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ON STOCHASTIC DIFFERENTIAL EQUATIONS IN A CONFIGURATION SPACE

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Abstract. Infinite systems of stochastic differential equations for randomly perturbed particle systems with pairwise interaction are considered. It is proved that under some reasonable assumption on the potential function there exists a local weak solution to the system and it is weakly locally unique for a wide class of initial conditions.

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1. Introduction

We consider a sequence of R^d-valued stochastic processes

$$x_k(t), \quad k = 1, 2, \dots ,$$

satisfying the system of stochastic differential equations of the form

$$dx_k(t) = \sum_{i \neq k} a(x_k(t) - x_i(t))dt + \sigma dw_k(t), \quad k = 1, 2, \dots , \tag{1}$$

where $a(x) = -U_x(x)$, and $U : R^d \to R$ is a smooth function for $|x| > 0$, and $\sigma > 0$ is a constant, $w_k(t)$, $k = 1, 2, \dots$, is a sequence of independent Wiener processes in R^d. System (1) describes the evolution of systems of pairwise interacting particles with the pairwise potential $U(x)$ which is perturbed by Wiener noises. The problem is to find conditions under which the system has a solution and this solution is unique.

Unperturbed systems were considered by many authors. We notice the recent articles of S. Albeverio, Yu. G. Kondratiev, and M. Röckner [1], [2] where a new powerful method for the investigation of unperturbed systems is proposed. Finite-dimensional perturbed systems were considered in my book [3] and my article [4]. The first general theorem on the existence and uniqueness of the solutions to infinite dimensional stochastic differential equations were obtained by Yu. L. Daletskii in [5]; he considered equations with smooth coefficients in a Hilbert space. The existence and uniqueness of the solution to system (1) for locally bounded smooth potentials and $d \leq 2$ was proved by J. Fritz in [6]. The main result of this article was published in [7] without a complete proof.

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2. The Space Γ

It is convenient to consider system (1) in the configuration space Γ which is the set of locally finite counting measures γ on the Borel σ-algebra $\mathcal{B}(R^d)$ of the space R^d. So a measure $\gamma \in \Gamma$ satisfies the condition: the support S_γ of the measure γ is a sequence of different points $\{x_k, k \in \mathcal{N}\}$ of R^d for which

$$|x_k| \to \infty, \quad \gamma(A) = \sum_k 1_A(x_k).$$

The topology in Γ is generated by the weak convergence of measures: $\gamma_n \to \gamma_0$ if

$$\int \phi(x)\gamma_n(dx) \to \int \phi(x)\gamma_0(dx)$$

for $\phi \in \mathcal{C}_f$ where $\mathcal{C}_f$ is the set of continuous functions $\phi : R^d \to R$ with bounded supports.

We use the notation

$$\langle \phi, \gamma \rangle = \int \phi(x)\gamma(dx), \quad \phi \in \mathcal{C}_f,$$

and

$$\langle \Phi, \gamma \times \gamma \rangle = \int \Phi(x, x')\gamma(dx)\gamma(dx') - \int \Phi(x, x)\gamma(dx),$$

where $\Phi : (R^d)^2 \to R$ is a continuous function with a bounded support.

We rewrite system (1) for Γ-valued function γ_t for which

$$\langle \phi, \gamma \rangle = \sum_k \phi(x_k(t)), \quad \phi \in \mathcal{C}_f.$$

Using the Itô's formula, and considering the function a as a function of two variables $a(x - x')$, we obtain the relation

$$\begin{aligned} d\langle \phi, \gamma_t \rangle &= \langle (\phi', a), \gamma_t \times \gamma_t \rangle dt + \frac{\sigma^2}{2}\langle \Delta\phi, \gamma_t \rangle \\ &\quad + \sum_k \sigma(\phi'(x_k(t)), dw_k(t)), \quad \phi \in \mathcal{C}_f^{(2)}, \end{aligned} \tag{2}$$

where $\Delta\phi(x) = \mathrm{Tr}\phi''(x)$, and $\mathcal{C}_f^{(2)}$ is the set of $\phi \in \mathcal{C}_f$ for which $\phi'(x)$ and $\phi''(x)$ are continuous bounded functions.

