Georgian Mathematical Journal
Volume 8 (2001), Number 2, 401–414

ON ACCURACY OF IMPROVED χ^2-APPROXIMATIONS

V. V. ULYANOV AND Y. FUJIKOSHI

Abstract. For a statistic S whose distribution can be approximated by χ^2-distributions, there is a considerable interest in constructing improved χ^2-approximations. A typical approach is to consider a transformation $T = T(S)$ based on the Bartlett correction or a Bartlett type correction. In this paper we consider two cases in which S is expressed as a scale mixture of a χ^2-variate or the distribution of S allows an asymptotic expansion in terms of χ^2-distributions. For these statistics, we give sufficient conditions for T to have an improved χ^2-approximation. Furthermore, we present a method for obtaining its error bound.

2000 Mathematics Subject Classification: 62E20.
Key words and phrases: Asymptotic expansion, error bound, improved χ^2-approximations, scale mixture of χ^2-variate, transformation.

1. Introduction

Suppose that a statistic S has an asymptotic χ^2-approximation as some parameter n tends to infinity. In this case, it is of considerable interest to construct improved χ^2-approximations for the statistic S. A typical approach is to consider a transformation $T = T(S)$ based on the Bartlett correction or a Bartlett type correction. For a Bartlett type correction, see, e.g., the works by Cordeiro and Ferrari [2] and by Fujikoshi [4]. In addition to that T has a limiting χ^2-distribution with q degrees of freedom, we can expect in some cases that

$$P(T \le x) = G(x) + O(n^{-2})$$

while

$$P(S \le x) = G(x) + O(n^{-1}),$$

where G is a distribution function of a χ^2-variate χ_q^2 with q degrees of freedom. We say that T has an improved χ^2-approximation.

Our aim is to construct an improved χ^2-approximation and to obtain its error bound. First we consider the case in which S is expressed as a scale mixture of a χ^2-variate, i.e.,

$$S = Y^{-1}\chi_q^2, \tag{1.1}$$

ISSN 1072-947X / \$8.00 /

where Y is a positive random variable independent of χ_q^2. In this case, the order of the remainder term depends on the closeness of Y to 1. Next, we consider the case in which S allows an asymptotic expansion such that

$$P(S \leq x) = G_q(x) + \frac{1}{n} \sum_{j=0}^{k} a_j G_{q+2j}(x) + R_k, \tag{1.2}$$

where R_k satisfies the inequality

$$|R_k| \leq c_k/n^2 \tag{1.3}$$

with some positive constant c_k. For these statistics, we give sufficient conditions for $T(S)$ to have an improved χ^2-approximation in terms of the inverse function to $T(x)$. Furthermore, we present a method for obtaining an error bound of the improved approximation.

In Section 5 we consider a special case in which $S = \chi_q^2/Y$ with $Y = n^{-1}\chi_n^2$ and Y, χ_n^2 are independent. We show what kind of results with computed values of absolute constants can be derived in this case from the general theorems of Section 2. We also consider examples of transformations of $S = n\chi_q^2/\chi_n^2$ with independent χ_q^2 and χ_n^2 which provide better approximations. We also compare the transformations.

Note that a more general approach to constructing transformed statistics $T(S)$ with improved Pearson type approximations but without error bounds can be found in [1].

2. Scale Mixtures of χ^2-Variates

In this section, let $S = \chi_q^2/Y$ be a mixture of a χ^2-variate defined by (1.1). Set $G(x) = \mathrm{P}\{\chi_q^2 \leq x\}$, and

$$\alpha_i = \mathrm{E}(Y-1)^i \ for\ i = 1, \dots, 4,\ and\ \beta = \max\{|\alpha_3|, \alpha_4\}.$$

Under the condition $\alpha_1 = 0$ it is easy to show (see, e.g. [7]) that

$$\Big|\mathrm{P}\{S \leq x\} - G(x)\Big| \leq c\alpha_2$$

with a constant c depending only on q.

In order to improve the appoximation we consider a transformation T that is an increasing non-negative function defined on $[0, +\infty)$. We denote by b the function which is inverse to $T(x)$, i.e.,

$$b\Big(T(x)\Big) = T\Big(b(x)\Big) = x \ for\ all\ x \geq 0.$$

Theorem 2.1. *Let $S = \chi_q^2/Y$, where Y is a positive random variable independent of χ_q^2. Suppose that $\alpha_1 = 0$ and that there exist positive constants $B_i = B_i(q)$, $i = 1, 2, 3$, depending only on q such that $B_1 \leq 1$ and, for all $x > 0$, one has*

$$\begin{aligned} b(x) &\geq B_1 x, && (2.1)\\ |b(x) - x| &\leq A(x)\sqrt{\beta}, && (2.2) \end{aligned}$$

and

$$\Big|G'(x)\Big(b(x)-x\Big)+\frac{\alpha_2}{2}G''(x)x^2\Big| \le B_3\beta, \tag{2.3}$$

where

$$A(x) = B_2 x \exp\left(B_1 x/16\right). \tag{2.4}$$

Then we have for $q \ge 2$

$$\Big|P\{T(S)\le x\} - G(x)\Big| \le c\beta, \tag{2.5}$$

where c *is a constant depending only on* q *and* B_i *with* $i = 1,2,3$ (*see* (3.24)).

