Gen. Math. Notes, Vol. 3, No. 2, April 2011, pp. 88-96
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Coefficient Inequality for Certain Sub-Classes of Analytic Functions with Respect to Symmetric Points

B.S. Mehrok[1], Harjinder Singh[2] and Deepak Gupta[3]

[1]Ex. Prof, Department of Mathematics, Khalsa College, Amritsar (Punjab)
E-mail: beantsingh.mehrok@gmail.com
[2]Department of Mathematics, Govt. Rajindra College, Bathinda (Punjab)
E-mail: harjindpreet@gmail.com
[3]Department of Mathematics, MM Engg. College, Mullana (Haryana)
E-mail: guptadeepak2003@yahoo.co.in

(Received: 8-2-11/ Accepted: 23-3-11)

Abstract

Let $\mathcal{A}$ *be the class of analytic functions* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *in the unit disc* $\mathrm{E} = \{z: |z| < 1\}$. *Denote by* $\mathrm{S}(\alpha; \delta, A, B)$ *the class of analytic functions in* $\mathcal{A}$ *satisfying the condition* $(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{\left(2zf'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}$, $(0 \le \alpha \le 1 \text{ and } 0 < \delta \le 1)$. *We obtain sharp upper bounds for the functional* $|a_3 - \mu a_2^2|$.

Keywords: *Analytic functions, Starlike functions with respect to symmetrical points, Subordination.*

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z: |z| < 1\}$.

Denote by U, the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1.2}$$

which are analytic in the unit disc $E = \{z: |z| < 1\}$ and satisfying there the conditions $w(0) = 0$ and $|w(z)| < 1$. It is known [5] that

$$|c_1| \le 1, \qquad |c_2| \le 1 - |c_1|^2. \tag{1.3}$$

Let $f(z)$ and $F(z)$ be two analytic functions in E. Then $f(z)$ is subordinate to $F(z)$ if there is an analytic function $w(z) \in U$ such that $f(z) = F\big(w(z)\big)$. Symbolically, we write

$$f(z) \prec F(z). \tag{1.4}$$

To avoid repetitions, it is admitted once that $0 \le \alpha \le 1$, $-1 \le B < A \le 1$, $0 < \delta \le 1$ and $E = \{z: |z| < 1\}$.

Sakaguchi [6] introduced the class ${S_s}^*$ of functions in $\mathcal{A}$ satisfying the condition

$$Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0. \tag{1.5}$$

These functions are called starlike with respect to symmetrical points in E.

The class K_s of convex functions with respect to symmetrical points in E was defined and studied by Das and Singh [2]. A function $f \in K_s$ if and only if

$$Re\left\{\frac{\big(2zf'(z)\big)'}{\big(f(z) - f(-z)\big)'}\right\} > 0. \tag{1.6}$$

Denote by C_s the class of functions f in $\mathcal{A}$ satisfying the following condition

$$Re\left\{\frac{2zf'(z)}{g(z) - g(-z)}\right\} > 0, \qquad g \in {S_s}^*. \tag{1.7}$$

Goel and Mehrok[3] introduced and studied the class ${S_s}^*(A, B)$ of functions of the form (1.1) satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}. \tag{1.8}$$

Janteng and Halim [4] considered the class $K_s(A, B)$ of functions of the form (1.1) satisfying the condition

$$\frac{(2zf'(z))'}{(f(z) - f(-z))'} \prec \frac{1 + Az}{1 + Bz}. \tag{1.9}$$

Denote by $C_s(A,B)$ and $C_s'(A,B)$ the classes of functions of the form (1.1) satisfying respectively the conditions

$$\frac{2zf'(z)}{g(z)-g(-z)} \prec \frac{1+Az}{1+Bz}, \qquad g \in S_s^* \tag{1.10}$$

and

$$\frac{(2zf'(z))'}{(g(z)-g(-z))'} \prec \frac{1+Az}{1+Bz}, \qquad g \in S_s^*. \tag{1.11}$$