A weak solution to equation (1). A Γ-valued stochastic process $\gamma_t(\omega)$ is called a weak solution to system (2) if, for all $\phi \in \mathcal{C}_f^{(2)}$, the stochastic process

$$\mu_\phi(\omega, t) = \langle \phi, \gamma_t \rangle - \int_0^t [\langle (\phi', a), \gamma_s \times \gamma_s \rangle + \frac{\sigma^2}{2}\langle \Delta\phi, \gamma_s \rangle] ds \tag{3}$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t = \sigma(\gamma_s, s \leq t)$, and the square characteristic of the martingale is

$$\langle \mu_\phi, \mu_\phi \rangle_t = \sigma^2 \int_0^t \langle (\phi', \phi'), \gamma_s \rangle ds. \tag{4}$$

If $\gamma_t(\omega)$ is a weak solution to system (2) and

$$\langle \phi, \gamma_t(\omega)\rangle = \sum \phi(x_k(t))$$

for all $\phi \in \mathcal{C}_f$, then the sequence $\{x_k(t), k \in \mathcal{N}\}$ is a weak solution to system (1).

Weak uniqueness. Let $\gamma_0 \in \Gamma$. System (2) has a unique weak solution with the initial value γ_0 if for any pair of weak solutions to system (2) $\gamma_t^1(\omega)$ and $\gamma_t^2(\omega)$ satisfying the condition $\gamma_0^1 = \gamma_0^2 = \gamma_0$, the following relations are fulfilled:

$$E\Phi(\xi_{11}^1, \ldots, \xi_{1m}^1, \ldots, \xi_{l1}^1, \ldots, \xi_{lm}^1) = E\Phi(\xi_{11}^2, \ldots, \xi_{1m}^2, \ldots, \xi_{l1}^2, \ldots, \xi_{lm}^2), \quad (5)$$

where $\Phi(y_{11}, \ldots, y_{lm})$ is a continuous bounded function on R^{lm} and,

$$\xi_{ij}^k = \langle \phi_i, \gamma_{t_j}^k(\omega)\rangle, \quad k = 1, 2, \quad i = 1, \ldots, l, \quad j = 1, \ldots, m,$$
$$\phi \in \mathcal{C}_f, \quad t_1, \ldots, t_m \in R_+.$$

This means that the distribution of the stochastic process γ_t^k does not depend on k, i.e., the distribution of the weak solution to system (2) is unique if it exists.

Local weak solutions. Let $\gamma_t(\omega)$ be a continuous Γ-valued stochastic process, and $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by it. $\gamma_t(\omega)$. is called a local weak solution to system (2) if there exists a stopping time τ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for which $P\{\tau > 0\} = 1$ and the stochastic process $\mu_\phi(\omega, t)$ which is determined by relation (3) is a martingale for $t < \tau$ with the square characteristic given by equality (4).

Local weak uniqueness. Let $\gamma_0 \in \Gamma$. System (2) has a locally unique weak solution with the initial value γ_0 if, for any pair of locally weak solutions to system (2) γ_t^1 and γ_t^2 satisfying the condition $\gamma_0^1 = \gamma_0^2 = \gamma_0$, relation (5) is fulfilled for

$$\xi_{ij}^k = \langle \phi_i, \gamma_{t_j}^k(\omega)\rangle I_{\{t_j < \tau^k\}}\,,$$

where τ^k, $k = 1, 2$, are stopping times with respect to the filtration $(\mathcal{F}_t^k)_{t\geq 0}$ generated by the stochastic process $\gamma_t^k(\omega)$.

Compacts in Γ. For any $\gamma \in \Gamma$ and a continuous decreasing function $\lambda(t) : (0, \infty) \to R_+$ for which $\lambda(0+) = +\infty, \lambda(+\infty) > 0$ there exists a continuous decreasing function $\Phi(t) : [0, \infty) \to R_+$ with $\Phi(+\infty) = 0$ such that

$$\iint \Phi(|x|)\Phi(|x'|)\lambda(|x - x'|)1_{\{x\neq x'\}}\gamma(dx)\gamma(dx') < \infty. \quad (6)$$

Denote

$$\Phi_\lambda(x, x') = \Phi(|x|)\Phi(|x'|)\lambda(|x - x'|). \quad (7)$$

For any compact set K from Γ and any fanction λ satisfying the conditions mentioned before there exists a function of the form given by relation (7) for which

$$\sup_{\gamma\in K}\langle \Phi_\lambda, \gamma \times \gamma\rangle < \infty.$$

Note that the set

$$\{\gamma : \langle \Phi_\lambda, \gamma \times \gamma \rangle \le c\}$$

is a compact in Γ for any Φ_λ of form (7) and $c > 0$. Denote by $\Gamma_{\Phi,\lambda}$ the set of those $\gamma \in \Gamma$ for which relation (6) is fulfilled. Set

$$d_{\Phi,\lambda}(\gamma_1, \gamma_2) = \sup\{|\langle \phi\Phi, \gamma_1\rangle - \langle \phi\Phi, \gamma_2\rangle : \phi \in \mathrm{Lip}^1|\} + |\langle \Phi_\lambda, \gamma_1 \times \gamma_1\rangle - \langle \Phi_\lambda, \gamma_2 \times \gamma_2\rangle|,$$

where

$$\mathrm{Lip}^1 = \left\{ \phi \in C_f : \sup_x |\phi(x)| \le 1, \quad \sup_{x,x'} \frac{|\phi(x) - \phi(x')|}{|x - x'|} \le 1 \right\}.$$

$\Gamma_{\Phi,\lambda}$ with the distance $d_{\Phi,\lambda}$ is a separable locally compact space.

