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MD-NUMBERS AND ASYMPTOTIC MD-NUMBERS OF OPERATORS

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Abstract. First, the basic properties of mean dilatation (MD-) numbers for linear operators acting from a finite-dimensional Hilbert space are investigated. Among other results, in terms of first and second order MD-numbers, a characterization of isometries is obtained and a dimension-free estimation of the p-th order MD-number by means of the first order MD-number is established. After that asymptotic MD-numbers for a continuous linear operator acting from an infinite-dimensional Hilbert space are introduced and it is shown that in the case of an infinite-dimensional domain the asymptotic p-th order MD-number, rather unexpectedly, is simply the p-th power of the asymptotic first order MD-number (Theorem 3.1).

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1. Introduction

For a linear operator T from a Hilbert space H with $1 \leq n = \dim H < \infty$ to an arbitrary Hilbert space Y and a natural number $p \leq n$, the p-th order mean dilatation number (briefly, the pMD-number) $\delta_p(T)$ is defined by the equality

$$\delta_p(T) = \left(\frac{1}{c_{n,p}} \int\limits_S \cdots \int\limits_S G(T_1, \dots, Tx_p) ds(x_1) \dots ds(x_p) \right)^{1/2}, \qquad (1.1)$$

where S is the unit sphere in H, s is the normalized isometrically invariant measure on it, $G(x_1, \dots, x_p)$ is the Grammian of $(x_1, \dots, x_p)$, and

$$c_{n,p} = \int\limits_S \cdots \int\limits_S G(x_1, \dots, x_p) ds(x_1) \dots ds(x_p) > 0$$

is a (natural) normalizing constant.

The definition of pMD-numbers was suggested in [6]. Afterwards they were considered also in [5].

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For simplicity, the first order MD-number $\delta_1(T)$ will be called the MD-number of T and, for it, (1.1) gives an expression

$$\delta_1(T) = \left(\int_S \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}. \tag{1.2}$$

Using the equality (1.2), the MD-number $\delta_1(T)$ can also be defined when Y is an arbitrary Banach space and in this case it has already been used as a tool of the local theory of Banach spaces (see, e.g., [7, p.81]).

In Section 2, after recalling some known facts concerning the Grammian (Remark 2.1) which motivate the definition of pMD-numbers, their properties in the finite-dimensional case are analyzed. In particular, a concrete expression for $\delta_p(T)$ in terms of the eigenvalues of the operator T^*T is found (Proposition 2.6) and the following characterization of isometric operators is obtained: T is an isometry if and only if $\delta_1(T) = \delta_2(T) = 1$ (Proposition 2.8). Although the equality $\delta_p(T) = \delta_1^p(T)$ does not hold in general, the validity of the following "dimension-free" estimate

$$\delta_p(T) \leq p^{\frac{p-1}{2}} \delta_1^p(T)$$

is proved (Lemma 2.10).

Possible infinite-dimensional extensions of pMD-numbers are introduced in Section 3. According to [6], we use the net $\mathcal{M}$ of all finite-dimensional subspaces of H to associate two quantities

$$\overline{\overline{\delta}}_p(T) := \limsup_{M \in \mathcal{M}} \delta_p(T|_M), \quad \underline{\underline{\delta}}_p(T) := \liminf_{M \in \mathcal{M}} \delta_p(T|_M)$$

to a given operator T (acting from H) and a natural number p. We call $\overline{\overline{\delta}}_p(T)$ the asymptotic upper pMD-number of T and $\underline{\underline{\delta}}_p(T)$ the asymptotic lower pMD-number of T. When they are equal, the operator is called asymptotically pMD-regular, their common value is denoted by $\overline{\delta}_p(T)$ and called the asymptotic pMD-number. Therefore for an asymptotic pMD-regular operator T we have

$$\overline{\delta}_p(T) := \lim_{M \in \mathcal{M}} \delta_p(T|_M).$$

When $p = 1$, we shall simply use the term "asymptotically MD-regular".

The main result of the section is Theorem 3.1, which asserts that, in the case of an infinite-dimensional domain, we have the following simple relation between the higher order and the first order asymptotic MD-numbers of a given operator T:

$$\overline{\overline{\delta}}_p(T) = (\overline{\overline{\delta}}_1(T))^p, \quad \underline{\underline{\delta}}_p(T) = (\underline{\underline{\delta}}_1(T))^p.$$

This result shows that in the case of the infinite-dimensional domain it is sufficient to study only the numbers $\overline{\overline{\delta}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$. Section 3 is concluded by Proposition 3.2, which implies, in particular, that not any operator is asymptotically MD-regular.

Notice, finally, that the numbers $\overline{\overline{\delta}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$ can also be defined when Y is a general Banach space. The properties of the corresponding quantities in this general setting are investigated in [2].

2. Finite Dimensional Case

Throughout this paper the considered Hilbert spaces is supposed to be defined over the same field $\mathbb{K}$, real $\mathbb{R}$ or complex $\mathbb{C}$ numbers. To simplify the notation, for the scalar product and the norm of different Hilbert spaces we use the same symbols $(\cdot|\cdot)$ and $\|\cdot\|$.