Let $F_s(A,B)$ and $F_s'(A,B)$ denote the classes of functions of the form (1.1) satisfying respectively the conditions

$$\frac{2zf'(z)}{h(z)-h(-z)} \prec \frac{1+Az}{1+Bz}, \qquad h \in K_s \tag{1.12}$$

and

$$\frac{(2zf'(z))'}{(h(z)-h(-z))'} \prec \frac{1+Az}{1+Bz}, \qquad h \in K_s. \tag{1.13}$$

It is easy to verify that $S_s^*(1,-1) \equiv S_s^*$, $K_s(1,-1) \equiv K_s$ and $C_s(1,-1) \equiv C_s$.
Also $S_s^*(1-2\beta,-1) \equiv S_s^*(\beta)$ is the class of starlike functions with respect to symmetrical points of order β defined by Das and Singh [1].

Let us denote by $S_s^*(\delta;A,B)$, $K_s(\delta;A,B)$, $C_s(\delta;A,B)$, $C_s'(\delta;A,B)$, $F_s(\delta;A,B)$ and $F_s'(\delta;A,B)$ the subclasses of functions $f(z)$ in $\mathcal{A}$ which satisfy respectively the following conditions

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}, \tag{1.14}$$

$$\frac{\left(2zf'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}, \tag{1.15}$$

$$\frac{2zf'(z)}{g(z)-g(-z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}; \quad g \in S_s^*, \tag{1.16}$$

$$\frac{\left(2zf'(z)\right)'}{\left(g(z)-g(-z)\right)'} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}; \quad g \in S_s^*, \tag{1.17}$$

$$\frac{2zf'(z)}{h(z)-h(-z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}; \quad h \in K_s, \tag{1.18}$$

and

$$\frac{(2zf'(z))'}{(h(z)-h(-z))'} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}; \quad h \in K_s. \tag{1.19}$$

We also introduce the following classes

$$S(\alpha;\delta;A,B)=\left\{f\in\mathcal{A};(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{\left(2zf'(z)\right)'}{\left(f(z)-f(-z)\right)'}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta}\right\} \tag{1.20}$$

$$T(\alpha;\delta;A,B)=\left\{f\in\mathcal{A};(1-\alpha)\frac{2zf'(z)}{g(z)-g(-z)}+\alpha\frac{\left(2zf'(z)\right)'}{\left(g(z)-g(-z)\right)'}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta},g\in S_s^{*}\right\} \tag{1.21}$$

$$H(\alpha;\delta;A,B)=\left\{f\in\mathcal{A};(1-\alpha)\frac{2zf'(z)}{h(z)-h(-z)}+\alpha\frac{\left(2zf'(z)\right)'}{(h(z)-h(-z))'}\prec\left(\frac{1+Az}{1+Bz}\right)^{\delta},h\in K_s\right\} \tag{1.22}$$

2. Coefficient Inequality

Lemma 2.1 [6]. *Let* $g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{A}$. *Then, for* $n\geq 2$,

$$|b_n|\leq\begin{cases}1 & \text{if } g\in S_s^{*} \quad (2.1)\\ \dfrac{1}{n} & \text{if } g\in K_s \quad (2.2)\end{cases}$$

Theorem 2.1. *Let* $f\in S(\alpha;\delta;A,B)$. *Then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\dfrac{\delta(A-B)}{2(1+2\alpha)}, & \text{if } |\lambda+\mu|\leq\nu, \quad (2.3)\\ \dfrac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda+\mu|, & \text{if } |\lambda+\mu|\geq\nu, \quad (2.4)\end{cases}$$

where

$$\lambda=\frac{(1+\alpha)^2[2B+(1-\delta)(A-B)]}{\delta(1+2\alpha)(A-B)}, \tag{2.5}$$

$$\nu=\frac{2(1+\alpha)^2}{\delta(1+2\alpha)(A-B)}. \tag{2.6}$$

Proof. Since $f\in S(\alpha;\delta;A,B)$, it follows that

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)}+\alpha\frac{\left(2zf'(z)\right)'}{\left(f(z)-f(-z)\right)'}=\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \tag{2.7}$$