3. An Extension of Girsanov's Formula

Assume that the potential function $U(x)$ satisfies the condition

(PC) $U(x) = u(|x|)$ where the function $u : (0, \infty) \to R$ is continuous, it has continuous derivatives u', u'', there exists a constant $r > 0$ for which $u(t) = 0$ for $t > r$, and

$$\int t^{d-1} |u(t)| dt < \infty.$$

Free particle processes. Let a measure γ_0 satisfy the condition **(IC)**

$$\langle \phi_\delta, \gamma_0 \rangle < \infty, \quad \phi_\delta(x) = \exp\{-\delta |x|^2\}.$$

Introduce Γ-valued stochastic processes by the relation

$$\langle \phi, \gamma_t^*(\gamma_0, \omega)\rangle = \sum_k \phi(x_k^0 + \sigma w_k(t)),$$

where

$$\sum_k \phi(x_k^0) = \langle \phi, \gamma_0 \rangle.$$

It is easy to check that $\gamma_t^*(\gamma_0, \omega)$ is a continuous Γ-valued stochastic process if $\gamma_0 \in \Gamma^0$, where Γ^0 is the set of finite measures from Γ. There exist functions Φ, λ for which $P\{\gamma_t^*(\gamma_0, \omega) \in \Gamma_{\Phi,\lambda}\} = 1$ for all $t > 0$. The stochastic process $\gamma_t^*(\gamma_0, \omega)$ is continuous in the space $\Gamma_{\Phi,\lambda}$, and for any $t_0 > 0$ the function

$$EF(\gamma^*(\gamma_0, \omega))$$

is a continuous function in $\gamma_0 \in \Gamma_{\Phi,\lambda}$ if F is a bounded continuous function on $C_{[0,t_0]}(\Gamma_{\Phi,\lambda})$ which is the space of continuous $\Gamma_{\Phi,\lambda}$-valued functions on the interval $[0, t_0]$.

Girsanov's formula for finite systems. Let γ_0^n be a sequence of finite measures from Γ satisfying the condition:

$$\gamma_0^n \to \gamma_0, \gamma_0^n \le \gamma_0^{n+1}.$$

It was proved in [4] that under the condition **(PC)**, for any n, there exists a unique strong solution to system (2) with the initial value γ_0^n. Denote it by $\gamma_t(\gamma_0^n, \omega)$.

Lemma 1. *Set* $a(x,x') = a(x-x')$,

$$G_1^n(t) = \sigma^{-1} \sum_{x_i \in S_{\gamma_0^n}} \int_0^t (\langle a(x_i + \sigma w_i(s), .), \gamma_s^*(\gamma_0^n, \omega)\rangle, dw_i(s)),$$

$$G_2^n(t) = \sigma^{-2} \int_0^t \int |\langle a(x,.), \gamma_s^*(\gamma_0^n, \omega)\rangle|^2 \gamma_s^*(\gamma_0^n, \omega, dx) ds,$$

$$\rho_n(t) = \exp\left\{G_1^n(t) - \tfrac{1}{2} G_2^n(t)\right\}.$$

Then

$$E\Phi(\langle \phi_1, \gamma_{t_1}(\gamma_0^n, \omega)\rangle, \dots, \langle \phi_k, \gamma_{t_k}(\gamma_0^n, \omega)\rangle)$$
$$= E\rho_n(t)\Phi(\langle \phi_1, \gamma_{t_1}^*(\gamma_0^n, \omega)\rangle, \dots, \langle \phi_k, \gamma_{t_k}^*(\gamma_0^n, \omega)\rangle)$$

for all $k = 1, 2, \dots$ *, bounded continuous functions* $\Phi : R^k \to R$ *and* $\phi_1, \dots, \phi_k \in C_f, t_1, \dots, t_k \in [0,t]$.

The proof of the lemma can be obtained using the approximation of the function $a(x,x')$ by smooth functions since for smooth $a(x,x')$ the proof is a consequence of Girsanov's formula [8].