For the Hilbert spaces H, Y we denote by $L(H,Y)$ the set of all continuous linear operators $T : H \to Y$.

For a given operator $T \in L(H,Y)$ we denote by

- $\|T\|$ the ordinary norm of T;
- T^* the (Hilbert) adjoint operator of T;
- $|T|$ the unique self-adjoint positive square root of $T^*T : H \to H$.

Moreover, for a Hilbert–Schmidt operator T we denote by $\|T\|_{\mathrm{HS}}$ the *Hilbert–Schmidt norm* of T.

Fix a (non-zero) Hilbert space H and a natural number p such that $p \le \dim H$. The p-th order *Grammian* G_p is the scalar-valued function, defined on H^p, which assigns the determinant of the matrix $(x_i|x_j)_{i,j=1}^p$ to any $(x_1,\dots,x_p) \in H^p$.

Evidently, $G_1(x) = \|x\|^2$ for each $x \in H$. Some other known properties of the Grammian are given in the next remark.

Remark 2.1. Fix a natural number $p > 1$ with $p \le \dim H$ and a finite sequence $(x_1,\dots,x_p) \in H^p$.

(a) We have $G_p(x_1,\dots,x_p) \ge 0$. Moreover, $G_p(x_1,\dots,x_p) > 0$ if and only if $(x_1,\dots,x_p)$ is linearly independent.

(b) $G_p(x_1,\dots,x_p) \le \|x_1\|^2 \dots \|x_p\|^2$. Moreover, if $x_1,\dots,x_p$ are nonzero elements, then we have $G_p(x_1,\dots,x_p) = \|x_1\|^2 \dots \|x_p\|^2$ iff $x_1,\dots,x_p$ are pairwise orthogonal.

(c) Let $P_{x_1,\dots,x_p} := \Big\{ \sum_{i=1}^p \alpha_i x_i : 0 \le \alpha_i \le 1,\ i = 1,\dots,p \Big\}$ be the parallelepiped generated by the sequence $(x_1,\dots,x_p)$. Then $G(x_1,\dots,x_p) = \mathrm{vol}^2(P_{x_1,\dots,x_p})$, where $\mathrm{vol}(P_{x_1,\dots,x_p})$ stands for the (p-dimensional) volume of $P_{x_1,\dots,x_p}$ (i.e., the Lebesgue measure of $P_{x_1,\dots,x_p}$ in a p-dimensional vector subspace containing $(x_1,\dots,x_p)$. For this reason the quantity $g_p(x_1,\dots,x_p) := G_p^{1/2}(x_1,\dots,x_p)$ is often simply called "the hypervolume determined by the vectors $(x_1,\dots,x_p)$."

(d) Let $[x_1,\dots,x_{p-1}]$ be the linear span of $\{x_1,\dots,x_{p-1}\}$. Then

$$g_p(x_1,\dots,x_p) = g_{p-1}(x_1,\dots,x_{p-1}) \cdot \mathrm{dist}(x_p,[x_1,\dots,x_{p-1}]).$$

(e) If Y is another Hilbert space and $T : H \to Y$ is a continuous linear operator, then $g_p(Tx_1,\dots,Tx_p) \le \|T\|^p g_p(x_1,\dots,x_p)$ (this is evident when $p = 1$, the rest follows from this and from (d) by induction, as $\mathrm{dist}(Tx_p,[Tx_1,\dots,Tx_{p-1}]) \le \|T\| \,\mathrm{dist}(x_p,[x_1,\dots,x_{p-1}])$.

(f) In the notation of the previous item suppose that $\dim H = p$. Then $g_p(Tx_1, \ldots, Tx_p) = \det(|T|) g_p(x_1, \ldots, x_p)$. It follows that if $\dim H = p$ and $(x'_1, \ldots, x'_p)$, $(x_1, \ldots, x_p)$ are algebraic bases of H, then

$$\frac{g_p(Tx'_1, \ldots, Tx'_p)}{g_p(x'_1, \ldots, x'_p)} = \frac{g_p(Tx_1, \ldots, Tx_p)}{g_p(x_1, \ldots, x_p)},$$

i.e., the "volume ratio" is independent of a particular choice of basis of H and equals to $\det(|T|)$.

For a Hilbert space with $\dim H \geq 1$, its unit sphere with center at the origin is denoted by S_H or, simply, by S. When $n = \dim H < \infty$, the *uniform distribution* on S, i.e. the unique isometrically invariant probability measure given on the Borel σ-algebra of S, is denoted by s. The following key equality follows directly from the isometric invariance of s:

$$\int_S \overline{(x|h_1)}(x|h_2) ds(x) = \frac{1}{n}(h_1|h_2), \quad \forall h_1, h_2 \in H, \tag{2.1}$$

where 'bar' stands for complex conjugation.

