**Georgian Mathematical Journal**
Volume 8 (2001), Number 4, 683–696

# SUSPENSION AND LOOP OBJECTS AND REPRESENTABILITY OF TRACKS

H.-J. BAUES AND M. JIBLADZE

**Abstract.** In the general setting of groupoid enriched categories, notions of *suspender* and *looper* of a map are introduced, formalizing a generalization of the classical homotopy-theoretic notions of suspension and loop space. The formalism enables subtle analysis of these constructs. In particular, it is shown that the suspender of a principal coaction splits as a coproduct. This result leads to the notion of *theories with suspension* and to the cohomological classification of certain groupoid enriched categories.

**2000 Mathematics Subject Classification:** 18D05, 55P35, 55P40.
**Key words and phrases:** Groupoid enriched category, suspension object, loop object, suspender, looper, principal coaction.

A category enriched in groupoids (termed a track category for short) is a special 2-category. A track category $\mathscr{T}$ consists of objects $A$, $B$,... and hom-groupoids $[\![A,B]\!]$ in which the objects are maps (1-arrows or 1-cells) and the morphisms are isomorphisms termed tracks (2-arrows or 2-cells). For each map $f : A \to B$ in $\mathscr{T}$ we have the group $\operatorname{Aut}(f)$ consisting of all tracks $\alpha : f \Rightarrow f$ in $\mathscr{T}$. This is an automorphism group in the hom-groupoid $[\![A,B]\!]$.

Our leading example is the track category $\mathbf{Top}^*$ consisting of spaces $A$, $B$,... with basepoint *, pointed maps $f : A \to B$ and tracks $\alpha : f \Rightarrow g$ which are homotopy classes (relative to the boundary) of homotopies $f \simeq g$; compare (1.3) in [5]. For the trivial map $0 : A \to * \to B$ in $\mathbf{Top}^*$ one has the well known isomorphism of groups

$$\operatorname{Aut}(A \to * \to B) = [\Sigma A, B]. \tag{$*$}$$

Here the left-hand side is the group of automorphisms of $0 : A \to B$ in the track category $\mathbf{Top}^*$ and the right-hand side is the group of homotopy classes of maps $\Sigma A \to B$ where $\Sigma A$ is the *suspension* of $A$. Dually we also have the canonical isomorphism

$$\operatorname{Aut}(A \to * \to B) = [A, \Omega B], \tag{$**$}$$

where $\Omega B$ is the *loop space* of $B$. Via $(*)$ and $(**)$ certain tracks in the track category $\mathscr{T} = \mathbf{Top}^*$ are *represented* by morphisms in the homotopy category $\mathscr{T}_\simeq$ of $\mathscr{T}$. We study in this paper the categorical aspects of such a representability of tracks which we call $\Sigma$-representability in $(*)$ and $\Omega$-representability in $(**)$. For this we introduce the notion of *suspender* generalizing the notion of

ISSN 1072-947X / $8.00 / 

suspension above by means of a universal property. The categorical dual of a suspender is a *looper* in a track category generalizing the notion of loop space. A track category $\mathscr{T}$ is $\Sigma$-representable, resp. $\Omega$-representable, if suspenders, resp. loopers exist in $\mathscr{T}$. Of course the track category $\mathbf{Top}^*$ of pointed spaces (more generally the track category associated to any Quillen model category) is both $\Sigma$-representable and $\Omega$-representable.

We describe basic properties of suspenders and loopers. In particular we show that the suspender of a principal coaction splits as a coproduct. This is a crucial result which leads to the notion of *theories with suspension* and the cohomological classification of certain $\Sigma$-representable track categories in [6].

In topology a typical example of a suspender of a pointed space $X$ is the space

$$\Sigma_* X = S^1 \times X / S^1 \times \{*\}.$$

Moreover a looper of $X$ is given by the free loop space

$$\Omega_* X = (X^{S^1}, 0)$$

with the function space topology and basepoint given by the trivial loop 0. Splitting results $\Sigma_* X \simeq X \vee \Sigma X$ (resp. $\Omega_* X \simeq X \times \Omega X$) are well known in case $X$ is a co-H-group (resp. H-group). For example Barcus and Barratt [5] or Rutter [12] use implicitly the splitting to obtain basic rules of homotopy theory. This paper and its sequel [6] specifies the categorical background of some of these rules. Suspenders and loopers are also responsible for the properties of *partial suspensions* and *partial loop operations* discussed in [6]; compare also [4, 3, 2].

The theory of $\Sigma$-representable track categories in this paper is also motivated by the approach of Gabriel and Zisman [7] who consider those properties of a track category $\mathscr{T}$ which imply existence of a Puppe sequence for mapping cones. The suspension $\Sigma A$ plays a crucial rôle in this sequence. Enriching the results in [7] we show that the main categorical nature of a suspension in a track category is described by the notion of suspender which is the link between tracks in $\mathscr{T}$ and homotopy classes of maps in $\mathscr{T}$. Such a link, for example, is needed in results of Hardie, Kamps and Kieboom [9, 8] and Hardie, Marcum and Oda [10] who study homotopy-theoretic secondary operations like Toda brackets in 2-categories. This paper does not aim at combining the theory of exact sequences in homotopy theory as considered in [7] and the theory of suspenders since the notion of suspenders, resp. loopers, is quite sophisticated and new.

## 1. $\Sigma$-representable track categories

We first introduce the following notation. In a groupoid $\mathbf{G}$ the composition of morphisms

$$\xleftarrow{f} \; \xleftarrow{g}$$

is denoted by $f+g$. Accordingly also the composition of tracks

$$\xLeftarrow{\alpha} \; \xLeftarrow{\beta}$$

in a track category is denoted by $\alpha+\beta$. We also write $\alpha : a \simeq b$ for a track $\alpha : a \Rightarrow b$. We say that a groupoid $\mathbf{G}$ is *abelian* if all automorphism groups of objects in $\mathbf{G}$ are abelian. For $\beta : y \to y$ and $\varphi : x \to y$ in $\mathbf{G}$ we obtain the *conjugate*

$$\beta^{\varphi} = -\varphi + \beta + \varphi : x \to x$$

so that $(-)^{\varphi} : \mathrm{Aut}(y) \to \mathrm{Aut}(x)$ is a homomorphism. The *loop groupoid* $\mathbf{G}$ of $\mathbf{G}$ is defined to have objects $\alpha : x \to x$, where $x$ is an object of $\mathbf{G}$; a morphism from $\alpha : x \to x$ to $\beta : y \to y$ is a morphism $\varphi : x \to y$ in $\mathbf{G}$ such that $\alpha = -\varphi + \beta + \varphi$.