Introduce the stochastic processes

$$w_k^c(t) = \int_0^t 1_{\{|x_k + w_k(s)| \le c\}} dw_k(s), \quad x_k \in S_{\gamma_0^n},$$

where $c > 0$ is a constant. Let $\mathcal{F}_t^{n,c}$ be the σ -algebra generated by

$$\{w_k^c(s), s \le t, x_k \in S_{\gamma_0^n}\}.$$

Lemma 2.

$$E(\rho_n / \mathcal{F}_t^{n,c}) = \rho_n(c,t),$$

where

$$\rho_n(c,t) = \exp\left\{G_1^n(c,t) - \tfrac{1}{2} G_2^n(c,t)\right\},$$

and

$$G_1^n(c,t) = \sigma^{-1} \sum_{x_i \in S_{\gamma_0^n}} \int_0^t (E(\langle a(x_i + \sigma w_i(s), .), \gamma_s^*(\gamma_0^n, \omega)\rangle / \mathcal{F}_s^{n,c}), dw_i^c(s)),$$

$$G_2^n(c,t) = \sigma^{-2} \int_0^t \int |E(\langle a(x,.), \gamma_s^*(\gamma_0^n, \omega)\rangle / \mathcal{F}_s^{n,c})|^2 1_{\{|x| \le c\}} \gamma_s^*(\gamma_0^n, \omega, dx) ds.$$

The proof rests on the statement below.

Statement. Let $\mathcal{F}_t, t \in R_+$ be a continuous filtration, and its subfiltration $\mathcal{F}_t^*, t \in R_+$, satisfy the condition

(RE)$E(\xi / \mathcal{F}_t)$ is a $\mathcal{F}_t^*$-measurable random variable if ξ is a bounded $\mathcal{F}_\infty^*$-measurable random variable.

Let μ_t be an $\mathcal{F}_t$ -martingale with the square characteristic $\langle\mu\rangle_t$ for which the stochastic process

$$\rho(t) = \exp\left\{\mu_t - \tfrac{1}{2}\langle\mu\rangle_t\right\}$$

is a martingale. Then

$$E(\rho(t)/\mathcal{F}_t^*) = \exp\left\{\mu_t^* - \tfrac{1}{2}\langle\mu^*\rangle_t\right\},$$

where $\mu_t^* = E(\mu_t/\mathcal{F}_t^*)$ is a martingale, and $\langle\mu^*\rangle_t$ is its square characteristic.
Proof. Set $\tilde{\mu}_t = \mu_t - \mu_t^*$. Condition **(RE)** implies the relation

$$E\left(\int_0^t g(s)d\tilde{\mu}_s/\mathcal{F}_\infty^*\right) = 0$$

for all $\mathcal{F}_t$-adapted functions $g(t)$ for which

$$E\left|\int_0^t g(s)d\tilde{\mu}_s\right| < \infty.$$

Set

$$\rho^*(t) = \exp\left\{\mu_t^* - \tfrac{1}{2}\langle\mu^*\rangle_t\right\}, \quad \tilde{\rho}(t) = \exp\left\{\tilde{\mu}_t - \tfrac{1}{2}\langle\tilde{\mu}\rangle_t\right\},$$

where $\langle\tilde{\mu}\rangle_t$ is the square characteristic of the martingale $\tilde{\mu}_t$. The proof follows from the relations $\rho(t) = \rho^*(t)\tilde{\rho}(t)$ and

$$E(\tilde{\rho}(t) - 1/\mathcal{F}_\infty^*) = E\left(\int_0^t \tilde{\rho}(s)d\tilde{\mu}(s)/\mathcal{F}_\infty^*\right) = 0. \quad \square$$

Remark 1. Assume that $\phi_i(x) = 0$ for $|x| \geq c, i = 1, 2, \ldots, k, \phi_1, \ldots, \phi_k \in \mathcal{C}_f, t_1, \ldots, t_k \in [0, t]$. Then for Φ satisfying the conditions of Lemma 1 we have the relation

$$\begin{aligned} &E\Phi(\langle\phi_1, \gamma_{t_1}(\gamma_0^n, \omega)\rangle, \ldots, \langle\phi_k, \gamma_{t_k}(\gamma_0^n, \omega)\rangle) \\ &= E\rho_n(c,t)\Phi(\langle\phi_1, \gamma_{t_1}^*(\gamma_0^n, \omega)\rangle, \ldots, \langle\phi_k, \gamma_{t_k}^*(\gamma_0^n, \omega)\rangle). \end{aligned}$$