In the rest of this section H will be a finite-dimensional Hilbert space with $\dim H = n \geq 1$, I will stand for the identity operator acting in H and Y will be another Hilbert space (not necessarily finite-dimensional).

Fix a number $p \in \{1, \ldots, n\}$ and introduce the functional $D_p : L(H, Y) \to \mathbb{R}_+$ defined by the equality

$$D_p(T) = \int_S \cdots \int_S G_p(Tx_1, \ldots, Tx_p) ds(x_1) \ldots ds(x_p), \quad T \in L(H, Y). \tag{2.2}$$

Therefore, for given T the number $D_p(T)$ can be viewed as the average of the squares of the volumes of the family of parallelepipeds

$$\left\{T(P_{x_1, \ldots, x_p}) : (x_1, \ldots, x_p) \in S \times \cdots \times S\right\}$$

with respect to the product $s \times \cdots \times s$ of the uniform distributions.[1]

Evidently, for $p = 1$ we get

$$D_1(T) = \int_S \|Tx\|^2 ds(x), \quad \forall T \in L(H, Y). \tag{2.3}$$

To make easier further references, we formalize some other easy observations in the next statement.

[1]The choice of s seems to be natural, since it gives no preference to any of the directions $(x_1, \ldots, x_p) \in S \times \cdots \times S$. Formally, the same definition can be given, taking any probability measure μ on S instead of s, but then the properties of $D_p(T)$ will depend on (the correlation operator of) μ.

Proposition 2.2. *Let $T \in L(H, Y)$.*

$$D_p(T) = D_p(|T|); \tag{2.4}$$
$$D_p(T) = 0 \iff \dim T(H) < p; \tag{2.5}$$
$$D_p(T) \le D_1^p(T); \tag{2.6}$$
$$D_2(T) = D_1^2(T) - \frac{1}{n} D_1(T^*T), \quad \textit{in particular,} \quad D_2(I) = 1 - \frac{1}{n}; \tag{2.7}$$
$$D_1(T^*T - I) = D_1(T^*T) - 2D_1(T) + 1. \tag{2.8}$$

Proof. (2.4) is true since $G_p(|T|x_1, \dots, |T|x_p) = G_p(Tx_1, \dots, Tx_p)$ for any $(x_1, \dots, x_p) \in H^p$.

(2.5) follows from Remark 2.1(a).

(2.6) follows from Remark 2.1(b) and (2.3).

(2.7): since $\forall x_1, x_2 \in H$

$$G_p(Tx_1, Tx_2) = \|Tx_1\|^2\|Tx_2\|^2 - |(Tx_1|Tx_2)|^2 = \|Tx_1\|^2\|Tx_2\|^2 - |(T^*Tx_1|x_2)|^2,$$

and since by (2.1)

$$\int\limits_S |(T^*Tx_1|x_2)|^2 ds(x_2) = \frac{1}{n}\|T^*Tx_1\|^2, \quad \forall x_1 \in H,$$

from (2.3) (using, of course, Fubini's theorem) we get (2.7).

(2.8) follows from (2.3) and from the evident equality $\|(T^*T - I)x\|^2 = \|T^*Tx\|^2 - 2\|Tx\|^2 + \|x\|^2$, $x \in H$. □

Remark 2.3. Fix $T \in L(H, Y)$.

(1) By means of the direct integration of the Grammian it is possible to get a "coordinate free" expression for $D_p(T)$ in terms of D_1 similar to (2.7) also for $p \ge 3$, e.g. we have

$$D_3(T) = D_1^3(T) - \frac{3}{n}\, D_1(T)D_1(T^*T) + \frac{2}{n^2}\, D_1(TT^*T).$$

However, the corresponding higher order combinatorial formula looks rather complicated. For this reason formula (2.12) from Proposition 2.4 is preferable.

(2) If $n \ge 2$ and $p = 2$, then (2.7) shows that in (2.6) we have the equality if and only if $T = 0$ (compare with Remark 2.1(b)).

Proposition 2.4. *Let $\dim H = n$, $p \le n$, $T \in L(H, Y)$ and $(e_1, \dots, e_n)$ be any orthonormal basis of H. Then:*

$$D_1(T) = \frac{1}{n}\left(\sum_{k=1}^{n} \|Te_k\|^2\right) = \frac{1}{n}\|T\|_{HS}^2, \tag{2.9}$$

$$D_p(T) = \frac{1}{n^p} \sum_{j_1, \dots, j_p = 1}^{n} G_p(Te_{j_1}, \dots Te_{j_p}), \tag{2.10}$$

$$D_p(I) = \frac{n!}{(n-p)!n^p} \tag{2.11}$$

and

$$D_p(T) = \frac{p!}{n^p} \sum_{1\le j_1<\cdots<j_p\le n} \lambda_{j_1}^2 \dots \lambda_{j_p}^2, \tag{2.12}$$

where $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of the operator* $|T| := (T^*T)^{\frac{1}{2}}$ *listed according to their multiplicities.*

Proof. Fix $x \in H$. We have $x = \sum_{k=1}^{n} (x|e_k)e_k$. Then $Tx = \sum_{k=1}^{n} (x|e_k)Te_k$ and

$$\|Tx\|^2 = \sum_{j,k=1}^{n} (x|e_j)\overline{(x|e_k)}(Te_j|Te_k).$$

Integrating this equality over S with respect to s and using (2.1) we get (2.9).