By expanding (2.7), we obtain

$$\begin{aligned}&1+2(1+\alpha)a_2z+2(1+2\alpha)a_3z^2+\cdots\\ &\quad=1+\delta(A-B)c_1z+\left(\delta(A-B)c_2-\delta(A-B)\left[B+\frac{1}{2}(1-\delta)(A-B)\right]c_1^2\right)z^2+\cdots\end{aligned}$$

Identifying the terms and solving, we get

$$a_2=\frac{\delta(A-B)}{2(1+\alpha)}c_1 \tag{2.8}$$

$$a_3 = \frac{\delta(A-B)}{2(1+2\alpha)}c_2 - \frac{\delta(A-B)[2B+(1-\delta)(A-B)]}{4(1+2\alpha)}c_1^2 \tag{2.9}$$

From (2.8) and (2.9), we have

$$a_3 - \mu a_2^2 = \frac{\delta(A-B)}{2(1+2\alpha)}c_2 - \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}(\lambda+\mu)c_1^2,$$

where λ is defined by (2.5).

Applying triangular inequality,

$$|a_3 - \mu a_2^2| \le \frac{\delta(A-B)}{2(1+2\alpha)}|c_2| + \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda+\mu||c_1|^2 \tag{2.10}$$

From (1.3) using $|c_2| \le 1 - |c_1|^2$, (2.10) leads us to

$$|a_3 - \mu a_2^2| \le \frac{\delta(A-B)}{2(1+2\alpha)} + \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}(|\lambda+\mu| - \nu)|c_1|^2,$$

where ν is defined by (2.6).

If $|\lambda+\mu| \le \nu$, then

$$|a_3 - \mu a_2^2| \le \frac{\delta(A-B)}{2(1+2\alpha)}.$$

The bound is sharp for $w(z) = z^2$.

If $|\lambda+\mu| \ge \nu$, then from (1.3) using $|c_1| \le 1$, we get

$$|a_3 - \mu a_2^2| \le \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda+\mu|.$$

This bound is sharp for $w(z) = z$.

Corollary 2.1. *If* $f \in S_s^*(\delta;\ A, B)$ *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\delta(A-B)}{2} & \text{if } |\lambda_1+\mu| \le \nu_1, \\ \dfrac{\delta^2(A-B)^2}{4}|\lambda_1+\mu| & \text{if } |\lambda_1+\mu| \ge \nu_1, \end{cases}$$

where

$$\lambda_1 = \frac{2B+(1-\delta)(A-B)}{\delta(A-B)} \quad and \quad \nu_1 = \frac{2}{\delta(A-B)}.$$

Corollary 2.2. *If* $f \in S_s^*(A, B)$, *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(A-B)}{2} & \text{if } |\lambda_2 + \mu| \le \nu_2, \\ \dfrac{(A-B)^2}{4}|\lambda_2 + \mu| & \text{if } |\lambda_2 + \mu| \ge \nu_2, \end{cases}$$

where

$$\lambda_2 = \frac{2B}{(A-B)} \quad \textit{and} \quad \nu_2 = \frac{2}{(A-B)}.$$

Corollary 2.3. *If* $f \in S_s^*$, *then*

$$|a_3 - \mu a_2^2| \le \max.\{1, |\mu - 1|\}.$$

Corollary 2.4. *If* $f \in K_s(\delta;\ A, B)$ *, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\delta(A-B)}{6} & \text{if } |\lambda_3 + \mu| \le \nu_3, \\ \dfrac{\delta^2(A-B)^2}{16}|\lambda_3 + \mu| & \text{if } |\lambda_3 + \mu| \ge \nu_3, \end{cases}$$

where

$$\lambda_3 = \frac{4\{2B + (1-\delta)(A-B)\}}{3\delta(A-B)} \quad \text{and} \quad \nu_3 = \frac{8}{3\delta(A-B)}.$$