Track functors between track categories and track transformations between track functors are the enriched versions of functors and natural transformations enriched in the category $\mathfrak{Gpd}$ of groupoids. Here $\mathfrak{Gpd}$ is also an example of a track category with functors between groupoids as 1-arrows and natural transformations as 2-arrows. Each object $C$ in a track category $\mathscr{T}$ yields the *representable track functor*

$$[\![C, -]\!] : \mathscr{T} \to \mathfrak{Gpd}$$

between track categories which carries $X \in \mathrm{Ob}(\mathscr{T})$ to the hom-groupoid $[\![C, X]\!]$ in $\mathscr{T}$.

In a track category $\mathscr{T}$, consider a map $f : A \to B$. For any object $X$, denote by $\mathbf{G}_f(X)$ the following groupoid: objects of $\mathbf{G}_f(X)$ are pairs $(g, \alpha)$, where $g : B \to X$ is a map and $\alpha : gf \Rightarrow gf$ is a track. A morphism from $(g', \alpha')$ to $(g, \alpha)$ is a track $\gamma : g' \Rightarrow g$ such that $\alpha' = \alpha^{\gamma f}$. Any map $h : X \to Y$ induces a functor $\mathbf{G}_f(h) : \mathbf{G}_f(X) \to \mathbf{G}_f(Y)$ sending $(g, \alpha)$ to $(hg, h\alpha)$ and $\gamma$ to $h\gamma$. Moreover any track $\eta : h \Rightarrow h'$ induces a natural transformation $\mathbf{G}_f(\gamma) : \mathbf{G}_f(h) \to \mathbf{G}_f(h')$ with components $\eta g : hg \Rightarrow h'g$ for objects $(g, \alpha)$ of $\mathbf{G}_f(X)$. Thus we have defined a track functor

$$\mathbf{G}_f : \mathscr{T} \to \mathfrak{Gpd}.$$

Any object $(g : B \to C, \alpha : gf \Rightarrow gf)$ of $\mathbf{G}_f(C)$ gives rise to a track transformation

$$(g, \alpha)^* : [\![C, -]\!] \to \mathbf{G}_f$$

consisting of functors

$$[\![C, X]\!] \to \mathbf{G}_f(X)$$

which assign to $h : C \to X$ the pair $(hg, h\alpha)$ and to $\eta : h' \Rightarrow h$ the track $\eta g$ (this indeed defines a morphism in $\mathbf{G}_f$ as $\eta gf + h'\alpha = h\alpha + \eta gf$, i. e. $h'\alpha = (h\alpha)^{\eta gf}$).

1.1. **Definition.** For a map $f : A \to B$ in a track category $\mathscr{T}$, a *suspender* for $f$ is a triple $(\Sigma_f, i_f, \upsilon_f)$ consisting of an object $\Sigma_f$, a map $i_f : B \to \Sigma_f$, and a track $\upsilon_f : i_f f \Rightarrow i_f f$ having the property that the induced track transformation

$$(i_f, \upsilon_f)^* : [\![\Sigma_f, -]\!] \to \mathbf{G}_f$$

induces a bijection of isomorphism classes of objects.

In other words, the following conditions must be satisfied:

(a) For any map $g : B \to C$ and any track $\eta : gf \Rightarrow gf$ there exists a map $\Sigma_\eta : \Sigma_f \to C$ and a track $\zeta_\eta : g \Rightarrow \Sigma_\eta i_f$ such that $\eta = (\Sigma_\eta \upsilon_f)^{\zeta_\eta f}$ (surjectivity);

(b) For any $h, h' : \Sigma_f \to C$ and any track $\gamma : h' i_f \Rightarrow h i_f$ with $h' \upsilon_f = (h \upsilon_f)^{\gamma f}$ one has $\gamma = \delta i_f$ for some track $\delta : h' \Rightarrow h$ (injectivity).

We point out that we do not assume for a suspender $\Sigma_f$ that the map $(i_f, \upsilon_f)^*$ is an equivalence of groupoids since this, in fact, does not hold in the example of topological spaces. Hence topology forces us to think of a weaker universal property, namely that $(i_f, \upsilon_f)^*$ induces only a bijection of isomorphism classes of objects. A track category $\mathscr{T}$ is $\Sigma$-*representable* if each map $f$ in $\mathscr{T}$ has a suspender $(\Sigma_f, i_f, \upsilon_f)$.

1.2. **Definition.** The dual notion of *looper* is obtained as a suspender in the opposite track category: a looper for $f : A \to B$ consists of a map $p_f : \Omega_f \to A$ and a track $\lambda_f : f p_f \Rightarrow f p_f$ satisfying conditions dual to the above ones for suspenders. A track category $\mathscr{T}$ is $\Omega$-representable if each map $f$ in $\mathscr{T}$ has a looper $(\Omega_f, p_f, \lambda_f)$.

Important particular cases are the suspenders and loopers for the identity map $1{=}\mathrm{id}_A : A \to A$ which will be denoted $\Sigma_*(A)$ and $\Omega_*(A)$ respectively; suspender for a map $0 : A \to *$ to the initial object will be called *suspension* of $A$ and denoted $\Sigma_0(A)$, or simply $\Sigma(A)$ if the map 0 is uniquely determined by the context; and dually the looper for a map $0 : 1 \to A$ from the terminal object to $A$ will be called *loop object* of $A$ and denoted $\Omega_0(A)$ or $\Omega(A)$.