In the proof of (2.10) we will use the standard notation, namely, $\mathfrak{S}_p$ will stand for the set of all permutations $\sigma : \{1,\dots,p\} \to \{1,\dots,p\}$, and we will denote by $\operatorname{sgn}\sigma$ the sign of a fixed permutation σ. Fix finite sequences $(a_1,\dots,b_p)$, $(b_1,\dots,b_p)$ of the elements of H. We have the next "Parseval equality" for determinants, which follows from [1, p. V. 34, formula (26) and Prop. 5]:

$$\det((a_i|b_j)_{i,j=1}^p) = \sum_{1\le j_1<\cdots<j_p\le n} \Big(\det((a_i|e_{j_k})_{i,k=1}^p)\Big) \overline{\Big(\det((b_i|e_{j_k})_{i,k=1}^p)\Big)}. \tag{2.13}$$

Fix now a finite sequence $(x_1,\dots,x_p)$ of the elements of H. Applying equality (2.13) to the matrix $(T^*Tx_i|x_j)_{i,j=1}^p$ we get

$$\begin{aligned} &G(Tx_1,\dots,Tx_p) \\ &= \sum_{1\le j_1<\cdots<j_p\le n} \sum_{\sigma,\pi\in\mathfrak{S}_p} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\pi \prod_{k=1}^{p} (x_k|T^*Te_{j_{\sigma(k)}})\overline{(x_k|e_{j_{\pi(k)}})}. \end{aligned} \tag{2.14}$$

Now integrating both sides of equality (2.14) with respect to the variables $x_1,\dots,x_p$ and measure $s\times\cdots\times s$ and using formula (2.1) we obtain

$$\begin{aligned} D_p(T) &= \frac{1}{n^p} \sum_{1\le j_1<\cdots<j_p\le n} \sum_{\sigma,\pi\in\mathfrak{S}_p} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\pi \prod_{k=1}^{p} (Te_{j_{\pi(k)}}|Te_{j_{\sigma(k)}}) \\ &= \frac{p!}{n^p} \sum_{1\le j_1<\cdots<j_p\le n} G(Te_{j_1},\dots,Te_{j_p}) = \frac{1}{n^p} \sum_{j_1,\dots,j_p=1}^{n} G(Te_{j_1},\dots Te_{j_p}). \end{aligned} \tag{2.15}$$

The second equality in (2.15) is true since it can be easily seen that the relation

$$\sum_{\sigma,\pi\in\mathfrak{S}_p} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\pi \prod_{k=1}^{p} (y_{\sigma(k)}|y_{\pi(k)}) = p!G(y_1,\dots,y_p)$$

holds for all finite sequences $(y_1,\dots,y_p)$ of elements of Y.

Equality (2.11) follows from (2.10).

Equality (2.12) also follows from (2.10) since in (2.10) the elements $e_k, k = 1,\dots,n$ can be taken the eigenvectors of T^*T. □

Remark 2.5. (1) The validity of the important equality (2.10) was pointed out in [6], while the proof is given in [5, p. 113]; the above-given proof is different and, in a sense, less "combinatorial".

(2) Let

$$|||T|||_{p,e.} := \left(\sum_{j_1,\ldots,j_p=1}^{n} G_p(Te_{j_1},\ldots,Te_{j_p}) \right)^{1/2}.$$

Then from equality (2.10), inequality (2.6) and equality (2.9) we get

$$|||T|||_{p,e.}^2 = n^p D_p(T) \leq \|T\|_{HS}^{2p}.$$

This relation implies, in particular, that the value of $|||T|||_{p,e.}$ does not depend on a particular choice of an orthonormal basis $e. := (e_1,\ldots,e_n)$ of H (compare with the next item).

(3) Suppose for a moment that H is an infinite-dimensional separable Hilbert space, $T : H \to Y$ is a continuous linear operator, $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H and put again

$$\begin{aligned} |||T|||_{p,e.} : &= \left(\sum_{j_1,\ldots,j_p=1}^{\infty} G_p(Te_{j_1},\ldots,Te_{j_p}) \right)^{1/2} \\ &= \lim_{n\to\infty} \left(\sum_{j_1,\ldots,j_p=1}^{n} G_p(Te_{j_1},\ldots Te_{j_p}) \right)^{1/2}. \end{aligned}$$

(3a) [3, p. 44, Prop. III.2.2] If T is a Hilbert–Schmidt operator, then

$$|||T|||_{p,e.} \leq \|T\|_{HS}^{p} \tag{2.16}$$

and the value of $|||T|||_{p,e.}$ does not depend on a particular choice of an orthonormal basis $e. := (e_n)_{n\in\mathbb{N}}$ of H (note that inequality (2.16) follows from Remark 2.1(b), however the second statement now needs a separate proof).