Remark 2.2. It is easy to see that the class of positive increasing functions b on $[0,+\infty)$ which satisfy (2.1)–(2.3) is not empty. It is enough to take

$$b(x) = x\Big(1-\frac{\alpha_2}{4}(q-2)\Big)+\frac{\alpha_2}{4}x^2. \tag{2.6}$$

If

$$\frac{\sqrt{\beta}}{4}(q-2) > 1,$$

then (2.5) easily follows with $c = \Big((q-2)/4\Big)^2$. Thus, we assume

$$\frac{\sqrt{\beta}}{4}(q-2) \le 1.$$

Therefore, since $\alpha_2 \le \sqrt{\beta}$, we obtain

$$1-\frac{\alpha_2}{4}(q-2) \ge 0$$

and $b(x)$ is increasing. Moreover, for b defined by (2.6), we have that (2.1) and (2.3) hold with

$$B_1 = 1-\frac{\alpha_2}{4}(q-2) \text{ and } B_3 = 0,$$

respectively. Condition (2.2) is also trivially satisfied (cf. Remark 4.2 after the proof of Theorem 4.1).

Remark 2.3. In fact, it is possible to obtain an inequality similar to (2.5) omitting the condition $q \ge 2$ and replacing two conditions (2.2) and (2.3) by only one condition. However, in this case we have also to replace the constant c in (2.5) by a larger one. Namely, the following theorem holds.

Theorem 2.4. *Let* $S = \chi_q^2/Y$, *where* Y *is a positive random variable independent of* χ_q^2. *Suppose that* $\alpha_1 = 0$ *and that there exist positive constants* B_1 *and* B_4 *depending only on* q *such that* $B_1 \le 1$ *and, for all* $x > 0$, *the condition* (2.1) *is satisfied and*

$$\Big|b(x)-x+\frac{\alpha_2}{4}x^2\Big(\frac{q-2}{x}-1\Big)\Big| \le B_4 x\exp(B_1 x/16)\beta. \tag{2.7}$$

Then we have for all $q \geq 1$ that

$$\Big|P\{T(S) \leq x\} - G(x)\Big| \leq c_1\beta, \tag{2.8}$$

where c_1 is a constant depending only on q, B_1, and B_4.

Remark 2.5. In Theorems 2.1 and 2.4, we give uniform error bounds. This means that the right-hand sides of (2.5) and (2.8) do not depend on x. However, by using an approach developed in [9], it is possible to construct the so-called non-uniform bounds when the right-hand sides tend to 0 as $x \to +\infty$. For example, the following theorem holds (see, e.g., Corollary 6 in [9]).

Theorem 2.6. *Let $S = \chi_q^2/Y$, where Y is a positive random variable independent of χ_q^2 and $EY^{-4} < \infty$. Suppose that $\alpha_1 = 0$ and we put*

$$T_0(x) = \frac{q-2}{2} - \frac{2}{\alpha_2} + \left(\frac{1}{4}\Big(q - 2 - \frac{4}{\alpha_2}\Big)^2 + \frac{4x}{\alpha_2}\right)^{1/2}.$$

Then we have for all $q \geq 2$ and $x > 0$ that

$$\Big|P\{T_0(S) \leq x\} - G(x)\Big| \leq \frac{c_2}{1+x^4}\left(\beta + EY^{-4} 1_{\{Y<1/2\}}\right),$$

where c_2 is a constant depending only on q and 1_A is the indicator function of an event A.

Remark 2.7. The proof of Theorem 2.6 and more general results with non-uniform bounds will be published separately.

3. Proofs of Theorems 2.1 and 2.4

Proof of Theorem 2.1. We have

$$\mathrm{P}\{T(\chi_q^2/Y) \leq x\} = \mathrm{P}\{\chi_q^2 \leq b(x) \cdot Y\} = \mathrm{E}G(Y \cdot b). \tag{3.1}$$

We consider the function $G(y \cdot b)$ as a function of two variables $F(y, b) = G(y \cdot b)$. Since $G(x)$ is smooth for $x > 0$, we can expand $F(y, b)$ at the point $(1, x)$ so that the remainder term is of order $O(\beta)$. Note that $F(1, x) = G(x)$ and

$$F(y,b) = F(y,x) + G'(yx)(b-x)y + \frac{1}{2}G''(yx')(b-x)^2y^2, \tag{3.2}$$

$$\begin{aligned} F(y,x) = {} & F(1,x) + G'(x)x(y-1) + \frac{1}{2}G''(x)x^2(y-1)^2 \\ & + \frac{1}{6}G'''(x)x^3(y-1)^3 + \frac{1}{24}G^{(4)}(y'x)x^4(y-1)^4, \end{aligned} \tag{3.3}$$

$$G'(yx) = G'(x) + G''(x)x(y-1) + \frac{1}{2}G'''(x)(y''x)x^2(y-1)^2, \tag{3.4}$$

where $x' \in (b \wedge x, b \vee x)$, $y' \in (y \wedge 1, y \vee 1)$, $y'' \in (y \wedge 1, y \vee 1)$ and as usual $b \wedge x = \min(b, x)$, $b \vee x = \max(b, x)$.

