n$ $(p>2)$, i.e., $r_1=1$, then in all above formulas
$$\sum_{d|r_1} d^7\prod_{p|d}\Big(1-\Big(\frac{\omega}{p}\Big)p^{-4}\Big)=1.$$

Proof. Equating the coefficients of Q^n in both sides of the identity from Theorem 1, by (1.9) and (1.10) we get
$$r(n;f)=\rho(n;f)+\frac{16\cdot 9}{17}\,w_1(n)+\frac{64\cdot 27}{17}\,w_2(n)$$
$$+\frac{32\cdot 21}{17}\,w_3(n)+\frac{896}{17}\,w_4(n),$$
where $w_1(n)$, $w_2(n)$, $w_3(n)$ and $w_4(n)$ respectively denote the coefficients of Q^n in the expansions of the functions
$$\frac{1}{512(\pi i)^3}\,\vartheta_{80}'^3(\tau;0,24),\quad \frac{1}{2048(\pi i)^3}\,\vartheta_{16,0}'^2(\tau;0,24)\vartheta_{80}'(\tau;0,24),$$
$$\frac{1}{1024(\pi i)^3}\,\vartheta_{80}'^2(\tau;0,24)\vartheta_{16,0}'(\tau;0,24),\quad \frac{1}{4096(\pi i)^3}\,\vartheta_{16,0}'^3(\tau;0,24)$$
in powers of Q.

The values of $\rho(n;f)$ are given in (2.12).

Further,

a) from (2.20) it follows that
$$\frac{1}{512(\pi i)^3}\,\vartheta_{80}'^3(\tau;0,24)$$
$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(6m_1+1)(6m_2+1)(6m_3+1)Q^{((6m_1+1)^2+(6m_2+1)^2+(6m_3+1)^2)/3}$$
$$=\sum_{n=1}^{\infty}\Big(\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3\Big)Q^n,$$
i.e.,
$$w_1(n)=\sum_{\substack{x_1^2+x_2^2+x_3^2=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 6}} x_1x_2x_3;$$
it is obvious that $w_1(n)=0$ for $n\equiv 0 \pmod 2$ and $n=3 \pmod 4$.

b) from (2.22) it follows that

$$\frac{1}{2048(\pi i)^3}\,\vartheta'^2_{16,0}(\tau;0,24)\vartheta'_{80}(\tau;0,24)$$

$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(3m_1+1)(3m_2+1)(6m_3+1)Q^{(4(3m_1+1)^2+4(3m_2+1)^2+(6m_3+1)^2)/3}$$

$$=\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{4(x_1^2+x_2^2)+x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 3),\ x_3\equiv 1 \pmod 6}} x_1x_2x_3\Bigg)Q^n,$$

i.e.,

$$w_2(n)=\sum_{\substack{4(x_1^2+x_2^2)+x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 3),\ x_3\equiv 1 \pmod 6}} x_1x_2x_3;$$

it is obvious that $w_2(n)=0$ for $n\equiv 0 \pmod 2$ and $n\equiv 1 \pmod 4$.

c) from (2.24) it follows that

$$\frac{1}{1024(\pi i)^3}\,\vartheta'^2_{80}(\tau;0,24)\vartheta'_{16,0}(\tau;0,24)$$

$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(6m_1+1)(6m_2+1)(3m_3+1)Q^{((6m_1+1)^2+(6m_2+1)^2+4(3m_3+1)^2)/3}$$

$$=\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{x_1^2+x_2^2+4x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 6),\ x_3\equiv 1 \pmod 3}} x_1x_2x_3\Bigg)Q^n,$$

i.e.,

$$w_3(n)=\sum_{\substack{x_1^2+x_2^2+4x_3^2=3n\\ x_1\equiv x_2\equiv 1 \pmod 6),\ x_3\equiv 1 \pmod 3}} x_1x_2x_3;$$

it is obvious that $w_3(n)=0$ for $n\equiv 1 \pmod 2$ and $n\equiv 0 \pmod 4$.

d) from (2.26) it follows that

$$\frac{1}{4096(\pi i)^3}\,\vartheta'^3_{16,0}(\tau;0,24)$$

$$=\sum_{m_1,m_2,m_3=-\infty}^{\infty}(3m_1+1)(3m_2+1)(3m_3+1)Q^{4((3m_1+1)^2+(3m_2+1)^2+(3m_3+1)^2)/3}$$

$$=\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3\Bigg)Q^n,$$

i.e.,

$$w_4(n)=\sum_{\substack{4(x_1^2+x_2^2+x_3^2)=3n\\ x_1\equiv x_2\equiv x_3\equiv 1 \pmod 3}} x_1x_2x_3;$$

it is obvious that $w_4(n)=0$ for $n\equiv 1 \pmod 2$ and $n\equiv 2 \pmod 4$. □

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