(3b) [3, p. 42, Prop. III.2.1] If for given T there are a natural number p and an orthonormal basis $e. := (e_n)_{n\in\mathbb{N}}$ of H such that $|||T|||_{p,e.} < \infty$, then T is a Hilbert–Schmidt operator.

Using the functional D_p, the definition of the *p-th order mean dilatation number* $\delta_p(T)$ *of an operator* $T \in L(H,Y)$, given in the introduction, can be rewritten as

$$\delta_p(T) = \left(\frac{D_p(T)}{D_p(I)} \right)^{\frac{1}{2}}. \tag{2.17}$$

Evidently, when $p = 1$, (2.17) gives

$$\delta_1(T) = D_1^{1/2}(T) = \left(\int_S \|Tx\|^2 ds(x) \right)^{\frac{1}{2}}. \tag{2.18}$$

The "normalized" version of Proposition 2.4 looks as follows:

Proposition 2.6. *Let* $\dim H = n$, $p \le n$, $T \in L(H,Y)$ *and* $(e_1, \dots, e_n)$ *be any orthonormal basis of* H. *Then:*

$$\delta_1(T) = \frac{1}{\sqrt{n}} \|T\|_{HS} = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{n} \|Te_k\|^2 \right)^{\frac{1}{2}}, \tag{2.19}$$

$$\delta_p(T) = \binom{n}{p}^{-\frac{1}{2}} \left(\sum_{1 \le j_1 < \cdots < j_p \le n} \lambda_{j_1}^2 \dots \lambda_{j_p}^2 \right)^{\frac{1}{2}}, \tag{2.20}$$

where $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of the operator* $|T| := (T^*T)^{\frac{1}{2}}$ *listed according to their multiplicities;*

$$\delta_n(T) = \det(|T|). \tag{2.21}$$

Proof. (2.19) follows from (2.9).

(2.20) follows from (2.12).

(2.21) follows from (2.20) by putting $p = n$. □

Remark 2.7. (a) Equality (2.21) can be derived directly from Remark 2.1(f), its validity was already noted in [6]. It shows that the normalization through $D_p(I)$ in the definition of the pMD-number is natural.

(b) Equality (2.19) implies that the functional $T \to \delta_1(T)$ is a norm on $L(H,Y)$ with the following property:

$$\frac{1}{\sqrt{n}} \|T\| \le \delta_1(T) \le \|T\|, \quad \forall\, T \in L(H,Y). \tag{2.22}$$

(c) Suppose $p > 1$, then the functional $T \to \delta_p(T)$ is absolutely p-homogeneous on $L(H,Y)$ (this follows, e.g., from (2.20)). Also (2.5) implies that $\delta_p(\cdot)$ vanishes on the operators with a rank $< p$.

(d) If for a given operator $T \in L(H,Y)$ we have $\|T\| \le 1$ and $\delta_1(T) = 1$, then T is an isometry (this is easy to see).

The following assertion shows that isometric operators can be characterized only in terms of δ_1 and δ_2.

Proposition 2.8. *Let* H *be a finite-dimensional Hilbert space with* $\dim H = n \ge 2$, Y *be any Hilbert space and* $T : H \to Y$ *be a linear operator. The following statements are equivalent:*

(i) T *is an isometry.*

(ii) $\delta_p(T) = 1 \quad \forall p \le n$.

(iii) $\delta_1(T) = \delta_2(T) = 1$.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are evident.

First proof of (iii) ⇒ (i). It is sufficient to verify that $T^*T = I$. For this it is enough to show that $\delta_1(T^*T - I) = 0$ (since δ_1 is a norm). Equalities (2.7) and (2.8) imply

$$\delta_2^2(T) = \frac{n}{n-1} \delta_1^4(T) - \frac{1}{n-1} \delta_1^2(T^*T) \tag{2.23}$$

and

$$\delta_1^2(T^*T - I) = \delta_1^2(T^*T) - 2\delta_1^2(T) + 1. \tag{2.24}$$

Now it is clear that (iii) and (2.23) imply $\delta_1(T^*T) = 1$. Thus the equality $\delta_1(T) = 1$ and (2.24) imply $\delta_1(T^*T - I) = 0$. Consequently, $T^*T = I$.