Combining (3.2)–(3.4), we arrive at the following representation:

$$
\begin{aligned}
F(y,b) \;=\;& G(x)+G'(x)\Big((b-x)y+x(y-1)\Big)\\
&+\frac{1}{2}G''(x)\Big(x^2(y-1)^2+2x(b-x)y(y-1)\Big)\\
&+\frac{1}{6}G'''(x)x^3(y-1)^3+\frac{1}{2}G''(yx')(b-x)^2y^2\\
&+\frac{1}{2}G'''(y''x)x^2(y-1)^2(b-x)y+\frac{1}{24}G^{(4)}(y'x)x^4(y-1)^4. \qquad (3.5)
\end{aligned}
$$

Our aim is to approximate $\mathrm{E}G(Y\cdot b)$ (see (3.1)) by $G(x)$ and to prove (2.5). Therefore, considering y in (3.5) as a random variable Y, we subtract from each term on the right-hand side of (3.5) its expectation excluding the terms of order $O(\beta)$. Then we arrive at an expression that we denote by $\triangle(y,b,x)$, namely

$$
\begin{aligned}
&\triangle(y,b,x)\equiv F(y,b)-G(x)-G'(x)(b-x)(y-1)-G'(x)x(y-1)\\
&-\frac{1}{2}G''(x)x^2(y-1)^2+\frac{\alpha_2}{2}G''(x)x^2-G''(x)x(b-x)y(y-1)\\
&+\alpha_2G''(x)x(b-x)-\frac{1}{6}G'''(x)x^3(y-1)^3+\frac{\alpha_3}{6}G'''(x)x^3. \qquad (3.6)
\end{aligned}
$$

It follows from (3.5) and (3.6) that $\triangle(y,b,x)$ can also be written in the form

$$
\begin{aligned}
\triangle(y,b,x) \;=\;& G'(x)(b-x)+\frac{\alpha_2}{2}G''(x)x^2+\alpha_2G''(x)x(b-x)\\
&+\frac{\alpha_3}{6}G'''(x)x^3+\frac{1}{2}G''(yx')(b-x)^2y^2\\
&+\frac{1}{2}G'''(y''x)x^2(b-x)(y-1)^2y+\frac{1}{24}G^{(4)}(y'x)x^4(y-1)^4. \qquad (3.7)
\end{aligned}
$$

Since $\alpha_1=0$ we have $\alpha_2=\mathrm{E}Y(Y-1)$. Then it follows from (3.1) and (3.6) that

$$
\mathrm{E}\triangle(Y,b,x)=\mathrm{E}F(Y,b)-G(x)=\mathrm{P}\{T(\chi_q^2/Y)\le x\}-G(x).
$$

Therefore, in order to prove the theorem, it is enough to show that

$$
\mathrm{E}|\triangle(Y,b,x)|\le c\cdot\beta. \qquad (3.8)
$$

Let us fix an arbitrary d such that $0<d<1$, and write

$$
\mathrm{E}|\triangle(Y,b,x)|=I_1+I_2,
$$

where $I_1=\mathrm{E}|\triangle(Y,b,x)|\cdot 1_{\{|Y-1|>d\}}$, $I_2=\mathrm{E}|\triangle(Y,b,x)|\cdot 1_{\{|Y-1|\le d\}}$. We prove the upper bounds for I_1 and I_2 with the help of (3.6) and (3.7), respectively.

First we consider I_1. By (3.6), we have, for all positive x and y with $|y-1| > d$, that

$$\begin{aligned}
|\triangle(y,b,x)| \leq\ & 1 + |y-1| \sup G'(x)x + \sup\left|G'(x)(b-x) + \frac{\alpha_2}{2}G''(x)x^2\right| \\
&+\frac{1}{2}(y-1)^2 \sup |G''(x)|x^2 + y \sup G'(x)|b-x| \\
&+(y|y-1| + \alpha_2) \sup |G''(x)(b-x)|x \\
&+\frac{1}{6}\Big(|y-1|^3 + |\alpha_3|\Big) \sup |G'''(x)|x^3 \\
\leq\ & (y-1)^4\Big(d^{-4} + d^{-3}\sup G'(x)x + \frac{1}{2}d^{-2}\sup|G''(x)|x^2\Big) \\
&+(1+d)d^{-2}(y-1)^2 \sup G'(x)|b-x| \\
&+\sup\left|G'(x)(b-x) + \frac{\alpha_2}{2}G''(x)x^2\right| \\
&+\Big(\frac{1+d}{d}(y-1)^2 + \alpha_2\Big) \sup |G''(x)(b-x)|x \\
&+\frac{1}{6}\Big(d^{-1}(y-1)^4 + |\alpha_3|\Big) \sup |G'''(x)|x^3,
\end{aligned} \tag{3.9}$$

where all supremums in (3.9) are taken over all positive x.