These examples are important in that sometimes suspenders or loopers of all maps can be constructed using solely $\Sigma_*$ and $\Omega_*$ – indeed sometimes just using $\Sigma_0$ and $\Omega_0$. See below.

We consider the following examples of $\Omega$-representable and $\Sigma$-representable track categories.

1.3. **Example.** The track category $\mathfrak{Gpd}$ of groupoids is $\Omega$-representable. In fact, for a functor $F : \mathbf{G} \to \mathbf{H}$ between groupoids the looper $\Omega_F$ is obtained by the pullback diagram

$$\begin{array}{ccc} \Omega_F & \longrightarrow & \mathbf{H} \\ \downarrow & & \downarrow \\ \mathbf{G} & \xrightarrow{F} & \mathbf{H}. \end{array}$$

Moreover the loop groupoid $\mathbf{H}$ itself has the universal property for $\Omega_*(\mathbf{H})$.

1.4. **Example.** The track category $\mathbf{Top}^*$ of pointed topological spaces is $\Sigma$-representable. Let $IA = (A \times [0,1]) \,/\, (\{*\} \times [0,1])$ be the cylinder in $\mathbf{Top}^*$.

Then the suspender $\Sigma_f$ of a map $f : A \to B$ is obtained by the pushout diagram

$$\begin{array}{ccc} A \vee A & \xrightarrow{(f,f)} & B \\ \downarrow{\scriptstyle (i_0,i_1)} & & \downarrow{\scriptstyle i_f} \\ IA & \xrightarrow{\upsilon_f} & \Sigma_f . \end{array}$$

Here $\upsilon_f$ yields the track $\upsilon_f : i_f \Rightarrow i_f$ for the suspender $\Sigma_f$. Next let $PB = B^I$ be the space of maps $[0,1] \to B$ with the compact open topology. Then the looper $\Omega_f$ of $f$ is obtained by the pullback diagram

$$\begin{array}{ccc} \Omega_f & \xrightarrow{\lambda_f} & PB \\ \downarrow{\scriptstyle p_f} & & \downarrow{\scriptstyle (q_0,q_1)} \\ A & \xrightarrow{(f,f)} & B \times B. \end{array}$$

Here $\lambda_f$ yields the track $\lambda_f : p_f \Rightarrow p_f$ for the looper $\Omega_f$. In the next example we show that the properties 1.1 are satisfied for $\Sigma_f$ and $\Omega_f$ respectively.

Let $\mathbf{C}$ be a cofibration category in the sense of Baues [3]. For each cofibrant object $X$ in $\mathbf{C}$ we choose a cylinder

$$X \vee X \rightarrowtail IX \xrightarrow{\sim} X$$

which is a factorization of $(1,1) : X \vee X \to X$. For a fibrant object $Y$ the homotopy classes relative to $X \vee X$ of maps $IX \to Y$ are the tracks in $\mathbf{C}$. Therefore the full subcategory $\mathbf{C}_{\mathrm{cf}}$ of cofibrant and fibrant objects in $\mathbf{C}$ is a track category; see [3, II §5].

1.5. **Lemma.** *For a cofibration category $\mathbf{C}$ the track category $\mathbf{C}_{\mathrm{cf}}$ is $\Sigma$-representable.*

*Proof.* For each cofibrant object $X$ in $\mathbf{C}$ a fibrant model $j : X \overset{\sim}{\rightarrowtail} RX$ can be chosen. Now the suspender $\Sigma_f$ of $f : A \to B$ in $\mathbf{C}_{\mathrm{cf}}$ is obtained by a fibrant model of the pushout $\Sigma'_f$ in the following diagram

$$\begin{array}{ccccc} A \vee A & \xrightarrow{(f,f)} & B & = & B \\ \downarrow & \text{push} & \downarrow{\scriptstyle i'_f} & & \downarrow{\scriptstyle i_f} \\ IA & \xrightarrow{\upsilon'_f} & \Sigma'_f & \overset{\sim}{\rightarrowtail} & \Sigma_f . \end{array}$$

The composite $\upsilon_f = j\upsilon'_f : IA \to \Sigma'_f \to \Sigma_f$ yields the track $\upsilon_f : i_f \Rightarrow i_f$. We now check that the properties (a) and (b) in 1.1 are satisfied. For a map $g : B \to C$ in $\mathbf{C}_{\mathrm{cf}}$ let $\eta : gf \simeq gf$ be a homotopy $\eta : IA \to C$. Then the pushout property of $\Sigma'_f$ yields a map $g \cup \eta : \Sigma'_f \to C$ which admits an extension $\Sigma_\eta : \Sigma_f \to C$ so that $\Sigma_\eta i_f = g$ and $\eta = \Sigma_\eta \upsilon_f$. Hence we can actually choose the track $\zeta_\eta$ in

(a) to be the identity isomorphism of $g$. This proves (a). Now we check (b) as follows. Let $\gamma : h' i_f \simeq h i_f$ with

$$h' \upsilon_f = (h \upsilon_f)^{\gamma f} = -\gamma f + hf + \gamma f \qquad (*)$$

as in (b). Here (*) is an equation of tracks. Now (*) implies that there is a map

$$\delta' : IIA \to C$$

with $\delta' i_0 = h\upsilon_f$, $\delta' i_1 = h'\upsilon_f$ and $\delta' I i_0 = \delta' I i_1 = \gamma I f$. Here we choose the cylinder $IB$ to be a fibrant object so that $If : IA \to IB$ is defined; see [3]. Now consider the following pushout diagram where $I(A \vee A) = IA \vee IA$.

$$\begin{array}{ccccc} I(A \vee A) & \xrightarrow{(If,If)} & IB & \xrightarrow{\gamma} & C \\ \downarrow & \text{push} & \downarrow{\scriptstyle Ii'_f} & & \\ IIA & \longrightarrow & I\Sigma'_f. & & \end{array}$$

Here the pushout $I\Sigma'_f$ is actually a cylinder for $\Sigma'_f$ and we define a cylinder $I\Sigma_f$ for $\Sigma_f$ by the pushout diagram

$$\begin{array}{ccc} \Sigma'_f \vee \Sigma'_f & \overset{j \vee j}{\rightarrowtail} & \Sigma_f \vee \Sigma_f \\ {\scriptstyle (i_0,i_1)}\downarrow & \text{push} & \downarrow \\ I\Sigma'_f & \overset{\sim}{\rightarrowtail} & I\Sigma_f. \end{array}$$

