Georgian Mathematical Journal
Volume 9 (2002), Number 1, 161–166

ON SOME PROPERTIES OF r-MAXIMAL SETS AND Q_{1-N}-REDUCIBILITY

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Abstract. It is shown that if M_1, M_2 are r-maximal sets and $M_1 \equiv_{Q_{1-N}} M_2$, then $M_1 \equiv_m M_2$. In addition, we prove that there exists a simultaneously Q_{1-N}- and W-complete recursively enumerable set which is not sQ-complete.

2000 Mathematics Subject Classification: 03D25, 03D30.
Key words and phrases: Recursively enumerable set, maximal set, r-maximal set, Q_{1-N}-reducibility, m-reducibility.

A set A is Q-reducible to a set B, written $A \leq_Q B$ (see [1, p. 207]), if there exists a general recursive function (GRF) f such that $(\forall x)\,(x \in A \Longleftrightarrow W_{f(x)} \subseteq B)$. If, in addition, there exists a GRF g such that $(\forall x)(\forall y)\,(y \in W_{f(x)} \Longrightarrow y < g(x))$, then a set A is sQ-reducible to a set B, written $A \leq_{sQ} B$.

A set A is Q_{1-N}-reducible to a set B, written $A \leq_{Q_{1-N}} B$, if there exists a GRF f such that the following relations hold:

1. $(\forall x)\,(x \in A \Longleftrightarrow W_{f(x)} \subseteq B)$,
2. $(\forall x)(\forall y)\,(x \neq y \Longrightarrow W_{f(x)} \cap W_{f(y)} = \varnothing)$,
3. $\bigcup_{x \in N} W_{f(x)}$ is recursive.

The notion of Q_{1-N}-reducibility was introduced by Bulitko in [2].

In this work some properties of Q_{1-N}-reducibility are investigated. In particular, it is proved that if M_1, M_2 are r-maximal sets and $M_1 \equiv_{Q_{1-N}} M_2$, then $M_1 \equiv_m M_2$. It is shown that there exists a simultaneously Q_{1-N}- and W-complete set which is not sQ-complete. All the notions and notation used without definition can be found in [1].

An infinite set A is cohesive if there is no recursively enumerable (RE) set W such that $W \cap A$ and $\overline{W} \cap A$ are both infinite.

An infinite set A is r-cohesive if there is no recursive set R such that $R \cap A$ and $\overline{R} \cap A$ are both infinite.

An RE set A is maximal (r-maximal) if M is cohesive (r-cohesive).

In [3] it is proved that if M is a maximal set and A is an arbitrary set, then

$$M \equiv_Q A \Longrightarrow M \leq_m A.$$

The following statement shows that a maximal set in this theorem cannot be replaced by an r-maximal one.

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Proposition 1. *There are r-maximal Q-complete sets M_1 and M_2 such that $M_1|_m M_2$.*

Proof. Let M_1 be an r-maximal Q-complete set. Then M_1 is not a hyperhypersimple set. Therefore there is a GRF f such that

$$(\forall x)\Big(W_{f(x)} \cap \overline{M}_1 \neq \varnothing\Big), \quad (\forall x)(\forall y)\Big(x \neq y \Longrightarrow W_{f(x)} \cap W_{f(y)} = \varnothing\Big).$$

Consider the set

$$M_2 = M_1 \bigcup \Big(\bigcup_{x \in K} W_{f(x)}\Big),$$

where K is a creative set. Then M_2 is an r-maximal Q-complete set and $|M_2 \backslash M_1| = \infty$. (Note that the sets M_1 and M_2 could be built using Theorem 1 from [4] and Proposition X.4.3 from [5], too.) In [6] it is shown that if M_1, M_2 are r-maximal sets, $M_1 \subset M_2$ and $|M_2 \backslash M_1| = \infty$, then $M_1|_m M_2$. □

Lemma 1. *Let M be an r-maximal set. Then*

$$(\forall f \text{ GRF})\bigg(\Big(|f(\overline{M})| = \infty \ \& \ f(\overline{M}) \subseteq \overline{M}\Big) \Longrightarrow \Big|\Big\{x : \ x \in \overline{M} \ \& \ f(x) \neq x\Big\}\Big| < \infty\bigg). \tag{1}$$

Proof. Kobzev [7] showed that if M is an r-maximal set, then

$$(\forall f \text{ GRF})\bigg(\Big|\Big\{f(x) : \ x \in \overline{M} \ \& \ f(x) \in \overline{M} \ \& \ x \neq f(x)\Big\}\Big| < \infty\bigg). \tag{2}$$