Next, we consider I_2. By (3.7), we obtain for all positive x and y with $|y-1| \leq d$ that

$$\begin{aligned}
|\triangle(y,b,x)| \leq\ & \sup\left|G'(x)(b-x) + \frac{\alpha_2}{2}G''(x)x^2\right| \\
&+\alpha_2 \sup |G''(x)(b-x)|x + \frac{|\alpha_3|}{6} \sup |G'''(x)|x^3 \\
&+\ \frac{(1+d)^2}{2} \sup |G''(yx')|(b-x)^2 \\
&+\ \frac{1+d}{2}(y-1)^2 \sup |G'''(y''x)(b-x)|x^2 \\
&+\ \frac{(y-1)^4}{24} \sup |G^{(4)}(y'x)|x^4,
\end{aligned} \tag{3.10}$$

where all supremums in (3.10) are taken over all positive x and y with $|y-1| \leq d$, $x' \in (b \wedge x, b \vee x)$, $y' \in (y \wedge 1, y \vee 1)$, $y'' \in (y \wedge 1, y \vee 1)$.

Let us recall that

$$G'(x) = \Big[2^{q/2}\Gamma(q/2)\Big]^{-1} x^{q/2-1}e^{-x/2} \quad \textit{for } x \geq 0.$$

Let $\gamma_i = \sup_{x>0} |G^{(i)}(x)x^i|$ for $i = 1, 2, 3$. For example,

$$\gamma_1 = \Gamma^{-1}(q/2)\Big(\frac{q}{2e}\Big)^{q/2}.$$

It is clear that γ_2 and γ_3 are also functions of q. Now we show how other supremums in (3.9) and (3.10) can be calculated by using the properties of b

described in (2.1)–(2.3). We consider the most complicated expression

$$\sup |G''(yx')|(b-x)^2.$$

Note that

$$G''(x)2^{q/2}\Gamma(q/2) = e^{-x/2}x^{q/2-2}\left(-\frac{x}{2}+\frac{q}{2}-1\right). \tag{3.11}$$

We shall employ below the following inequality: for any positive numbers a and b, we have $|a-b| \le a \vee b$. We consider two cases.

Case 1: $b(x) \le x$. Then $x' \in (b,x)$ and, by (2.1), we have for $q \ge 2$

$$\exp\Big(-\frac{yx'}{2}\Big) \le \exp\Big(-\frac{(1-d)B_1}{2}x\Big), \tag{3.12}$$

$$\begin{aligned}(yx')^{q/2-2}\Big|-\frac{yx'}{2}+\frac{q}{2}-1\Big| &\le \frac{1}{2}(1+d)^{q/2-2}x^{q/2-2}\\ &\quad\times[((1+d)x)\vee(q-2)]. \end{aligned}\tag{3.13}$$

Case 2: $b(x) > x$. Then $x' \in (x,b)$ and we have for $q \ge 2$

$$\exp\Big(-\frac{yx'}{2}\Big) \le \exp\Big(-\frac{(1-d)}{2}x'\Big), \tag{3.14}$$

$$\begin{aligned}(yx')^{q/2-2}\Big|-\frac{yx'}{2}+\frac{q}{2}-1\Big| &\le \frac{1}{2}(1+d)^{q/2-2}(x')^{q/2-2}\\ &\quad\times\Big[((1+d)x')\vee(q-2)\Big]. \end{aligned}\tag{3.15}$$

Since the function $A(x)$ in condition (2.2) of the theorem is increasing, we have in Case 2 that

$$\begin{aligned}&\sup|G''(yx')|(b-x)^2 \le \frac{\beta}{2}(1+d)^{q/2-2}\\ &\times \sup_x \exp\Big(-\frac{(1-d)}{2}x\Big)A^2(x)x^{q/2-2}\Big[\big[((1+d)x)\vee(q-2)\Big]. \end{aligned}\tag{3.16}$$

Let us take $d = 1/2$. Combining (2.4), (3.11)–(3.16) we obtain

$$\begin{aligned}&2^{q/2}\Gamma(q/2)\sup|G''(yx')|(b-x)^2 \le \frac{\beta B_2^2}{2}\Big(\frac{3}{2}\Big)^{q/2-2}\\ &\times\Big[\Big(\frac{3}{2}c(B_1/8,q/2+1)\Big)\vee\Big((q-2)c(B_1/8,q/2)\Big)\Big]\\ &\equiv c_1(q,B_1)B_2^2\beta, \end{aligned}\tag{3.17}$$

where

$$c(\alpha,\gamma) = \sup_{x>0}\Big(x^\gamma \exp\{-\alpha x\}\Big) = \Big(\frac{\gamma}{\alpha e}\Big)^\gamma \textit{ for positive } \alpha \textit{ and } \gamma.$$