Now the map $\delta' \cup \gamma : I\Sigma'_f \to C$ is defined with $(\delta' \cup \gamma) i_0 = h' j$ and $(\delta' \cup \gamma) i_1 = hj$. Hence a map $\delta = (\delta' \cup \gamma) \cup (h', h) : I\Sigma_f \to C$ is defined. The track defined by $\delta$ satisfies $\delta : h' \Rightarrow h$. Moreover $\delta i_f = \gamma$ since $\delta i_f$ is represented by $(\delta' \cup \gamma)(I i'_f) = \gamma$. □

For a model category $\mathbf{Q}$ as in [11] let $\mathbf{Q}_{\mathrm{c}}$ and $\mathbf{Q}_{\mathrm{f}}$ denote the full subcategory of cofibrant and fibrant objects, respectively. Then $\mathbf{Q}_{\mathrm{c}}$ is a cofibration category and $\mathbf{Q}_{\mathrm{f}}$ is a fibration category in the sense of Baues [3]. Here fibration category is the categorical dual of cofibration category. Therefore 1.5 above shows:

1.6. **Corollary.** *Let $\mathbf{Q}_{\mathrm{cf}}$ be the track category of cofibrant and fibrant objects in a Quillen model category. Then $\mathbf{Q}_{\mathrm{cf}}$ is $\Sigma$-representable and $\Omega$-representable.*

The examples in 1.4 are also consequences of 1.5 since $\mathbf{Top}^*$ is a cofibration category and also a fibration category in which all objects are cofibrant and fibrant, compare [3]. Moreover using [3, Remark I.8.15] we see that also $\mathbf{Top}^*_0$ is a fibration category in which all objects are fibrant and cofibrant. Here we use the structure [3, I.3.3] and [3, I.4.6].

## 2. Functorial properties of suspenders

We consider functorial properties of suspenders. This implies a kind of uniqueness and compatibility with sums. For a category $\mathbf{T}$ the category $\mathrm{Pair}(\mathbf{T})$ is the usual category of pairs in $\mathbf{T}$. Objects of $\mathrm{Pair}(\mathbf{T})$ are morphisms $A \to B$ and morphisms from $(A \to B)$ to $(X \to Y)$ are commutative diagrams in $\mathbf{T}$

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y. \end{array}$$

2.1. **Lemma.** *For any commutative diagram of unbroken arrows*

$$\begin{array}{ccc} A & \xrightarrow{p} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{q} & Y \\ {\scriptstyle i_f}\downarrow & \nearrow\!\!\!\nearrow{\scriptstyle \zeta_{(p,q)}} & \downarrow{\scriptstyle i_g} \\ \Sigma_f & \underset{\Sigma_*(p,q)}{\dashrightarrow} & \Sigma_g \end{array}$$

*there exist a map* $\Sigma_*(p,q)$ *and a track* $\zeta_{(p,q)}$ *as indicated, with*

$$\Sigma_*(p,q)\upsilon_f = (\upsilon_g p)^{\zeta_{(p,q)} f}. \tag{a}$$

*Choosing such maps for each commutative square as above gives a functor*

$$\Sigma_- : \mathrm{Pair}(\mathscr{T}) \to \mathrm{Pair}(\mathscr{T}_{\simeq})$$

*carrying* $f : A \to B$ *to* $[i_f] \in [B, \Sigma_f]$ *and the commutative square* $(p,q) : f \to g$ *as above to* $([q], [\Sigma_*(p,q)]) : [i_f] \to [i_g]$.

*Proof.* By the definition of suspenders, the track $\upsilon_g p$ considered as an automorphism of $i_g g p = i_g q f$ produces a map $\Sigma_{\upsilon_g p} : \Sigma_f \to \Sigma_g$ and a track $\zeta_{\upsilon_g p} : \Sigma_{\upsilon_g p} i_f \Rightarrow i_g q$ such that (a) holds. So one can define

$$\Sigma_*(p,q) = \Sigma_{\upsilon_g p}, \quad \zeta_{(p,q)} = \zeta_{\upsilon_g p}.$$

Then the injectivity condition for suspenders guarantees that there are tracks

$$\Sigma_*(\mathrm{id}_A, \mathrm{id}_B) \simeq \mathrm{id}_{\Sigma_f}$$

and

$$\Sigma_*(p,q)\Sigma_*(p',q') \simeq \Sigma_*(pp', qq')$$

for any two matching commutative squares. The lemma follows. □

2.2. **Lemma.** *Let* $(\Sigma_f, i_f, \upsilon_f)$ *and* $(\Sigma'_f, i'_f, \upsilon'_f)$ *be two suspenders of a map* $f : A \to B$*. Then they are equivalent in* $\mathscr{T}$*. More precisely, there exist maps* $l : \Sigma_f \to \Sigma'_f$ *and* $l' : \Sigma'_f \to \Sigma_f$ *such that* $li_f = i'_f$*,* $l'i'_f = i_f$*,* $l\upsilon_f = \upsilon'_f$*, and* $l'\upsilon'_f = \upsilon_f$*. Moreover there exist tracks* $\lambda : l'l \simeq \mathrm{id}_{\Sigma_f}$*,* $\lambda' : ll' \simeq \mathrm{id}_{\Sigma'_f}$ *with* $\lambda i_f = \mathrm{id}_{i_f}$*,* $\lambda' i'_f = \mathrm{id}_{i'_f}$*.*