Second proof of (iii) $\Rightarrow$ (i).[2] Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities. Using (iii) and (2.20) we get

$$\sum_{k=1}^{n} {\lambda_k}^2 = n{\delta_1}^2(T) = n, \qquad \sum_{1 \le j_1 < j_2 \le n} \lambda_{j_1}^2 \lambda_{j_2}^2 = \binom{n}{2}\delta_2^2(T) = \binom{n}{2}.$$

Then

$$\sum_{k=1}^{n} {\lambda_k}^4 = \left(\sum_{k=1}^{n} {\lambda_k}^2\right)^2 - 2\sum_{1 \le j_1 < j_2 \le n} \lambda_{j_1}^2 \lambda_{j_2}^2 = n^2 - 2\binom{n}{2} = n$$

and

$$\sum_{k=1}^{n} ({\lambda_k}^2 - 1)^2 = \sum_{k=1}^{n} {\lambda_k}^4 - 2\sum_{k=1}^{n} {\lambda_k}^2 + n = 0.$$

Consequently, $\lambda_k = 1,\ k = 1, \dots, n$ and $|T| = I$. □

Remark 2.9. It is interesting to note that in [6] only the validity of the implication (ii) $\Rightarrow$ (i) was conjectured. We see that even (iii) implies (i).

The following assertion will be used in the next section.

Lemma 2.10. *Let* $\dim H = n$, $T : H \to Y$ *be a linear operator and* $p \in \{1, \dots, n\}$. *Put*

$$c_{n,p} := D_p(I) = \frac{n!}{(n-p)!n^p}$$

and

$$r_n(p, T) := D_1^p(T) - D_p(T).$$

Then:

$$\frac{1}{p^{p-1}} \le c_{n,p} \le 1, \tag{2.25}$$

$$\delta_p(T) \le p^{\frac{p-1}{2}} \delta_1^p(T), \tag{2.26}$$

$$0 \le r_n(p, T) \le \frac{p!}{n}\|T\|^{2p}, \tag{2.27}$$

$$\delta_p^2(T) = \frac{1}{c_{n,p}}(\delta_1^{2p}(T) - r_n(p, T)). \tag{2.28}$$

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[2]Suggested by the Referee.

Proof. Relation (2.25) is easy to verify. Inequality (2.26) follows from (2.6) and (2.25).

(2.27): Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the operator $|T| := (T^*T)^{\frac{1}{2}}$ listed according to their multiplicities. Using (2.9), Newton's (polynomial) formula and (2.10) we can write:

$$\begin{aligned}(D_1(T))^p - D_p(T) &= \frac{1}{n^p}(\sum_{k=1}^{n}\lambda_k^2)^p - \frac{p!}{n^p}\sum_{1\le j_1<\cdots<j_p\le n}\lambda_{j_1}^2\dots\lambda_{j_p}^2\\ &= \frac{p!}{n^p}\sum_{k_1+\cdots+k_n=p}\frac{1}{k_1!\dots k_n!}\lambda_1^{2k_1}\dots\lambda_n^{2k_n}\\ &-\frac{p!}{n^p}\sum_{k_1+\cdots+k_n=p,\,\max k_i=1}\frac{1}{k_1!\dots k_n!}\lambda_1^{2k_1}\dots\lambda_n^{2k_n}\\ &= \frac{p!}{n^p}\sum_{k_1+\cdots+k_n=p,\,\max k_i>1}\frac{1}{k_1!\dots k_n!}\lambda_1^{2k_1}\dots\lambda_n^{2k_n}.\end{aligned}$$

Consequently,

$$(D_1(T))^p - D_p(T) = \frac{p!}{n^p}\sum_{k_1+\cdots+k_n=p,\,\max k_i>1}\frac{1}{k_1!\dots k_n!}\lambda_1^{2k_1}\dots\lambda_n^{2k_n}.$$

From this equality, as $\lambda_k \le \|T\|$, $k = 1, \dots, n$, and

$$\sum_{k_1+\cdots+k_n=p,\,\max k_i>1}\frac{1}{k_1!\dots k_n!} \le n^{p-1},$$

we obtain (2.27).

(2.28) follows from (2.27). □

Remark 2.11. (1) By (2.20) it can be seen that for a given operator T the equality $\delta_p(T) = \delta_1^p(T)$ holds if and only if T is a scalar multiplier of an isometry. In this connection the asymptotic versions behave better (see the next section).

(2) A simple relation between $\delta_p(T)$ and $\delta_1^p(T)$, can be written using the notation from the theory of outer products of Hilbert spaces. Namely, let H, Y be finite-dimensional Hilbert spaces, $T \in L(H, Y)$ and $\bigwedge^p T$ be the antisymmetric outer p-th power of the operator T. Then

$$\delta_p(T) = \binom{n}{p}^{-\frac{1}{2}}\|\textstyle\bigwedge^p T\|_{\mathrm{HS}} \ \text{ and } \ \delta_p(T) = \delta_1(\textstyle\bigwedge^p T). \tag{2.29}$$

These formulas follow easily from the following relation, established in [5, p. 112–113, the proof of Prop. VI.2.2]: $\|\bigwedge^p T\|_{\mathrm{HS}}^2 = \frac{n^p}{p!}D_p(T)$. In [5] it is also shown that the antisymmetric outer p-th power of a Hilbert–Schmidt (or a nuclear) operator T acting between infinite-dimensional Hilbert spaces is again a Hilbert–Schmidt (a nuclear) operator (cf. also [4, Th. 2]).