Since

$$2^{q/2}\Gamma(q/2)G'''(x) = e^{-x/2}x^{q/2-3}\left(\frac{x^2}{4} - \left(\frac{q}{2}-1\right)x + \left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)\right) \tag{3.18}$$

and

$$2^{q/2}\Gamma(q/2)G^{(4)}(x) = e^{-x/2}x^{q/2-4}\Bigg(-\frac{x^3}{8} + \frac{3}{4}\left(\frac{q}{2}-1\right)x^2 - \frac{3}{2}\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)x + \left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)\left(\frac{q}{2}-3\right)\Bigg), \tag{3.19}$$

similarly to (3.17) we obtain

$$\sup G'(x)|b-x| \le \left(2^{q/2}\Gamma(q/2)\right)^{-1}B_2\sqrt{\beta}\; c\big(7/16, q/2\big), \tag{3.20}$$

$$\begin{aligned} &2^{q/2}\Gamma(q/2)\sup|G''(x)(b-x)|x \\ &\le \frac{1}{2}\Big[c\big(7/16, q/2+1\big) \vee \Big((q-2)c\big(7/16, q/2\big)\Big)\Big]B_2\sqrt{\beta} \\ &\equiv c_2(q)B_2\sqrt{\beta}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &2^{q/2}\Gamma(q/2)\sup|G'''(y''x)(b-x)|x^2 \\ &\le \Bigg(\frac{1}{4}c\big(3/16, q/2+2\big) + \left|\frac{q}{2}-1\right| c\big(3/16, q/2+1\big) \\ &+ \left|\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)\right| c\big(3/16, q/2\big)\Bigg)B_2\sqrt{\beta} \equiv c_3(q)B_2\sqrt{\beta}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &2^{q/2}\Gamma(q/2)\sup|G^{(4)}(y'x)|x^4 \\ &\le \frac{1}{8}c\big(1/4, q/2+3\big) + \frac{3}{4}\left|\frac{q}{2}-1\right| c\big(1/4, q/2+2\big) \\ &+ \frac{3}{2}\left|\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)\right| c\big(1/4, q/2+1\big) \\ &+ \left|\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)\left(\frac{q}{2}-3\right)\right| c\big(1/4, q/2\big) \equiv c_4(q). \end{aligned} \tag{3.23}$$

Since $\mathrm{E}(y-1)^2 = \alpha_2 \le \sqrt{\beta}$, we obtain from (3.9), (3.10), and (3.17)–(3.23) that (3.8) holds with

$$\begin{aligned} c &= 16 + 8\gamma_1 + 2\gamma_2 + \frac{1}{2}\gamma_3 + B_3 \\ &+ B_2\left(2^{q/2}\Gamma(q/2)\right)^{-1}\Big[6c(7/16, q/2) + 4c_2(q) \\ &+ \frac{9}{8}B_2c_1(q, B_1) + \frac{3}{4}c_3(q) + \frac{1}{24}c_4(q)\Big], \end{aligned} \tag{3.24}$$

where $c_1(q, B_1), c_2(q), c_3(q)$ and $c_4(q)$ are defined in (3.17), (3.21), (3.22), and (3.23), respectively. Thus, Theorem 2.1 is proved.

Proof of Theorem 2.4. It is enough to verify that (2.7) implies (2.2) and (2.3). Note that

$$x \exp(B_1 x/16) \exp(-x/2) x^{q/2-1} \leq c\Big(1/2 - B_1/16, q/2\Big).$$

Therefore, (2.7) implies (2.3) with

$$B_3 = B_4[2^{q/2}\Gamma(q/2)]^{-1} c(1/2 - B_1/16, q/2).$$

Furthermore, without loss of generality, we may assume that $\beta \leq 1$. Otherwise (2.8) holds with $c_1 = 1$. It is easy to see that (2.2) follows from (2.7) if we take in (2.4)

$$B_2 = B_4 + \frac{1}{4}(|q-2| + c(B_1/16, 1)).$$

Moreover, now q can be equal to 1. In this case, the expression $q-2$ and the symbol $\vee$ in (3.13), (3.15)–(3.17), and (3.21) must be replaced by 1 and the symbol $+$, respectively. Thus, Theorem 2.4 is proved. $\square$

4. Statistic with an Error Estimate

In this section we consider a statistic satisfying (1.2). Certain transformations which improve approximation for S have been proposed by Cordeiro and Ferrari [2], Kakizawa [6], Fujisawa [5], Fujikoshi [4], and others. We give sufficient conditions under which there exists a transformation T that improves approximation for S. We shall find a transformation T in the class of positive increasing functions defined on $[0, +\infty)$. Fujisawa [5] has shown that a monotone increasing transformation improving approximation exists. Sufficient conditions will be formulated in terms of the function $b(x)$ inverse to T. Our aim is also to give a method of finding an error bound for this improved approximation.