Let

$$(\exists f_1 \text{ GRF})\bigg(|f_1(\overline{M})| = \infty \ \& \ f_1(\overline{M}) \subseteq \overline{M} \ \& \ \Big|\Big\{x : \ x \in \overline{M} \ \& \ f(x) \neq x\Big\}\Big| = \infty\bigg). \tag{3}$$

By (2) and (3), we have $|\{f_1(x) : \ x \in \overline{M} \ \& \ f_1(x) = x\}| = \infty$. From this it follows that

$$\Big|\Big\{x : \ x \in \overline{M} \ \& \ f_1(x) = x\Big\}\Big| = \infty. \tag{4}$$

If the statement of Lemma 1 is false, then from (4) we have that the recursive sets $R_1 = \{x : \ f_1(x) \neq x\}$ and $R_2 = \{x : \ f_1(x) = x\}$ give a splitting of $\overline{M}$ into two infinite sets, which is impossible. □

By using the construction of Theorem 1 [3], we shall prove the following statement.

Theorem 1. *Let M_1 and M_2 be r-maximal sets. Then*

$$M_1 \equiv_{Q_{1-N}} M_2 \Longrightarrow M_1 \equiv_m M_2.$$

Proof. Let M_1 and M_2 be r-maximal sets, $M_1 \leq_{Q_{1-N}} M_2$ via a GRF f and $M_2 \leq_{Q_{1-N}} M_1$ via a GRF g. Using the recursively enumerability of the sets M_1 and M_2, we can assume that

$$(\exists f_1 \text{ GRF})(\forall x)\left(W_{f_1(x)} = M_2 \cup W_{f(x)}\right),$$
$$(\exists g_1 \text{ GRF})(\forall x)\left(W_{g_1(x)} = M_1 \cup W_{g(x)}\right).$$

By r-maximality of the sets M_1 and M_2, from the relations above, in particular, it follows that

$$\bigcup_{x\in N} W_{f_1(x)} =^* N \quad \text{and} \quad \bigcup_{x\in N} W_{g_1(x)} =^* N, \tag{5}$$

where $X =^* Y$ stands for $|(X\backslash Y) \cup (Y\backslash X)| < \infty$.

Let us define a partial recursive function (PRF) φ as follows. We compute simultaneously $\{W_{g_1(i)}\}_{i\in N}$ and $\{W_{f_1(j)}\}_{j\in N}$ and, for given z, seek for first integers x, y (if they exist) such that $z \in W_{g_1(y)}$ & $y \in W_{f_1(x)}$. If we can find such x and y, then we let $\varphi(z) = x$. It is clear that if $z \in \overline{M}_1$, and $\varphi(z)$ is defined, then $\varphi(z) \in \overline{M}_1$.

From the definition of the function φ and from (5) it is clear, that φ is defined for almost all points of the set N.

Lemma 2. $|\varphi(\overline{M}_1)| = \infty$.

Proof. From the recursiveness of the set $\bigcup_{i\in N} W_{g(i)}$ and from the condition that $(\forall x)(\forall y)\,(x \neq y \Longrightarrow W_{g(x)} \cap W_{g(y)} = \varnothing)$ it follows that $(\forall i)\,(W_{g(i)}$ is recursive).

Let us show that $(\forall x)\,(|W_{g(x)} \cap \overline{M}_1| < \infty)$ and, hence,

$$(\forall x)\left(|W_{g_1(x)} \cap \overline{M}_1| < \infty\right). \tag{6}$$

Let us assume the contrary and let

$$(\exists x_1)\left(|W_{g(x_1)} \cap \overline{M}_1| = \infty\right). \tag{7}$$

Then by the nonrecursiveness of the set $\overline{M}_2$,

$$\left|\overline{M}_1\backslash\left(W_{g(x_1)} \cap \overline{M}_1\right)\right| = \infty. \tag{8}$$

Conditions (7) and (8) yield a contradiction to the r-maximality of the set M_1 since the set $W_{g(x_1)}$ is recursive.

Thus, condition (6) holds. Now from the definition of the function φ it is clear that $|\varphi(\overline{M}_1)| = \infty$. $\square$

Hence the function φ is defined for almost all points of the set N, $\varphi(\overline{M}_1) \subseteq \overline{M}_1$ and $|\varphi(\overline{M}_1)| = \infty$.