3. Infinite-Dimensional Case

In this section H will be an infinite-dimensional Hilbert space and Y will be another Hilbert space. As before, $L(H,Y)$ will denote the set of all continuous linear operators $T : H \to Y$.

Fix a natural number p and consider the collection $\mathcal{M}_p$ of all finite-dimensional vector subspaces $M \subset H$ with $\dim M \geq p$. This collection is a directed set by set-theoretic inclusion.

Fix an operator $T \in L(H,Y)$. Then for any $M \in \mathcal{M}_p$ and the restriction $T|_M$ of T to M, the pMD-number $\delta_p(T|_M)$ is defined. In this way T generates with the net $(\delta_p(T|_M))_{M\in\mathcal{M}_p}$. Let us denote the upper limit of this net by $\overline{\overline{\delta}}_p(T)$ and call it the *asymptotic upper pMD-number* of T. In a similar manner, let us denote the lower limit of this net by $\underline{\underline{\delta}}_p(T)$ and call it the *asymptotic lower pMD-number* of T. Therefore

$$\overline{\overline{\delta}}_p(T) := \limsup_{M\in\mathcal{M}_p} \delta_p(T|_M) \quad \text{and} \quad \underline{\underline{\delta}}_p(T) := \liminf_{M\in\mathcal{M}_p} \delta_p(T|_M).$$

By formula (2.26) of Lemma 2.10 we have

$$\delta_p(T|_M) \leq p^{\frac{p-1}{2}} \|T\|^p, \quad \forall\ M \in \mathcal{M}_p.$$

This inequality shows that the net $(\delta_p(T|_M))_{M\in\mathcal{M}_p}$ is bounded and, consequently, always

$$\underline{\underline{\delta}}_p(T) \leq \overline{\overline{\delta}}_p(T) \leq p^{\frac{p-1}{2}} \|T\|^p < \infty.$$

In general, the net $(\delta_p(T|_M))_{M\in\mathcal{M}_p}$ may not be convergent (as we will see below). In the case of convergence we shall call the operator T *asymptotically pMD-regular*. For an asymptotically pMD-regular T we put

$$\overline{\delta}_p(T) = \lim_{M\in\mathcal{M}_p} \delta_p(T|_M),$$

and call $\overline{\delta}_p(T)$ the *asymptotic pMD-number* of T.

An asymptotically 1MD-regular operator T will be called *asymptotically MD-regular* and its 1MD-number will be called *asymptotic MD-number*.

The next result is somewhat unexpected. Its validity was not predicted in [6].

Theorem 3.1. *Let H be an infinite-dimensional Hilbert space, Y be a Hilbert space, $T \in L(H,Y)$ and $p > 1$ be a natural number. Then*

$$\overline{\overline{\delta}}_p(T) = (\overline{\overline{\delta}}_1(T))^p \quad \textit{and} \quad \underline{\underline{\delta}}_p(T) = (\underline{\underline{\delta}}_1(T))^p; \tag{3.1}$$

moreover, the operator T is asymptotically pMD-regular if and only if it is asymptotically MD-regular and in the case of asymptotic MD-regularity the equality

$$\overline{\delta}_p(T) = (\overline{\delta}_1(T))^p \tag{3.2}$$

holds.

Proof. Let $M \in \mathcal{M}_p$. We can write using formula (2.28) of Lemma 2.10:

$$\delta_p^2(T|_M) = \frac{1}{p!}\binom{\dim M}{p}^{-1} (\dim M)^p \left(\delta_1^{2p}(T|_M) - r_{\dim M}(p, T|_M)\right). \quad (3.3)$$

Evidently,

$$\lim_{M \in \mathcal{M}_p} \frac{1}{p!}\binom{\dim M}{p}^{-1} (\dim M)^p = 1. \quad (3.4)$$

Since according to inequality (2.27) of Lemma 2.10

$$r_{\dim M}(p, T|_M)| \le \frac{p!}{\sqrt{\dim M}} \|T|_M\|^{2p} \le \frac{p!}{\sqrt{\dim M}} \|T\|^{2p},$$

we also have

$$\lim_{M \in \mathcal{M}_p} r_{\dim M}(p, T|_M) = 0. \quad (3.5)$$

From (3.3) via (3.4) and (3.5) the we get (3.1). The "moreover" part now is clear. □

This theorem shows that in the case of an infinite-dimensional domain it is sufficient to study only the numbers $\overline{\overline{\delta}}_1(T)$, $\underline{\underline{\delta}}_1(T)$ and $\overline{\delta}_1(T)$.

To simplify the notation, for a given operator $T \in L(H, Y)$ let us denote its

- asymptotic upper 1MD-number $\overline{\overline{\delta}}_1(T)$ by $\overline{\overline{\delta}}(T)$,
- asymptotic lower 1MD-number $\underline{\underline{\delta}}_1(T)$ by $\underline{\underline{\delta}}(T)$,
- asymptotic MD-number $\overline{\delta}_1(T)$ by $\overline{\delta}(T)$.