*Proof.* Existence of $l = \Sigma_{\upsilon'_f}$ and $l' = \Sigma_{\upsilon_f}$ satisfying the required identities is clear from the definition of suspenders. Then further by the uniqueness property of suspenders, for the identity track $\Sigma_{\upsilon'_f}\Sigma_{\upsilon_f} i_f = \Sigma_{\upsilon'_f} i'_f = i_f = \mathrm{id}_{\Sigma_f} i_f$ one has $\Sigma_{\upsilon'_f}\Sigma_{\upsilon_f}\upsilon_f = \Sigma_{\upsilon'_f}\upsilon'_f = \upsilon_f$, hence there is a track $\lambda : \Sigma_{\upsilon'_f}\Sigma_{\upsilon_f} \simeq \mathrm{id}_{\Sigma_f}$ with $\lambda i_f = \mathrm{id}_{i_f}$. In an exactly symmetric way one has $\lambda'$ with required properties. □

Also the converse is true:

2.3. **Lemma.** *Let* $(\Sigma_f, i_f, \upsilon_f)$ *be a suspender for the map* $f : A \to B$ *and let the maps* $l : \Sigma_f \to \Sigma$*,* $l' : \Sigma \to \Sigma_f$ *and tracks* $\lambda : l'l \simeq \mathrm{id}_{\Sigma_f}$*,* $\lambda' : ll' \simeq \mathrm{id}_{\Sigma}$ *realise a homotopy equivalence. Then* $(\Sigma, li_f, l\upsilon_f)$ *is another suspender for* $f$*.*

*Proof.* Consider the composite functor

$$[\![\Sigma, -]\!] \xrightarrow{[\![l,-]\!]} [\![\Sigma_f, -]\!] \xrightarrow{(i_f,\upsilon_f)^*} \mathbf{G}_f.$$

Clearly it coincides with $(li_f, l\upsilon_f)^*$. Moreover $(i_f, \upsilon_f)^*$ induces bijection on isomorphism classes by the universal property of suspenders, and so does $[\![l, -]\!]$ – in fact the latter is an equivalence, with inverse $[\![l', -]\!]$. Hence the lemma. □

2.4. **Lemma.** *For any object* $A$*, a suspender for the map* $!_A : * \to A$ *from the (possibly weak) initial object to* $A$ *is given by* $(A, \mathrm{id}_A, \mathrm{id}_{!_A})$*. Given suspenders* $(\Sigma_f, i_f, \upsilon_f)$ *and* $(\Sigma_{f'}, i_{f'}, \upsilon_{f'})$ *for the maps* $f : A \to B$ *and* $f' : A' \to B'$*, respectively,* $(\Sigma_f \vee \Sigma_{f'}, i_f \vee i_{f'}, \upsilon_f \vee \upsilon_{f'})$ *is a suspender for* $f \vee f' : A \vee A' \to B \vee B'$*.*

*Proof.* The first assertion follows easily from the fact that the functor $\mathbf{G}_{!_A}$ coincides with the covariant representable functor $[\![A, -]\!]$.

For the second, consider the functors

$$[\![\Sigma_f \vee \Sigma_{f'}, -]\!] \overset{\cong}{\to} [\![\Sigma_f, -]\!] \times [\![\Sigma_{f'}, -]\!] \xrightarrow{(i_f,\upsilon_f)^* \times (i_{f'},\upsilon_{f'})^*} \mathbf{G}_f \times \mathbf{G}_{f'} \to \mathbf{G}_{f\vee f'},$$

where the rightmost functor is the one assigning to $((g, \alpha), (g', \alpha'))$ with $g : B \to C$, $\alpha : gf \simeq gf$, $g' : B' \to C'$, $\alpha' : g'f' \simeq g'f'$ the pair $(\binom{g}{g'}, \binom{\alpha}{\alpha'})$, where $\binom{g}{g'} : B \vee B' \to C$ and $\binom{\alpha}{\alpha'} : \binom{gf}{g'f'} \simeq \binom{gf}{g'f'} \simeq \binom{g}{g'}(f \vee f')$ are obtained from the equivalences $[\![B \vee B', C]\!] \simeq [\![B, C]\!] \times [\![B', C]\!]$. It is clear how to define this functor on morphisms. One sees directly that this functor induces bijection on isomorphism classes of objects; hence so does the composite, which is easily seen to coincide with $(i_f \vee i_{f'}, \upsilon_f \vee \upsilon_{f'})^*$. The lemma follows. □

## 3. Suspensions

Let $*$ be the initial object of a track category $\mathscr{T}$ in the strong sense so that the hom-groupoid $[\![*, X]\!]$ is the trivial groupoid for any $X$. Then the suspender $\Sigma_0 A$ of a map $0 : A \to *$ is termed a *suspension* (associated to 0) of $A$.

3.1. **Proposition.** *For any map $0 : A \to *$ to the initial object, the corresponding suspension $\Sigma_0(A)$ is canonically equipped with a cogroup structure in the homotopy category $\mathscr{T}_{\simeq}$. Moreover for any $a : A' \to A$ the induced map* (*see* 2.1) $\Sigma_*(f, \mathrm{id}_*) : \Sigma_{0a}(A) \to \Sigma_0(A)$ *respects this cogroup structure.*

*Proof.* Recall that the initial object is understood in the strong sense, so that $[\![*, X]\!]$ is a trivial groupoid for any $X$. It then follows that the groupoid $\mathbf{G}_0(X)$ has as many objects as there are tracks $\alpha :!_X 0 \simeq !_X 0$, and only identity morphisms. In other words, it is the discrete groupoid on the set $\mathrm{Aut}(!_X 0)$. Let us equip this set with a group structure coming from the obvious one on $\mathrm{Aut}(!_X 0)$. Then moreover the functor $\mathbf{G}_0(X) \to \mathbf{G}_0(Y)$ induced by a map $f : X \to Y$ is given on objects by $\alpha \mapsto f\alpha$, hence is a homomorphism of groups. One so obtains a lifting of the functor $\mathbf{G}_0$ to groups. But by the universal property of the suspender, this functor coincides with $[\Sigma_0, -]$. So considered as an object of $\mathscr{T}_{\simeq}$, the suspension $\Sigma_0$ has the property that its covariant representable functor lifts to the category of groups. It then follows by the standard categorical argument that this object has a cogroup structure in $\mathscr{T}_{\simeq}$. Explicitly, the cozero of this cogroup is $\Sigma_{\mathrm{id}_0}$, i. e. the map $\Sigma_0 \to *$ induced by the pair $(\mathrm{id}_* : * \to *, \mathrm{id}_0 : 0\mathrm{id}_* = 0 \simeq 0 = 0\mathrm{id}_*)$. The coaddition map $+ : \Sigma_0 \to \Sigma_0 \vee \Sigma_0$ is induced by the pair $(!_{\Sigma_0 \vee \Sigma_0} : * \to \Sigma_0 \vee \Sigma_0, i_1 \upsilon_0 + i_2 \upsilon_0)$, where $\upsilon_0 \in \mathrm{Aut}(!_{\Sigma_0} 0)$ is the universal track and $i_1, i_2 : \Sigma_0 \to \Sigma_0 \vee \Sigma_0$ are the coproduct inclusions. The inverse map $\Sigma_0 \to \Sigma_0$ is induced by $(!_{\Sigma_0} 0, -\upsilon_0)$.