Remark 2. Denote by $\mathcal{F}_t^c$ the σ-algebra generated by $\{w_k^c(s), s \leq t, x_k \in S_{\gamma_0}\}$. Then there exist the limits in probability

$$\lim_{n\to\infty} G_1^n(c,t) = G_1(c,t), \quad \lim_{n\to\infty} G_2^n(c,t) = G_2(c,t),$$

$$\lim_{n\to\infty} \rho_n(t) = \rho(c,t) = \exp\left\{G_1(c,t) - \frac{1}{2}G_2(c,t)\right\},$$

where

$$G_1(c,t) = \sigma^{-1}\sum_{x_i \in S_{\gamma_0}} \int_0^t (E(\langle a(x_i + \sigma w_i(s), .), \gamma_s^*(\gamma_0, \omega)\rangle/\mathcal{F}_s^c), dw_i^c(s))),$$

$$G_2(c,t) = \sigma^{-2}\int_0^t \int |E(\langle a(x,.), \gamma_s^*(\gamma_0,\omega)\rangle/\mathcal{F}_s^c)|^2 1_{\{|x|\leq c\}}\gamma_s^*(\gamma_0,\omega,dx)ds.$$

Let $w(t)$ be a standard Wiener process . Introduce the functions

$$Q_c(s,x,B) = P\{x+\sigma w(s)\in B, \inf_{u\le s}|x+\sigma w(u)|>c\},$$
$$x\in V_c,\ \ B\in\mathcal{B}(V_c),\ \ V_c=\{x\in R_d : |x|>c\},$$
$$Q_c^*(s,x,B)=\lim_{\lambda\downarrow 1}\frac{Q_c(s,x,B)}{Q_c(s,\lambda x,V_c)},\ \ |x|=c,\ \ B\in\mathcal{B}(V_c).$$

Set

$$\theta_i(c,s)=\inf\Delta_i(c,s),\ \ \zeta_i(c,s)=\sup\Delta_i(c,s),$$

where

$$\Delta_i(c,s)=\{u\le s : |x_i^*(u)|\le c\},\ \ x_i^*(u)=x_i+w_i(u).$$

Then the following statement holds.

Lemma 3.

$$E(a(x,x_i^*(s))1_{\{|x_i^*(s)|\le c\}}/\mathcal{F}_s^c)=a(x,x_i^*(s))1_{\{|x_i^*(s)|\le c\}}$$
$$+1_{\{\theta_i(c,s)<\infty\}}\int a(x,z)Q_c^*(s-\zeta_i(c,s),x_i^*(\zeta_i(c,s)),dz)$$
$$+1_{\{\theta_i(c,s)=+\infty\}}\int a(x,z)Q_c(s,x_i,dz).$$

Corollary 1. *Introduce the measures*

$$\nu_s(A)=\sum_k 1_{\{x_k\in A\}}1_{\{\theta_k(c,s)=+\infty\}},\ \ A\in\mathcal{B}(V_c),$$
$$\nu_s^*(\Lambda,A)=\sum_k 1_{\{\zeta_k(c,s)\in\Lambda\}}1_{\{x_k(\zeta_k(c,s))\in A\}},$$
$$\Lambda\in\mathcal{B}([0,s]),\ \ A\in\mathcal{B}(V_c'),\ \ V_c'=\{x\in R_d : |x|=c\}.$$

Then

$$E(\langle a(x,\cdot),\gamma_s^*(\gamma_0,\omega)\rangle/\mathcal{F}_s^c)=a_c(s,x,\omega)+\int a(x,x')1_{\{|x'|\le c\}}\gamma_s^*(\gamma_0,\omega,dx'),$$

where

$$a_c(s,x,\omega)=\iint a(x,z)Q_c(s,x',dz)\nu_s(dx')$$
$$+\iiint a(x,z)Q_c^*(s-u,x',dz)\nu_s^*(du,dx').$$

Corollary 2. *The functions $G_k(c,t)$, $k=1,2$, can be represented in the form*

$$G_1(c,t)=\sigma^{-1}\int_0^t\sum_i(a_c(s,x_i^*(s),\omega),dw_i^c(s)),$$

and

$$G_2(c,t)=\sigma^{-2}\int_0^t H(c,s)ds,$$

where

$$H_c(s) = \sum_i \Big| \sum_{j\neq i} a(x_i^*(s), x_j^*(s)) 1_{\{|x_j^*(s)|\leq c\}} + a_c(s, x_i^*(s), \omega)\Big|^2 1_{\{|x_i^*(s)|\leq c\}}.$$

Remark 3. Denote by $(\mathcal{F}_t^c(i))$ the filtration generated by the stochastic process $w_i^c(t)$, and by $(\mathcal{F}_t^c(i,j))$ the filtration generated by the pair of stochastic processes $(w_i^c(t); w_j^c(t))$, $i \neq j$. Set

$$\rho^i(c,t) = E(\rho(c,t)/\mathcal{F}_t^c(i)), \quad \rho^{i,j}(c,t) = E(\rho(c,t)/\mathcal{F}_t^c(i,j).$$