Using formula (2.19) from Proposition 2.6 the definition of these numbers can be formulated directly in terms of the Hilbert–Schmidt norm as follows:

$$\overline{\overline{\delta}}(T) = \limsup_{M \in \mathcal{M}_1} \frac{1}{\sqrt{\dim M}} \|T|_M\|_{HS}, \quad \underline{\underline{\delta}}(T) = \liminf_{M \in \mathcal{M}_1} \frac{1}{\sqrt{\dim M}} \|T|_M\|_{HS}.$$

For an operator $T \in L(H, Y)$ let us put

$$m(T) := \inf\{\|Tx\| : x \in H, \|x\| = 1\}.$$

The number $m(T)$ is sometimes called the lower bound of T.

The next statement implies, in particular, that a given operator T may not be asymptotically MD-regular.

Proposition 3.2. *Let H, Y be infinite-dimensional Hilbert spaces and $T : H \to Y$ be a continuous linear operator. Then:*

(a) *For any infinite-dimensional closed vector subspace $X \subset H$ the inequality*

$$\overline{\overline{\delta}}(T) \ge m(T|_X)$$

holds.

(b) *If $\ker(T)$ is infinite-dimensional, then $\underline{\underline{\delta}}(T) = 0$.*

(c) *If T is a partial isometry such that* $\ker(T)$ *and* $T(H)$ *are both infinite-dimensional, then* $\overline{\overline{\delta}}(T) = 1$, *while* $\underline{\underline{\delta}}(T) = 0$ *and so T is not asymptotically MD-regular.*

Proof. (a) Fix an infinite-dimensional $X \subset H$, a finite-dimensional vector subspace $M \subset H$ and put

$$\beta_M := \sup\{\delta_1(T|_N) : N \in \mathcal{M}_1, N \supset M\}.$$

Let us show that

$$\beta_M \geq m(T|_X). \tag{3.6}$$

To prove (3.6), fix a natural number n and an n-dimensional vector subspace X_n of X such that $X_n \cap M = \{0\}$ (such a choice is possible because M is finite-dimensional and X is infinite-dimensional). Let also $M_n := M + X_n$ and let M' be the vector subspace of M_n orthogonal to X_n. Using formula (2.19) from Proposition 2.6 we can write:

$$\delta_1^2(T|_{M_n}) = \frac{\|T|_{M'}\|_{HS}^2 + \|T|_{X_n}\|_{HS}^2}{\dim(M') + n} \geq \frac{\|T|_{X_n}\|_{HS}^2}{\dim(M') + n} \geq \frac{n}{\dim(M') + n} m(T|_X)^2.$$

As $\beta_M \geq \delta_1(T|_{M_n})$ and $\dim(M') \leq \dim(M)$, we get

$$\beta_M^2 \geq \frac{n}{\dim(M) + n}\, m^2(T|_X). \tag{3.7}$$

Since n is arbitrary, from (3.7) we have

$$\beta_M^2 \geq \sup_n \frac{nm^2(T|_X)}{\dim(M) + n} \geq \lim_n \frac{nm^2(T|_X)}{\dim(M) + n} = m^2(T|_X).$$

This relation, together with (3.6) and the definition of $\overline{\overline{\delta}}(T)$, implies (a).

(b) Let $X = \ker(T)$, fix a finite-dimensional vector subspace $M \subset H$ and put

$$\alpha_M := \inf\{\delta_1(T|_N) : N \in \mathcal{M}_1, N \supset M\}.$$

Let us show that

$$\alpha_M = 0. \tag{3.8}$$

To prove (3.8), fix a natural number n and an n-dimensional vector subspace X_n of X such that $X_n \cap M = \{0\}$ (such a choice is possible because M is finite-dimensional and X is infinite-dimensional). Let also $M_n := M + X_n$ and M' be the subspace of M_n orthogonal to X_n. Using formula (2.19) from Proposition 2.6 and taking into account that $T|_{X_n} = 0$ we can write:

$$\delta_1^2(T_{M_n}) = \frac{\|T|_{M'}\|_{HS}^2 + \|T|_{X_n}\|_{HS}^2}{\dim(M') + n} = \frac{\|T|_{M'}\|_{HS}^2}{\dim(M') + n}.$$

From this we get

$$\alpha_M^2 \leq \frac{\|T|_{M'}\|_{HS}^2}{\dim(M') + n}.$$

Observe now that since $\dim(M') \le \dim(M)$ and $\|T|_{M'}\|_{HS} \le \|T\|\sqrt{\dim(M')}$, we have

$$\alpha_M^2 \le \lim_n \frac{\|T|_{M'}\|^2}{\dim(M')+n} = 0.$$

This relation together with the definition of $\underline{\underline{\delta}}(T)$ implies (b).

(c) follows from (a) and (b). □

The last proposition motivates the following

Problem. Give a characterization of asymptotically MD-regular operators in terms of some other known parameters (in terms of the spectrum, in terms of the diagonal (for the diagonal operators), etc.).

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