Theorem 4.1. *Suppose that there exists a positive, increasing function b defined on $[0, +\infty)$ such that, for some positive constants $D_i = D_i(q, k)$, $i = 1, 2, 3, 4$: $D_1 \leq 1$, $D_3 < D_1/4$, the following conditions are satisfied for all $x > 0$:*

$$b(x) \geq D_1 x \tag{4.1}$$

$$|b(x) - x| \leq \frac{D_2 x}{n} \exp(D_3 x) \tag{4.2}$$

$$G'_q(x) \left| b(x) - x + \frac{x}{n} \sum_{j=1}^{k} a_j \sum_{m=0}^{j-1} \frac{(x/2)^m}{\prod_{l=0}^{m}(q/2 + l)} \right| \leq D_4/n^2, \tag{4.3}$$

where a_j's are the same as in (1.2).

If $P(S \leq x)$ can be written in form (1.2), *then*

$$|P(T(S) \leq x) - G_q(x)| \leq \frac{c}{n^2}, \tag{4.4}$$

where T is the inverse function to b and c is a positive constant depending on q, $D_j,\ j=1,\dots,4,$ and c_k in (1.3) (*see* (4.12)).

Proof. Since $G_q(x)$ is smooth for all $x>0$, we can write

$$G_q\Big(b(x)\Big) = G_q(x) + G_q'(x)(b-x) + \frac{1}{2}G''(x')(b-x)^2, \tag{4.5}$$

where $x' \in (b\wedge x, b\vee x)$. It is known that

$$G_{q+2}(x) = G_q(x) + \frac{(x/2)^{q/2}e^{-x/2}}{\Gamma(q/2+1)}. \tag{4.6}$$

Note that in (1.2), it is necessary that $\sum_{j=0}^{k} a_j = 0$. By using (1.2), (1.3), (4.5), and (4.6), we obtain

$$\begin{aligned}\mathrm{P}\Big(T(S)\le x\Big) &= \mathrm{P}\Big(S\le b(x)\Big) = G_q(x) + G_q'(x)(b-x)\\ &+ \frac{1}{n}\sum_{j=1}^{k} a_j e^{-x/2}(x/2)^{q/2}\sum_{m=1}^{j}\frac{(x/2)^{m-1}}{\Gamma(q/2+m)}\\ &+ \frac{1}{2}G_q''(x')(b-x)^2 + \frac{1}{n}\sum_{j=0}^{k} a_j G_{q+2j}'(x'')(b-x) + R_k,\end{aligned} \tag{4.7}$$

where $x'' \in (b\wedge x, b\vee x)$.

Now we construct a uniform bound for $G_q''(x')(b-x)^2$. We consider two cases.

Case 1: $b\le x$. Then $x'\in(b,x)$. Since

$$G_q''(x) = \Big(2^{q/2}\Gamma(q/2)\Big)^{-1} e^{-x/2}x^{q/2-2}\left(-\frac{x}{2}+\frac{q}{2}-1\right),$$

we obtain from (4.1) and (4.2) that

$$\begin{aligned}\sup{}_1|G_q''(x')|(b-x)^2 \;&\le\; \frac{\Big(2^{q/2}\Gamma(q/2)\Big)^{-1}}{2n^2}D_2^2\\ &\quad\times\sup{}_2\exp\Big(-x(D_1/2-2D_3)\Big)x^{q/2}\left(x+|q-2|\right)\\ &\equiv\; c_1(k,q,D_1,D_3)/n^2,\end{aligned} \tag{4.8}$$

where the first supremum on the left-hand side is taken over x such that $b(x)\le x$ and the second supremum is taken over all $x>0$. By the hypotheses of the theorem, we have $c_1(k,q,D_1,D_3)<\infty$.

Case 2: $b>x$. Then $x'\in(x,b)$ and, by (4.2), we obtain

$$\begin{aligned}&\sup{}_1|G_q''(x')|(b-x)^2 \le \frac{1}{n^2}\sup{}_2|G_q''(x')|D_2^2(x')^2\exp(2D_3x')\\ &\le \frac{\Big(2^{q/2}\Gamma(q/2)\Big)^{-1}}{2n^2}D_2^2\sup{}_3\exp\Big(-x(1/2-2D_3)\Big)|x|^{q/2}(|x|+|q-2|)\\ &\equiv c_2(k,q,D_1,D_3)/n^2,\end{aligned} \tag{4.9}$$

where the first two supremums are taken over x such that $b(x) > x$ and the third one is taken over all $x > 0$. By the hypotheses of the theorem, we have

$$c_2(k, q, D_1, D_3) < \infty.$$

Since $c_2(k, q, D_1, D_3) \le c_1(k, q, D_1, D_3)$, we obtain

$$\sup G_q''(x')(b - x)^2 \le c_1(k, q, D_1, D_3)/n^2, \tag{4.10}$$

where the supremum is taken over all $x > 0$. By using (4.6), we can obtain similarly to (4.10) that, for all $x > 0$, one has