Then

$$\rho^i(c,t) = \exp\Big\{ \int_0^t (g^i(c,s), dw_i^c(s)) - \frac{1}{2}\int_0^t |g^i(c,s)|^2 ds \Big\},$$

where

$$\sigma g^i(c,s) = \Big[E(a_c(s,x) + \sum_{k\neq i} a(x, x_k^*(s)) 1_{\{|x_k^*(s)|\leq c\}}) \Big]_{\{x = x_i^*(s)\}},$$

and

$$\rho^{i,j}(c,t) = \exp\Big\{ \int_0^t [(g_{ij}^i(c,s), dw_i^c(s)) + (g_{ij}^j(c,s), dw_j^c(s))] \Big\}$$
$$= \exp\Big\{ -\frac{1}{2}\int_0^t [|g_{ij}^i(c,s)|^2 + |g_{ij}^j(c,s)|^2] ds \Big\},$$

where

$$\sigma g_{ij}^i(c,s) = a(x_i^*(s), x_j^*(s)) 1_{\{|x_i^*(s)|\wedge|x_j^*(s)|\leq c\}}$$
$$+ \Big[E(a_c(s,x) + \sum 1_{\{k\neq i\}} 1_{\{k\neq j\}} a(x, x_k^*(s)) 1_{\{|x_k^*(s)|\leq c\}}) \Big]_{\{x = x_i^*(s)\}}.$$

Remark 4. Assume that τ_c is a stopping time with respect to the filtration $(\mathcal{F}_t^c)_{t\geq 0}$ satisfying the condition $G_2(c,\tau_c) \leq c_1$, where c_1 is a constant. Then $E\rho(c,\tau_c) = 1$ and $E(\rho(c,\tau_c))^2 \leq \exp\{c_1\}$.

Introduce the stochastic processes

$$w_i^*(c,t) = w_i(t) - \sigma^{-1}\int_0^{t\wedge\tau_c} a_c(s, x_i^*(s), \omega) 1_{\{|x_i^*(s)|\leq c\}} ds$$
$$-\sigma^{-1}\int_0^{t\wedge\tau_c} \sum_{i\neq j} a(x_i^*(s), x_j^*(s)) 1_{\{|x_i^*(s)|\vee|x_j^*(s)|\leq c\}} ds. \qquad (8)$$

Lemma 4. *Denote by $\{\Omega, \mathcal{F}, P\}$, the probability space generated by the sequence $\{w_k(t), k = 1, 2, \dots\}$ and let P_c be the measure on $\{\Omega, \mathcal{F}\}$ for which*

$$\frac{dP_c}{dP}(\omega) = \rho(c,\tau_c).$$

Then $\{w_k(c,t), k = 1,2,\dots\}$ *is the sequence of independent Wiener processes on the probability space* $\{\Omega, \mathcal{F}.P_c\}$.

The proof is a consequence of Girsanov's results [8].

Remark 5. Let $c_1 < c_2$ and τ_{c_k} be stopping time with respect to the filtration $(\mathcal{F}_t^{c_k})_{t\geq 0}, k = 1,2, \tau_{c_1} < \tau_{c_2}$ and $G_2(c_1,\tau_{c_1}) + G_2(c_2,\tau_{c_2}) \leq c_3$, where c_3 is a constant. Then

$$E(\rho(c_2,\tau_{c_2})/\mathcal{F}^{c_1}_{\tau_{c_1}}) = \rho(c_1,\tau_{c_1}).$$

This formula is a consequence of the relation

$$E(G_1(c_2,\tau_{c_2})/\mathcal{F}^{c_1}_{\tau_{c_1}}) = G_1(c_1,\tau_{c_1}).$$

Lemma 5. *Let* $\{c_k, k = 1,2,\dots\}$ *be a sequence of positive numbers, for which* $\lim_{k\to+\infty} c_k = +\infty$*. Then there exists a sequence of positive numbers* $\{a_k\}$ *for which*

$$P\Big\{\sum_k a_k G_2(c_k,t) < \infty\Big\} = 1$$

for all $t > 0$.