Now given any $a : A' \to A$, it obviously respects counit. To show that it respects coaddition, one must find a track $+\Sigma_*(a, \mathrm{id}_*) \simeq (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*))+$. According to the uniqueness property of the suspender $\Sigma_{0a}$, for this it is enough to find a track $\alpha : +\Sigma_*(a, \mathrm{id}_*) i_{0a} \simeq (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) + i_{0a}$ satisfying $+\Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} = ((\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) + \upsilon_{0a})^{\alpha 0a}$. There is a unique choice for such $\alpha$ – namely the identity track, as * is initial in the strong sense. Then $+\Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} = +\upsilon_0 a = (i_1 \upsilon_0 + i_2 \upsilon_0) a = i_1 \upsilon_0 a + i_2 \upsilon_0 a = i_1 \Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} + i_2 \Sigma_*(a, \mathrm{id}_*) \upsilon_{0a} = ((\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) i_1 \upsilon_{0a} + (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) i_2 \upsilon_{0a} = (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*))(i_1 \upsilon_{0a} + i_2 \upsilon_{0a}) = (\Sigma_*(a, \mathrm{id}_*) \vee \Sigma_*(a, \mathrm{id}_*)) + \upsilon_{0a}$ as required. □

3.2. **Corollary.** *Suppose that an object $A$ has a co-$H$-structure, i. e. a coaddition $a : A \to A \vee A$ with a two-sided cozero $0 : A \to *$ in $\mathscr{T}_{\simeq}$. Then the above canonical cogroup structure on $\Sigma_0$* (*see* 3.1) *is coabelian.*

*Proof.* By 2.1 and 2.4, there are maps $0' = \Sigma_*(0, \mathrm{id}_*) : \Sigma_0 \to *$ and $+' = \Sigma_*(a, \mathrm{id}_*) : \Sigma_0 \to \Sigma_0 \vee \Sigma_0$ which equip $\Sigma_0$ with a co-$H$-structure in $\mathscr{T}_{\simeq}$. On the other hand it has a canonical cogroup structure $(\Sigma_{\mathrm{id}_0}, +, -)$ in $\mathscr{T}_{\simeq}$ by 3.1.

But in fact $\Sigma_{\mathrm{id}_0}$ and $\Sigma_*(0, \mathrm{id}_*)$ coincide in $\mathscr{T}_\simeq$, so it follows that these cogroup structures have the same cozero.

Moreover the fact that $+'$ respects the cogroup structure means commutativity of

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{+} & \Sigma_0 \vee \Sigma_0 \\ {\scriptstyle +'}\downarrow & & \downarrow{\scriptstyle +'\vee+'} \\ \Sigma_0 \vee \Sigma_0 & \xrightarrow{+_2} & \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \end{array}$$

in $\mathscr{T}_\simeq$, where $+_2$ is the coaddition for the canonical cogroup structure on $\Sigma_0 \vee \Sigma_0$ considered as $\Sigma_{\binom{0}{0}}$. It is clear that this cogroup structure coincides with the coproduct of cogroup structures on $\Sigma_0$. In general, for two cogroups $X$ and $Y$ the coaddition on their coproduct is given by

$$X \vee Y \xrightarrow{+_X \vee +_Y} X \vee X \vee Y \vee Y \xrightarrow{X \vee \binom{i_X}{i_Y} \vee Y} X \vee Y \vee X \vee Y,$$

so that $+_2$ is given by the composite

$$\Sigma_0 \vee \Sigma_0 \xrightarrow{+\vee+} \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \xrightarrow{(23)} \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0,$$

where (23) denotes the map permuting second and third summands.

Composing the above diagram with

$$\mathrm{id}_{\Sigma_0} \vee \begin{pmatrix} 0' \\ 0' \end{pmatrix} \vee \mathrm{id}_{\Sigma_0} : \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \to \Sigma_0 \vee * \vee * \vee \Sigma_0 \cong \Sigma_0 \vee \Sigma_0$$

then gives that there is a track $+' \simeq +$, whereas composing it with

$$\begin{pmatrix} 0' \vee \mathrm{id}_{\Sigma_0} \\ \mathrm{id}_{\Sigma_0} \vee 0' \end{pmatrix} : \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \vee \Sigma_0 \to * \vee \Sigma_0 \vee \Sigma_0 \vee * \cong \Sigma_0 \vee \Sigma_0$$

gives that there is a track $+' \simeq (12)+$. This means that $+$ is coabelian in $\mathscr{T}_\simeq$. □

## 4. Suspenders of coactions

We show that the suspender $\Sigma_f$ of a map $f : A \to B$ splits as a coproduct if $A$ has the structure of a principal coaction in the homotopy category. Recall that a *theory* $\mathbf{C}$ is a category with finite sums $A \vee B$.

4.1. **Definition.** A *principal coaction* in a theory $\mathbf{C}$ consists of an object $A$ together with a cogroup object $S$ in $\mathbf{C}$ and a right coaction

$$a : A \to A \vee S$$

on $A$ such that the map

$$(i_A, q) : A \vee A \to A \vee S$$

is an isomorphism in $\mathbf{C}$. The cogroup structure of $S$ is given by maps $\mu : S \to S \vee S$, $\nu : S \to S$ and $e : S \to *$. The inverse of $(i_A, a)$ yields the map $d : S \to A \vee A$.