Corollary 2.5. *If* $f \in K_s(A, B)$, *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(A-B)}{6} & \text{if } |\lambda_4 + \mu| \le \nu_4, \\ \dfrac{(A-B)^2}{16}|\lambda_4 + \mu| & \text{if } |\lambda_4 + \mu| \ge \nu_4, \end{cases}$$

where

$$\lambda_4 = \frac{8B}{3(A-B)} \quad \textit{and} \quad \nu_4 = \frac{8}{3(A-B)}.$$

Corollary 2.6. *If* $f \in K_s$, *then*

$$|a_3 - \mu a_2^2| \le \text{Max.}\left\{\frac{1}{3}, \frac{1}{4}\left|\mu - \frac{4}{3}\right|\right\}.$$

Theorem 2.2. *If* $f \in T(\alpha; \delta; A, B)$ *and* μ *is a complex number, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\delta(A-B) + 1 + 2\alpha}{3(1+2\alpha)} & \text{if } |\lambda + \mu| \le \nu, & (2.11) \\ \dfrac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda + \mu| + \dfrac{1}{3} & \text{if } |\lambda + \mu| \ge \nu, & (2.12) \end{cases}$$

where

$$\lambda = \frac{2(1+\alpha)^2[2B+(1-\delta)(A-B)]}{3\delta(1+2\alpha)(A-B)}, \tag{2.13}$$

$$\nu = \frac{4(1+\alpha)^2}{3\delta(1+2\alpha)(A-B)}\,. \tag{2.14}$$

Proof. Since $f \in T(\alpha;\delta;A,B)$, by definition of subordination

$$(1-\alpha)\frac{2zf'(z)}{g(z)-g(-z)} + \alpha\frac{\left(2zf'(z)\right)'}{\left(g(z)-g(-z)\right)'} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} \tag{2.15}$$

for some $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_s^*$.
By expanding (2.15), we obtain

$$1 + 2(1+\alpha)a_2 z + (1+2\alpha)(3a_3 - b_3)z^2 + \cdots$$

$$= 1 + \delta(A-B)c_1\, z + \left[\delta(A-B)c_2 - \delta(A-B)\left\{B + \frac{1}{2}(1-\delta)(A-B)\right\}c_1^2\right]z^2 + \cdots$$

After equating the terms and solving, we get

$$a_2 = \frac{\delta(A-B)}{2(1+\alpha)}c_1 \tag{2.16}$$

$$a_3 = \frac{\delta(A-B)}{3(1+2\alpha)}c_2 - \frac{\delta(A-B)[2B+(1-\delta)(A-B)]}{6(1+2\alpha)}c_1^2 + \frac{1}{3}b_3 \tag{2.17}$$

From (2.16) and (2.17), we have

$$a_3 - \mu a_2^2 = \frac{\delta(A-B)}{3(1+2\alpha)}c_2 - \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}(\lambda+\mu)c_1^2 + \frac{1}{3}b_3,$$

where λ is defined by (2.13).

Applying triangular inequality, we obtain

$$|a_3 - \mu a_2^2| \le \frac{\delta(A-B)}{3(1+2\alpha)}|c_2| + \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda+\mu||c_1|^2 + \frac{1}{3}|b_3|. \tag{2.18}$$

Using lemma 2.1 and $|c_2| \le 1 - |c_1|^2$, (2.18) leads us to

$$|a_3 - \mu a_2^2| \le \frac{\delta(A-B)}{3(1+2\alpha)} + \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}(|\lambda+\mu| - \nu)|c_1|^2 + \frac{1}{3},$$

where ν is defined by (2.14).