It is easy to show that Lemma 1 is valid for every PRF $\widetilde{\varphi}$ which is defined for almost all points of the set N. Therefore it is possible apply Lemma 1 for φ and, consequently, we have

$$\Big|\Big\{x:\ x\in\overline{M}_1\ \&\ \varphi(x)\ \text{defined}\ \&\ \varphi(x)\neq x\Big\}\Big|<\infty.$$

Therefore for almost all x we get

$$x\in\overline{M}_1\Longrightarrow\Big|\Big\{y:\ y\in W_{f_1(x)}\ \&\ x\in W_{g_1(y)}\Big\}\Big|=1.$$

Then, for almost all x, we have:

$$x\in\overline{M}_1\Longrightarrow x\in W_{g_1(y)}\ \&\ y\in W_{f_1(x)}\Longrightarrow y\in\overline{M}_2,$$
$$x\in M_1\Longrightarrow x\in W_{g_1(y)}\ \&\ y\in W_{f_1(x)}\Longrightarrow y\in M_2.$$

Let, for all x, $\widetilde{f}(x)$ be the first element which appears in the computation of the set $\{y:\ y\in W_{f_1(x)}\ \&\ x\in W_{g_1(y)}\}$. Then with the help of $\widetilde{f}$ it is possible to construct a GRF which m-reduces the set M_1 to the set M_2.

By symmetry the conditions of the theorem for the sets M_1 and M_2 yield $M_2\le{}_mM_1$. □

Theorem 2. *Let A be a maximal set and B be an r-maximal set. Then*

$$A\le{}_{Q_{1-N}}B\le{}_QA\Longrightarrow A\equiv{}_mB.$$

Proof. Let the conditions of the theorem be satisfied. Then by Theorem 1 [3] $A\le{}_mB$. Note here that it is possible to prove Theorem 2 without appealing to Theorem 1 [3]. If it is shown that $B\le{}_mA$, then by the well-known Young theorem (see [1, Theorem XV]) we will have $B\equiv{}_mA$. Thus, for the proof of Theorem 2 it is sufficient to show that $B\le{}_mA$.

Let $A\le{}_{Q_{1-N}}B$ via a GRF f and $B\le{}_QA$ via a GRF g. Using the recursive enumerability of the sets A and B, we can assume that

$$(\exists f_1\ \text{GRF})(\forall x)\Big(W_{f_1(x)}=B\cup W_{f(x)}\Big),$$
$$(\exists g_1\ \text{GRF})(\forall x)\Big(\Big(x\in B\Longleftrightarrow W_{g_1(x)}\subseteq A\Big)$$
$$\&\ \Big(x\in\overline{B}\Longrightarrow|W_{g_1(x)}\cap\overline{A}|<\infty\Big)\ \&\ A\subseteq W_{g_1(x)}\Big).$$

From the last relations, in particular, we obtain

$$\bigcup_{x\in N}W_{f_1(x)}={}^*N\ \text{ and }\ \bigcup_{x\in N}W_{g_1(x)}={}^*N.$$

Let us define a PRF φ as follows. We compute simultaneously $\{W_{f(i)}\}_{i\in N}$ and $\{W_{g_1(j)}\}_{j\in N}$ and, for given z, we seek (if they exist)for first integers x, y such that

$$z\in W_{f(y)}\ \&\ y\in W_{g_1(x)}.$$

If we can find such x and y, then we let $\varphi(z)=x$. It is clear that if $z\in\overline{B}$ and $\varphi(z)$ is defined, then $\varphi(z)\in\overline{B}$.

Similarly to the proof of Lemma 2 we can prove

Lemma 3. $|\varphi(\overline{B})| = \infty$.

Now the proof of Theorem 2 can be completed in exactly the same way as that of Theorem 1. □

Remark. Kobzev [7] proved that if M_1 and M_2 are r-maximal sets and $M_1 \equiv_{\mathrm{btt}} M_2$, then $M_1 \equiv_m M_2$.

An RE set A is finitely strongly hypersimple if it is coinfinite and if there is no GRF f such that:
(1) $(\forall x)\,(W_{f(x)} \cap \overline{A} \neq \varnothing)$,
(2) $(\forall x)(\forall y)\,(x \neq y \Longrightarrow W_{f(x)} \cap W_{f(y)} = \varnothing)$,
(3) $(\forall x)\,(|W_{f(x)}| < \infty)$,
(4) $\bigcup_{x \in N} W_{f(x)} = N$.
It is easy to show that condition (4) can be replaced by the condition
(4′) $\bigcup_{x \in N} W_{f(x)}$ is recursive.