A principal coaction is *trivial* if $A$ is isomorphic to $S$ in such a way that $\mu$ corresponds to the cogroup structure of $S$. It is well known that a principal coaction $(A, a)$ is trivial if and only if there exists a map $A \to *$ in $\mathbf{C}$, where $*$ is the initial object of $\mathbf{C}$.

4.2. **Remark.** A *principal action* in a category with finite products consists of an object $T$ together with an internal group $G$ in this category and a right action $a : T \times G \to T$ of $G$ on $T$ such that the map $(p_T, a) : T \times G \to T \times T$ is an isomorphism ($p_T$ being the product projection). Of course a principal action is the categorical dual of a principal coaction.

For our purposes we need a weaker notion which we call principal *quasi* action or *quasi torsor*. It consists of objects $T$, $G$ and morphisms $T \times G \to T$, $d$ : $T \times T \to G$, $1 \to G$, denoted via $(x, g) \mapsto x \cdot g$, $(x, y) \mapsto x\backslash y$, and $e$ respectively, for $x, y :? \to T$, $g :? \to G$, such that the following identities hold:

- $x\backslash x = e$;
- $x \cdot (x\backslash y) = y$.

(Note that the above conditions imply also $x \cdot e = x$.)

Clearly, any principal action is a particular case of this, as one can define $d$ to be the composite map

$$T \times T \xrightarrow{(p_T,a)^{-1}} T \times G \xrightarrow{p_G} G\ .$$

The categorical dual of a principal quasi action is a *principal quasi coaction* which generalizes the principal coaction in 4.1.

Recall that a *track theory* is a track category with coproducts $A \vee B$ in the weak sense (see [5]) so that for all $X$ one has the equivalence of hom-groupoids

$$[\![A \vee B, X]\!] \xrightarrow{\sim} [\![A, X]\!] \times [\![B, X]\!].$$

Let $\omega$ be an inverse of this equivalence.

4.3. **Theorem.** *Let $\mathscr{T}$ be a track theory and let $f : A \to B$ be a map in $\mathscr{T}$. Assume $A$ has the structure of a principal (quasi)coaction in the homotopy category $\mathscr{T}_{\simeq}$ represented by a map $a : A \to A \vee S$ in $\mathscr{T}$, where $S$ is a cogroup in $\mathscr{T}_{\simeq}$. Let $\Sigma S = \Sigma_e S$ be a suspension of $S$ in $\mathscr{T}$ associated to a map $e : S \to *$ in $\mathscr{T}$ representing the counit of $S$. Then there is a suspender of $f$ with*

$$\Sigma_f = B \vee \Sigma S$$

*and $i_f = i_B : B \to B \vee \Sigma S$ the coproduct inclusion and $\upsilon_f : i_B f \Rightarrow i_B f$ a certain canonically defined track.*

The theorem shows that existence of certain suspensions in a track category implies existence of a wider class of suspenders. Moreover by 2.2 we get the following corollary.

4.4. **Corollary.** *Let $\mathscr{T}$ be a $\Sigma$-representable track theory and let $f : A \to B$ be a map in $\mathscr{T}$ where $A$ admits the structure of a principal (quasi)coaction $A \to A \vee S$ in $\mathscr{T}_{\simeq}$. Then there exists a homotopy equivalence $\Sigma_f \simeq B \vee \Sigma S$ where $\Sigma S$ is a suspension associated to a map $S \to *$ representing the counit of $S$.*

*Proof of* 4.3. To simplify exposition, let us introduce the following notation. The given principal coaction gives rise, for each object $X$, to functors

$$[\![A,X]\!] \times [\![S,X]\!] \xrightarrow{\omega} [\![A \vee S, X]\!] \xrightarrow{[\![a,X]\!]} [\![A,X]\!]$$

and

$$[\![A,X]\!] \times [\![A,X]\!] \xrightarrow{\omega} [\![A \vee A, X]\!] \xrightarrow{[\![d,X]\!]} [\![S,X]\!],$$

whose actions on both objects and morphisms will be denoted by

$$(x,s) \mapsto a \cdot s, \quad (x,y) \mapsto x\backslash y,$$

respectively. The principal coaction structure in $\mathscr{T}_{\simeq}$ implies existence of tracks $\varkappa$, $\lambda$ which for any $x, y : A \to X$ induce tracks

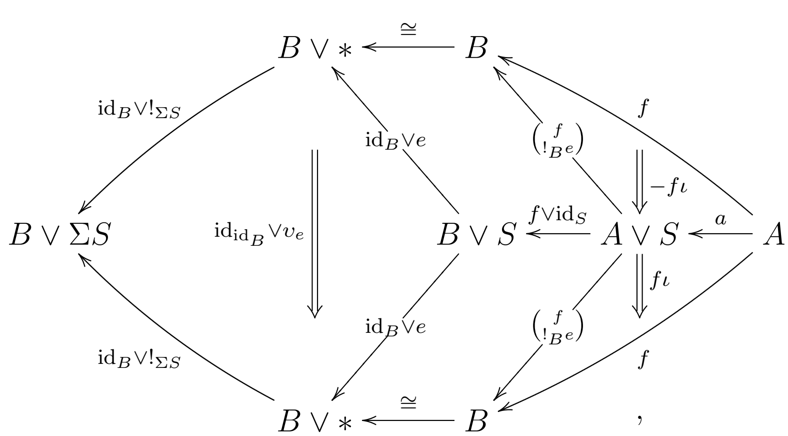
$$x\varkappa : e \Rightarrow x\backslash x,$$

$$\binom{x}{y}\lambda : x \cdot (x\backslash y) \Rightarrow y.$$

Let us define another track $\iota$ by

$$\begin{array}{ccccc} & & x \cdot (x\backslash x) & & \\ & \overset{\mathrm{id}_x \cdot \varkappa}{\nearrow} & & \searrow^{x\binom{\mathrm{id}_A}{\mathrm{id}_A}\lambda} & \\ x \cdot e & & \xrightarrow{\quad x\iota \quad} & & x. \end{array}$$