$$\begin{aligned}
&\left|\sum_{j=0}^{k} a_j G_{q+2j}'(x'')(b - x)\right| = |b - x|\left|\sum_{j=1}^{k} a_j \sum_{m=1}^{j} \frac{((x''/2)^{\frac{q+2m}{2}-1} e^{-x''/2})'}{\Gamma\left(\frac{q+2m}{2}\right)}\right| \\
&\le \frac{D_2 x}{2^{q/2}\Gamma(q/2)n}(x'')^{q/2-1}\exp\left(D_3 x - \frac{x''}{2}\right) \\
&\quad \times \sum_{j=1}^{k} |a_j| \sum_{m=1}^{j} \frac{x'' + q + 2m - 2}{\prod_{l=0}^{m-1}(q/2 + l)} \left(\frac{x''}{2}\right)^{m-1} \\
&\le \frac{D_2}{n}\left(2^{q/2}\Gamma(q/2)\right)^{-1} \sup_{x>0}\Bigg[\exp(-x(D_1/2 - D_3)x^{q/2} \\
&\quad \times \sum_{j=1}^{k} |a_j| \sum_{m=0}^{j-1} \frac{x + q + 2m}{\prod_{l=0}^{m}(q/2 + l)} \left(\frac{x}{2}\right)^{m}\Bigg] \\
&\equiv c_3(k, q, D_1, D_3)/n.
\end{aligned} \tag{4.11}$$

Combining (4.7), (4.10), (4.3), (4.12), and (1.3), we obtain (4.4) with

$$c = D_4 + \frac{1}{2}c_1(k, q, D_1, D_3) + c_3(k, q, D_1, D_3) + c_k. \tag{4.12}$$

This brings our proof to the end. □

Remark 4.2. It is clear that a positive function b that satisfies (4.1)–(4.3) may not be increasing. Therefore, we have to require the existence of an increasing function b in Theorem 4.1. We have shown in the previous sections that the required function b exists.

Theorem 4.3. *Suppose that there exist a positive, increasing function $b(x)$ defined on $[0, +\infty)$ and positive constants $D_i = D_i(q, k)$, $j = 1, 3, 5$, such that* (4.1) *holds and, for all $x > 0$, one has*

$$\left|b(x) - x + \frac{x}{n}\sum_{j=1}^{k} a_j \sum_{m=0}^{j-1} \frac{(x/2)^m}{\prod_{l=0}^{m}(q/2 + l)}\right| \le \frac{D_5}{n^2} x \exp(D_3 x) \tag{4.13}$$

with $D_3 < D_1/4$. If a statistic S admits the representation (1.2), *then we have* (4.4), *where T is the inverse function to b and c is a positive constant depending on q, k, and D_1, D_3, D_5.*

Proof. The claim is proved similarly to Theorem 2.4. □

5. Examples

At first we consider the case in which $S = \chi_q^2/Y$ with $Y = n^{-1}\chi_n^2$ and Y, χ_n^2 are independent. In this case S can be represented in form (1.2) with

$$k = 2, \quad a_0 = -\frac{1}{4}q(q-2), \; a_1 = \frac{1}{2}q^2, \quad a_2 = -\frac{1}{4}q(q+2),$$

and a uniform bound for the remainder term of type (1.3) can be obtained (see, e.g., [7]). Therefore we can apply Theorem 4.1. However, it would be preferable to apply Theorem 2.1, since it yields a better constant c in a bound of type (4.4). This is due to the fact that in Theorem 2.1 we have used the properties of a χ^2-variate mixture, but not representation (1.2). At the same time, it is easily seen that some parts of the proofs of Theorem 2.1 and Theorem 4.1 are similar. By using Theorem 2.1, we can obtain the following result.

Corollary 5.1. *Let* $Y = n^{-1}\chi_n^2$. *Suppose that conditions* (2.1) *and* (2.2) *of Theorem* 2.1 *are satisfied and that* (2.3) *is replaced by the following:*

$$\left| b(x) - x + \frac{x^2}{2n}((q-2)/x - 1) \right| \le \frac{1}{n^2} 14.824 B_3 2^{q/2}\Gamma(q/2)x^{1-q/2}e^{x/2}, \qquad (5.1)$$

where B_3 *is the same as in* (2.3). *Then we have*

$$\left| P\{T(\chi_q^2/Y) \le x\} - G(x) \right| \le 14.824c/n^2, \qquad (5.2)$$

provided that Y *and* χ_q^2 *are independent and* c *is defined by* (3.24).