Proposition 2. *A coinfinite* RE *set A is finitely strongly hypersimple if and only if it has no Q_{1-N}-complete superset.*

Proof. Let a coinfinite RE set A be not finitely strongly hypersimple. Then there is a GRF f such that conditions (1)–(3) and (4′) are valid. Consider the set

$$B = A \bigcup \Big(\bigcup_{x \in K} W_{f(x)} \Big),$$

where K is a creative set and, hence, Q_{1-N}-complete. Then the set B is Q_{1-N}-complete.

Let A be a finitely strongly hypersimple set, B be a Q_{1-N}-complete set and $A \subseteq B$, C be a RE set such that there is an infinite recursive set $R \subseteq \overline{C}$, $C \leq_{Q_{1-N}} B$ via a GRF g, h is a one-to-one GRF, Val $h = R$. Let us define a GRF f as follows:

$$(\forall x)\,\Big(W_{f(x)} = W_{gh(x)}\Big).$$

Then the function f satisfies the conditions (1)–(3) and (4′), hence, the set A is not a finitely strongly hypersimple set, which is a contradiction. □

Corollary. *There is a Q-complete, but not a Q_{1-N}-complete* RE *set.*

Proof. It is known [4] that there is a Q-complete finitely strongly hypersimple set. □

Theorem 1 from [8] asserts that there is a simultaneously Q- and W-complete RE set, which is not sQ-complete. Since there is a Q-complete, but not Q_{1-N}-complete RE set, the following theorem is a stronger statement than Theorem 1 [8].

Theorem 3. *There is a simultaneously Q_{1-N}- and W-complete RE set which is not sQ-complete.*

Proof. First, let us show that there is a Q_{1-N}-complete, but not a W-complete RE set. Let A be a hypersimple, but not finitely strongly hypersimple set. Then there is a GRF f such that conditions (1)–(3) and (4′) are valid. Consider the set

$$B = A \bigcup \Big(\bigcup_{x \in K} W_{f(x)} \Big),$$

where K is a creative set. Then the set B is a Q_{1-N}-complete hypersimple set. It is known [1], that a hypersimple set is not W-complete. Thus there is a Q_{1-N}-complete, but not a W-complete RE set.

It is known [9] that every RE W-degree contains a nowhere simple set.

Let A_1 be a Q_{1-N}-complete but not a W-complete RE set, B_1 be a W-complete nowhere simple set. The set $A_1 \oplus B_1 = \{2x : \ x \in A_1\} \cup \{2x+1 : x \in B_1\}$ is simultaneously Q_{1-N}- and W-complete. Let us assume that the set $A_1 \oplus B_1$ is sQ-complete and S is a simple sQ-complete set. Then $S \leq_{sQ} A_1 \oplus B_1$. From this, by Theorem 2 [8], we have $S \leq_{sQ} A_1$, i.e., the set A_1 is sQ-complete, which is a contradiction. □

References

1. H. Rogers, Jr. Theory of recursive functions and effective computability. *McGraw-Hill, New York*, 1967.
2. V. K. Bulitko, On ways for characterizing complete sets. (Russian) *Izv. Akad. Nauk SSSR, Ser. Math.* **55**(1991), No. 2, 227–253; English transl.: *Math. USSR, Izv.* **38**(1992), No. 2, 225–249.
3. R. Sh. Omanadze, Upper semilattice of recursively enumerable Q-degrees. (Russian) *Algebra i Logika* **23**(1984), No. 2, 175–184; English transl.: *Algebra and Logic* **23**(1984), 124–130.
4. R. Sh. Omanadze, Major sets, classes of simple sets, and Q-complete sets. (Russian) *Mat. Zametki* **71**(2002), No. 1, 100-108; English transl.: *Math. Notes* **71**(2002), No. 1, 90–97.
5. R. I. Soare, Recursively enumerable sets and degrees. *Springer-Verlag, Berlin*, 1987.
6. A. N. Degtev, m-degrees of simple sets. (Russian) *Algebra i Logika* **11**(1972), No. 2, 130–139; English transl.: *Algebra and Logic* **11**(1972), 74–80.
7. G. N. Kobzev, On btt-reducibilities. (Russian) *Algebra i Logika* **12**(1973), No. 4, 433–444; English transl.: *Algebra and Logic* **12**(1973), 242–248.
8. R. Sh. Omanadze, Relations between some reducibilities. (Russian) *Algebra i Logika* **33**(1994), No. 6, 681–688; English transl.: *Algebra and Logic* **33**(1994), 381–385.
9. R. A. Shore, Nowhere simple sets and the lattice of recursively enumerable sets. *J. Symb. Logic* **43**(1978), No. 2, 322–330.

(Received 1.05.2001)

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