Proof. Choose a_k satisfying the inequality

$$P(G_2(c_k,k) > (k^2 a_k)^{-1}) < k^{-2}.$$

Then for any $t > 0$ we have the relation

$$\sum_k P(a_k G_2(c_k,t) > k^{-2}) < t + \sum_{k\geq t} k^{-2} < \infty.$$

This completes the proof. $\square$

Corollary 3. *Let a sequence* $\{a_k\}$ *satisfy the statement of Lemma* 5*. Set*

$$G(t) = \sum_k a_k G_2(c_k,t).$$

With probability 1 $G(t)$ *is an increasing continuous function satisfying the relations* $G(0) = 0, \lim_{t\to\infty} G(t) = \infty$*. Set*

$$\tau^* = \inf\{t : G(t) > b\}, \tag{9}$$

where b *is a positive number. Then*

$$E\rho(c_k,\tau^*) = 1, \;\; E(\rho(c_k,\tau^*))^2 \leq \exp\{b a_k^{-1}\}.$$

4. A Theorem on Existence of a Local Weak Solution

Theorem 1. *Let conditions* **(PC)** *and* **(IC)** *be fulfillrd,and let τ^* be a stopping time introduced by relation* (9). *Then the following statements hold:*

(i) *there exists a probability measure P^* on $(\Omega, \mathcal{F})$ for which*

$$P^*(A) = \lim_{c\to\infty} E1_A\rho(c,\tau*), \quad A \in \bigvee_k \mathcal{F}_{\tau^*}^{c_k}, \quad P^*(A) = P(A), A \in \mathcal{F}^*,$$

where the σ-algebra $\mathcal{F}^*$ is generated by the processes

$$\{x_k^*(t \vee \tau^*) - x_k^*(t), \quad t \geq 0, \quad k = 1, 2, \dots\}.$$

(ii) *the stochastic processes given by the formula*

$$\sigma w_k^*(t) = x_k^*(t) - \int\limits_0^{t\wedge\tau^*} \sum_{i\neq k} a(x_k^*(s), x_i^*(s))ds, \quad k = 1, 2, \dots,$$

are independent Wiener processes with respect to the filtration $(\mathcal{F}_t)_{(t\geq 0)}$ on the probability space $(\Omega, \mathcal{F}, P^*)$.

Proof. Since τ^* is a stopping time, for $A \in \mathcal{F}_{\tau^*}^{c_k}$ we can write, using Remark 5, the relations

$$E1_A\rho(c,\tau^*) = E1_A\rho(c_k,\tau^*)$$

if $c > c_k$. This implies the existence of limits

$$\lim_{c\to\infty} 1_{A\times B}\rho(c,\tau^*), \quad A \in \bigvee_k \mathcal{F}_{\tau^*}^{c_k}, \quad B \in \mathcal{F}^*.$$

It follows from the last formula that there exists a limit

$$\lim_{c\to\infty} \Phi(\xi_1,\dots,\xi_m)\rho(c,\tau^*), \tag{10}$$

where $\Phi \in \mathcal{C}(R^m)$ and

$$\xi_i = \langle f_i, \gamma_{t_i}^*(\gamma_0,\omega)\rangle, \quad i = 1,\dots,m,$$

where $f_i \in \mathcal{C}(R^d)$, $f_i(x) = 0$ for $|x|$ large enough.

Now we prove that there exists a probability measure P^* on $(\Omega, \mathcal{F})$ for which a limit in the formula (10) is represented in the form $E^*\Phi(\xi_1,\dots,\xi_m)$, where E^* is the expectation with respect to the probability P^*. Set

$$f_n(x,x') = \Big(1 - \frac{|x|}{c_n}\Big)\Big(1 - \frac{|x'|}{c_n}\Big)\Big(1 + \frac{1}{|x-x'|}\Big) \vee 0.$$

Using Corollary 3 we can write the inequality

$$\begin{aligned} &E\langle f_n, \gamma_t^*(\gamma_0,\omega) \times \gamma_t^*(\gamma_0,\omega)\rangle\rho(c,\tau^*) \\ &\leq (E\langle f_n, \gamma_t^*(\gamma_0,\omega) \times \gamma_t^*(\gamma_0,\omega)\rangle^2)^{\frac12} \exp\{\tfrac12 ba_n^{-1}\}. \end{aligned} \tag{11}$$

Denote

$$A_n(t) = E\langle f_n, \gamma_t^*(\gamma_0,\omega) \times \gamma_t^*(\gamma_0,\omega)\rangle^2.$$

The functions $A_n(t)$ are non-negative and continuous. There exists a sequence of positive numbers $\{b_n\}$ for which

$$\sum_n b_n(1 + A_n(t)) \exp\{\tfrac{1}{2}ba_n^{-1}\} < \infty, \quad t > 0. \tag{12}$$

Formulas (11) and (12) imply the relation

$$\limsup_{c\to\infty} E\sum_n b_n\langle f_n, \gamma_t^*(\gamma_0,\omega) \times \gamma_t^*(\gamma_0,\omega)\rangle\rho(c,\tau^*) < \infty.$$

This formula implies that the set of measures $\{P_c^*, c > 0\}$ for which

$$\frac{dP_c^*}{dP}(\omega) = \rho(c,\tau^*)$$

is weakly compact, so there exists a sequence $c_k', c_k' \to \infty$ for which $P_{c_k'}^*$ converges weakly to a measure P^*. The statement (i) is proved.