If $|\lambda+\mu| \le \nu$, then

$$|a_3 - \mu a_2^2| \le \frac{\delta(A-B)+1+2\alpha}{3(1+2\alpha)}.$$

This is the sharp bound for $w(z) = z^2$.

If $|\lambda + \mu| \geq \nu$, then from (1.3) using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda + \mu| + \frac{1}{3}.$$

This bound is sharp for $w(z) = z$.

Corollary 2.7. *If* $f \in C_s(\delta;\ A, B)$ *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{\delta(A-B)+1}{3} & \text{if } |\lambda_5 + \mu| \leq \nu_5, \\ \dfrac{3\delta^2(A-B)^2|\lambda_5 + \mu| + 4}{12} & \text{if } |\lambda_5 + \mu| \geq \nu_5, \end{cases}$$

where

$$\lambda_5 = \frac{2\{2B + (1-\delta)(A-B)\}}{3\delta(A-B)} \quad and \quad \nu_5 = \frac{4}{3\delta(A-B)}.$$

Corollary 2.8. *If* $f \in C_s(A, B)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{A-B+1}{3} & \text{if } |\lambda_6 + \mu| \leq \nu_6, \\ \dfrac{3(A-B)^2|\lambda_6 + \mu| + 4}{12} & \text{if } |\lambda_6 + \mu| \geq \nu_6, \end{cases}$$

where

$$\lambda_6 = \frac{4B}{3(A-B)} \quad and \quad \nu_6 = \frac{4}{3(A-B)}.$$

Corollary 2.9. *If* $f \in C_s$, *then*

$$|a_3 - \mu a_2^2| \leq \max.\left\{1, \frac{1}{3} + \left|\mu - \frac{2}{3}\right|\right\}.$$

Corollary 2.10. *If* $f \in C_s'(\delta;\ A, B)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{\delta(A-B)+3}{9} & \text{if } |\lambda_7 + \mu| \leq \nu_7, \\ \dfrac{3\delta^2(A-B)^2|\lambda_7 + \mu| + 16}{48} & \text{if } |\lambda_7 + \mu| \geq \nu_7, \end{cases}$$

where

$$\lambda_7 = \frac{8\{2B + (1-\delta)(A-B)\}}{9\delta(A-B)} \quad and \quad \nu_7 = \frac{16}{9\delta(A-B)}.$$

Corollary 2.11. *If* $f \in C_s'(A,B)$ *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{A-B+3}{9} & \text{if } |\lambda_8 + \mu| \leq \nu_8\,, \\ \dfrac{3(A-B)^2|\lambda_8+\mu| + 16}{48}\,, & \text{if } |\lambda_8+\mu| \geq \nu_8\,, \end{cases}$$

where

$$\lambda_8 = \frac{16B}{9(A-B)} \quad \textit{and} \quad \nu_8 = \frac{16}{9(A-B)}.$$

Corollary 2.12. *If* $f \in C_s'$, *then*

$$|a_3 - \mu a_2^2| \leq \max.\left\{\frac{5}{9}, \frac{1}{3} + \frac{1}{4}\left|\mu - \frac{8}{9}\right|\right\}.$$

On the same lines as in Theorem 2.2, we have the following:

Theorem 2.3. *If* $f \in H(\alpha;\delta;A,B)$ *and* μ *is a complex number, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{3\delta(A-B)+2\alpha+1}{9(1+2\alpha)}\,, & \text{if } |\lambda+\mu| \leq \nu, \\ \dfrac{\delta^2(A-B)^2}{4(1+\alpha)^2}|\lambda+\mu| + \dfrac{1}{9}, & \text{if } |\lambda+\mu| \geq \nu, \end{cases}$$

where

$$\lambda = \frac{2(1+\alpha)^2[2B + (1-\delta)(A-B)]}{3\delta(1+2\alpha)(A-B)} \quad \textit{and} \quad \nu = \frac{4(1+\alpha)^2}{3\delta(1+2\alpha)(A-B)}.$$

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