Proof. It is enough to note that under the hypotheses of this corollary, we have

$$\alpha_2 = \mathrm{E}(Y-1)^2 = 2/n, \quad \alpha_3 = \mathrm{E}(Y-1)^3 = 8/n^2,$$
$$\alpha_4 = \mathrm{E}(Y-1)^4 = 12/n^2 + 48/n^3. \qquad (5.3)$$

Since c in (3.24) is not less than 18.94, we may assume that $n \ge 17$. Then, by (5.3), we have

$$\beta = \max\{|\alpha_3|, \alpha_4\} = \alpha_4 \le 14.824n^2.$$

Therefore, this corollary follows from Theorem 2.1. □

Now we consider examples of transformations of $S = n\chi_q^2/\chi_n^2$ with independent χ_q^2 and χ_n^2 which provide better approximations:

$$\begin{aligned} T_1(x) &= (n + (q-2)/2)\log(1 + x/n), \\ T_2(x) &= n\exp\left(\frac{q-2}{2n}\right)\left(1 - \exp\left(-\frac{x}{n}\right)\right), \\ T_3(x) &= \frac{q-2}{2} - n + \left(\left(n - \frac{q-2}{2}\right)^2 + 2nx\right)^{1/2}. \end{aligned}$$

The transformations $T_1(x)$ and $T_2(x)$ have been introduced, respectively, in [4] and [3]. The transformation $T_3(x)$ is a new one (cf. $T_0(x)$ in Section 2. We show what bounds of type (5.2) can be obtained for these transformations.

In what follows we assume that $q = 4$. Then we obtain

$$\begin{aligned} b_1(x) &= n\Big(\exp\Big(\frac{x}{n+1}\Big) - 1\Big), \\ b_2(x) &= -n\log\Big(1 - \frac{x}{n}\exp\Big(-\frac{1}{n}\Big)\Big), \\ b_3(x) &= x\Big(1 - \frac{1}{n}\Big) + \frac{x^2}{2n} \end{aligned}$$

as the inverse functions for T_1, T_2, and T_3, respectively. Note that $b_2(x)$ is defined only for

$$x\colon\ x < n\exp(1/n),$$

whereas $b_1(x)$ and $b_3(x)$ are defined for all x.

Moreover, if we take $x_\varepsilon = n\exp(1/n)(1-\varepsilon)$, then $b_2(x_\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$. This means that condition (2.2) is not satisfied for $b_2(x)$, and the approach proposed in Theorem 2.1 does not lead to a uniform estimate of type (5.2) in this case. Therefore, we exclude T_2 from our considerations here.

In the case of T_1, it is easy to verify that (2.1), (2.2), and (5.2) are satisfied with

$$B_1 = 79/80,\ B_2 = 0.97065,\ B_3 = 0.67556,$$

provided that $n > 78$. Therefore (5.2) holds for $T = T_1$ with

$$c = c(T_1) = 424.16.$$

However, in the case of the transformation T_3, we obtain

$$B_1 = 64/65,\ B_2 = 0.77633,\ B_3 = 0.$$

Therefore (5.2) holds for $T = T_3$ with

$$c = c(T_3) = 321.46.$$

Thus T_3 yields better bounds for the remainder term than T_1.

Moreover, by using the above simple expression for b_3 and arguing similarly to the proof of Theorem 2.1, we can prove (5.2) with

$$c(T_3) = 132.34.$$

In particular, one has to take $d = 0.34$ in the proof.

Acknowledgement

Research was supported by RFBR (grant No. 00–01–00406) and JSPS.

References

1. L. N. Bolshev, Asymptotically Pearson transformations. (Russian) *Teor. Veroyant. i Primen.* **8**(1963), No. 2, 129–155, English translation: *Theory Probab. Appl.* **8**(1963), 121–146.
2. G. M. Cordeiro and S. P. Ferrari, A modified score test statistic having chi-squared distribution to order n^{-1}. *Biometrika* **78**(1991), 573–582.
3. G. M. Cordeiro, S. P. Ferrari, and A. H. M. A. Cysneiros, A formula to improve score test statistics. *J. Statist. Comput. Simulation* **62**(1998), No. 1–2, 91–104.
4. Y. Fujikoshi, A method for improving the large-sample chi-squared approximations to some multivariate tests statistics. *Amer. J. Math. Management Sci.* **17**(1997), 15–29.
5. H. Fujisawa, Improvement on chi-squared approximation by monotone transformation. *J. Multivariate Anal.* **60**(1997), 84–89.
6. Y. Kakizawa, Higher order monotone Bartlett-type adjustment for some multivariate test statistics. *Biometrika* **83**(1996), 923–927.
7. R. Shimizu and Y. Fujikoshi, Sharp error bounds for asymptotic expansions of the distribution functions of scale mixtures. *Ann. Inst. Statist. Math.* **49**(1997), 285–297.
8. M. Siotani, T. Hayakawa, and Y. Fujikoshi, Modern multivariate statistical analysis: a graduate course and handbook. *American Sciences Press, Inc., Ohio*, 1985.
9. V. Ulyanov, Y. Fujikoshi, and R. Shimizu, Non-uniform error bounds in asymptotic expansions for scale mixtures under mild moment conditions. *J. Math. Sci.* **93**(1999), No. 4, 600–608.

(Received 15.12.2000)

Authors' addresses:

V. V. Ulyanov
Faculty of Computational Mathematics and Cybernetics
Moscow State University
Moscow 119899, Russia
E-mail: vladim195@mtu-net.ru

Y. Fujikoshi
Faculty of Science, Hiroshima University
Higashi–Hiroshima
739–8526 Japan