We now turn to the construction of the universal track $\upsilon_f$. It is the composite track in the diagram




where the two parallelograms commute. More formally, $\upsilon_f = (\upsilon a)^{-i_B f\iota}$, where the track

$$\upsilon = (\mathrm{id}_{\mathrm{id}_B} \vee \upsilon_e)(f \vee \mathrm{id}_S) \in \mathrm{Aut}((\mathrm{id}_B \vee (!_{\Sigma S} e))(f \vee \mathrm{id}_S))$$

is considered as an automorphism of the map

$$i_B\binom{f}{!_Be} = (\mathrm{id}_B \vee !_{\Sigma S})(f \vee e)$$
$$= (\mathrm{id}_B \vee !_{\Sigma S})(\mathrm{id}_B \vee e)(f \vee \mathrm{id}_S)$$
$$= (\mathrm{id}_B \vee (!_{\Sigma S} e))(f \vee \mathrm{id}_S).$$

To show that $\upsilon_f$ is indeed universal, we must show that, for each object $X$, the functor

$$[\![B, X]\!] \times [\![\Sigma S, X]\!] \cong [\![B \vee \Sigma S, X]\!] \xrightarrow{(i_B, \upsilon_f)^*} \mathbf{G}_f(X)$$

induces bijection on isomorphism classes of objects. Now $\upsilon_f$ is chosen in such a way that given $x : B \to X$ and a track $\varepsilon \in \mathrm{Aut}(!_X e)$ with the corresponding map $\Sigma_\varepsilon : \Sigma S \to X$, one has

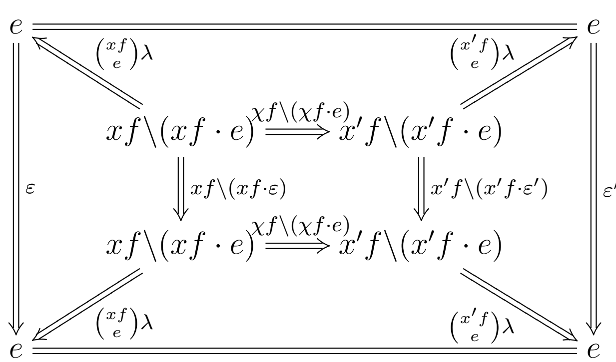
$$(i_B, \upsilon_f)^*(x, \Sigma_\varepsilon) = (\mathrm{id}_{xf} \cdot \varepsilon)^{-xf\iota} : xf \simeq xf.$$

Taking into account the universal property of $\Sigma S$, we may replace isomorphism classes of $[\![\Sigma S, X]\!]$ by those of $\mathbf{G}_e(X)$. We thus must show

- For any $x : B \to X$ and any track $\alpha \in \mathrm{Aut}(xf)$ there is a track $\varepsilon \in \mathrm{Aut}(!_X e)$ such that $\mathrm{id}_{xf} \cdot \varepsilon = \alpha^{xf\iota}$;
- For any $x, x' : B \to X$, any $\varepsilon, \varepsilon' \in \mathrm{Aut}(!_X e)$ and any $\chi : x \simeq x'$ with $\mathrm{id}_{x'f} \cdot \varepsilon' = (\mathrm{id}_{xf} \cdot \varepsilon)^{\chi f}$ there is a track $\eta : \Sigma_{\varepsilon'} \to \Sigma_\varepsilon$.

For the first, define, for $\alpha \in \mathrm{Aut}(xf)$, the track $\varepsilon = (\mathrm{id}_{xf} \backslash \alpha)^{xf\varkappa}$. Then because of our special choice of $\iota$ the required identity will be satisfied.

For the second, note that if $\chi : x \simeq x'$ satisfies the hypothesis, then in the diagram




all inner squares commute, hence the outer square commutes too, i. e. actually $\varepsilon = \varepsilon'$. □

## Acknowledgement

The second author gratefully acknowledges hospitality of the Max Planck Institut für Mathematik, Bonn and of the Université Catholique de Louvain, Louvain-la-Neuve.

## References


1. W. D. Barcus and M. G. Barratt, On the homotopy classification of the extensions of a fixed map. *Trans. Amer. Math. Soc.* **88**(1958), 57–74.
2. H.-J. Baues, Obstruction theory. *Lecture Notes in Math.,* 628, *Springer, Berlin–N. Y.* 1977.
3. H.-J. Baues, Algebraic homotopy. *Cambridge studies in advanced mathematics,* **15**, *Cambridge University Press, Cambridge* 1989.
4. H.-J. Baues, On the cohomology of categories, universal Toda brackets and homotopy pairs. *K-theory* **11**(1997), 259–285.
5. H.-J. Baues and M. Jibladze, Classification of abelian track categories. *Preprint.*
6. H.-J. Baues and M. Jibladze, Suspension and loop objects in theories and cohomology. *Georgian Math. J.* **8**(2001), No. 4, 697–712.
7. P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory. *Ergebnisse der Math. und ihrer Grenzgebiete* 35, *Springer, N. Y.* 1967.
8. K. A. Hardie, K. H. Kamps and R. W. Kieboom, A homotopy category of maps and tracks. *Preprint.*
9. K. A. Hardie, K. H. Kamps and R. W. Kieboom, A homotopy bigroupoid of a topological space. *Preprint.*
10. K. A. Hardie, H. J. Marcum and N. Oda, Bracket operations in the homotopy theory of a 2-category. *Preprint.*
11. D. G. Quillen, Homotopical algebra. *Lecture Notes in Math.* 43 *Springer, Berlin–N. Y.* 1967.
12. J. W. Rutter, A homotopy classification of maps into an induced fibre space. *Topology* **6**(1967), 379–403.


(Received 28.11.2000)

Authors' addresses:

H.-J. Baues
Max-Planck-Institut fuer Mathematik
Vivatsgasse 7, D-53111 Bonn
Germany
E-mail: baues@mpim-bonn.mpg.de

M. Jibladze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia
E-mail: jib@rmi.acnet.ge