Introduce the stopping times

$$\tau_n^* = \inf\Big\{t : \sum_{k\le n} a_k G_2(c_k,t) \ge b\Big\},$$

and set

$$P_n^*(A) = E1_A\rho(c_n,\tau_n^*), \quad A \in \mathcal{F}.$$

Let $\Phi(\xi_1,\dots,\xi_m)$ be the same as before. Then

$$\lim_{n\to\infty} E_n^*\Phi(\xi_1,\dots,\xi_m) = E^*\Phi(\xi_1,\dots,\xi_m);$$

here E_n^* is the expectation with respect to probability P_n^*.This formula is a consequence of the relations

$$\tau_n^* > \tau^*, \quad \tau_n^* \to \tau^*$$

in probability and

$$E(\rho(c_n,\tau_n^*)/\mathcal{F}_{\tau^*}^{c_n}) = \rho(c_n,\tau*).$$

Let $w_i^*(c,t)$ be given by formula (8). Then for fixed n the sequence

$$\{w_i^*(c_n,t), i = 1,2,\dots\}$$

represents independent Wiener processes on the probability space $\{\Omega,\mathcal{F},P_n^*\}$. Introduce stopping times

$$\zeta_i^n = \inf\{t : |x_i(t)| > c_n - r\}.$$

Since $a_c(x,\omega) = 0$ for $|x| < c_n - r$,we have

$$w_i^*(c_n, t\vee\zeta_i^n) = w_i^*(t\vee\zeta_i^n).$$

Using Remark 3 we can prove that the relation

$$\lim_{c\to\infty}\limsup_{n\to\infty} P_n^*\{\sup_{s\le t}|x_i^*(s)| > c\} = 0 \tag{13}$$

is fulfilled for any i and $t>0$. Let $\Phi, h_1, \dots, h_m$ be the same as before . Set

$$\xi_i^* = \int_0^t h_i(s)dw_i^*(s), \quad \xi_i^n = \int_0^t h_i(s)dw_i^*(c_n, s), \;\; i = 1, \dots .$$

Note that

$$E_n^*\Phi(\xi_1^n, \dots, \xi_m^n) = E\Phi\bigg(\int_0^t h_1(s)dw_1(s), \dots, \int_0^t h_m(s)dw_m(s)\bigg) \tag{14}$$

for all n. Denote the expression in the right hand side of formula (14) by $\bar{\Phi}$. Then

$$E_n^*\Phi(\xi_1^*, \dots, \xi_m^*) = \bar{\Phi} + E_n^* O\bigg(\sum_{i\le m} 1_{\{\sup_{s\le t}|x_i^*(s)|>c_n-r\}}\bigg). \tag{15}$$

Formulas (13) and (15) imply the relation

$$E^*\Phi(\xi_1^*, \dots, \xi_m^*) = \lim_{n\to\infty} E_n^*\Phi(\xi_1^*, \dots, \xi_m^*) = \bar{\Phi}.$$

The statement (ii) is proved. $\square$

References

1. S. Albeverio, Yu. G. Kondratiev, and Röckner, Analysis and geometry on configuration spaces. *J. Funct. Anal.* **154**(1998), 444–500.
2. S. Albeverio, Yu. G. Kondratiev, and Röckner, Analysis and geometry on configuration spaces: the Gibbsian case. *J. Funct. Anal.* **157**(1998), 242–291.
3. A. V. Skorokhod, Stochastic equations for a complex system. *D. Reidel Publishing Co., Dordrecht-Boston, Mass.,* 1988.
4. A. V. Skorokhod, On the regularity of many-particle dynamical systems perturbed by white noise. *J. Appl. Math. Stoch. Anal.* **9**(1996), 427–437.
5. Yu. L. Daletskii, Infinite-dimensional elliptic operators and related parabolic equations. (Russian) *Uspekhi. Mat. Nauk* **22**(1967), 3–54.
6. J. Fritz, Gradient dynamics of infinite point systems. *Ann. Probab.* **15**(1987), 478–514.
7. A. V. Skorokhod, On infinite systems of stochastic differential equations. *Methods Funct. Anal. Topology* (1999), No. 4, 54–61.
8. I. V. Girsanov, On a transformation of some class of stochastic processes by absolute continuous changing of measures. *Theory Probab. Appl.* **5**(1960), 314–330.

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