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# New Types of Hardy-Hilbert's Integral Inequality

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**Abstract**

*Two new form inequalities similar to Hardy-Hilbert's integral inequality are given.*

**Keywords:** *Hardy-Hilbert,s Integral inequality, Integral inequality.*

## 1 Introduction

Let $f, g \geq 0$ satisfy

$$0<\int_0^\infty f^2(t)\,dt<\infty \ \ and \ \ 0<\int_0^\infty g^2(t)\,dt<\infty,$$

then

(1) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dxdy<\pi\left(\int_0^\infty f^2(t)\,dt\int_0^\infty g^2(t)\,dt\right)^{1/2},$$

where the constant factor $\pi$ is the best possible (cf. Hardy et al. [2]).Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by
Hardy [1] as follows

If p>1, $\frac{1}{p}+\frac{1}{q}=1,\ f,g\geq 0$ satisfy

$$0<\int_0^\infty f^p(t)dt<\infty \ and \ \int_0^\infty g^q(t)dt<\infty ,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)dt\right)^{1/p}\left(\int_0^\infty g^q(t)dt\right)^{1/q}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [3]).
B. Yang gave the following extension of (2) as follows :

Theorem [4]. *If* $\lambda>2-\min\{p,q\},\ f,g\geq 0,$ satisfy

$$0<\int_0^\infty t^{1-\lambda}f^p(t)dt<\infty \ and \ \int_0^\infty t^{1-\lambda}g^q(t)dt<\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy<k_\lambda(p)\left(\int_0^\infty t^{1-\lambda}f^p(t)dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda}g^q(t)dt\right)^{1/q}, \tag{3}$$

where the constant factor $k_\lambda(p)=B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.
The object of this paper is to give some new inequalities similar to that of Hardy-Hilbert’s inequality.

## 2 Lemma

The following lemma is needed for our result

**Lemma** Let $1<1+\alpha<\lambda<2.$ Then

$$\int_0^{\infty} \frac{t^{\alpha-1}}{|1-t|^{\lambda-1}} dt = B(\alpha, 2-\lambda) + B(\lambda-\alpha-1, 2-\lambda)$$

**Proof**

$$\int_0^{\infty} \frac{t^{\alpha-1}}{|1-t|^{\lambda-1}} dt = \int_0^{1} \frac{t^{\alpha-1}}{(1-t)^{\lambda-1}} dt + \int_1^{\infty} \frac{t^{\alpha-1}}{(t-1)^{\lambda-1}} dt$$

$$= \int_0^{1} \frac{t^{\alpha-1}}{(1-t)^{\lambda-1}} dt + \int_0^{1} \frac{t^{\lambda-\alpha-2}}{(1-t)^{\lambda-1}} dt$$

$$= B(\alpha, 2-\lambda) + B(\lambda-\alpha-1, 2-\lambda).$$

## 3 Main Result

We state and prove the following new results

**Theorem 1.** *Let* $f, g, h \geq 0,\ p, q, r > 1,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,\ 0<\alpha<1,\ \beta<\lambda<2,$ *where* $\alpha \in \{a, b, c\}$, *and* $\beta \in \{a+b+1,\ a+c+1,\ b+c+1\}$. *Let*

$$0 < \int_0^{\infty} t^{b+c+1-\lambda+(1-a)(p-1)} f^p(t)\, dt < \infty, \quad 0 < \int_0^{\infty} t^{a+c+1-\lambda+(1-b)(q-1)} g^q(t)\, dt < \infty,$$

$$0 < \int_0^{\infty} t^{a+b+1-\lambda+(1-c)(r-1)} h^r(t)\, dt < \infty.$$

*Then we have*

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)h(z)}{|x-y-z|^{\lambda-1}} dxdydz \leq K_1K_2K_3 \left(\int_0^{\infty} t^{b+c+1-\lambda-(1-a)(p-1)} f^p(t)\, dt\right)^{1/p} \times$$

$$\left(\int_0^{\infty} t^{a+c+1-\lambda+(1-b)(q-1)} g^q(t)\, dt\right)^{1/q} \left(\int_0^{\infty} t^{a+b+1-\lambda+(1-c)(r-1)} h^r(t)\, dt\right)^{1/r},$$

*where*

$$K_1 = [B(c, 2-c) + B(\lambda-c-1, 2-\lambda)]^{1/p} [B(b, 2+c-\lambda) + B(\lambda-b-c-1, 2+c-\lambda)]^{1/p}$$

$$K_2 = [B(a, 2-\lambda) + B(\lambda-a-1, 2-\lambda)]^{1/q} [B(c, \lambda-a-c-1)]^{1/q}$$

$$K_3 = [B(b, 2-\lambda) + B(\lambda-b-1, 2-\lambda)]^{1/r} [B(a, 2+b-\lambda) + B(\lambda-a-b-1, 2+b-\lambda)].$$

**Proof.**

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)\,h(z)}{|x-y-z|^{\lambda-1}}\,dx\,dydz$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty \frac{f(x)\,y^{\frac{b-1}{p}}\,z^{\frac{c-1}{p}}}{x^{(a-1)\left(\frac{1}{q}+\frac{1}{r}\right)}\,|x-y-z|^{\frac{\lambda-1}{p}}}\times\frac{g(y)\,z^{\frac{c-1}{q}}\,x^{\frac{a-1}{q}}}{y^{(b-1)\left(\frac{1}{p}+\frac{1}{r}\right)}\,|x-y-z|^{\frac{\lambda-1}{q}}}$$

$$\times\frac{h(z)\,x^{\frac{a-1}{r}}\,y^{\frac{b-1}{r}}}{z^{(c-1)\left(\frac{1}{p}+\frac{1}{q}\right)}\,|x-y-z|^{\frac{\lambda-1}{r}}}\;dxdydz$$

$$\le\left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{f^p(x)y^{b-1}z^{c-1}}{x^{(a-1)(p-1)}|x-y-z|^{\lambda-1}}\,dxdydz\right)^{1/p}\left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{g^q(y)z^{c-1}x^{a-1}}{y^{(b-1)(q-1)}|x-y-z|^{\lambda-1}}\,dxdydz\right)^{1/q}$$

$$\times\left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{h^r(z)x^{a-1}\,y^{b-1}}{z^{(c-1)(r-1)}|x-y-z|^{\lambda-1}}\,dxdydz\right)^{1/r}$$

$$=P^{1/p}\,Q^{1/q}\,R^{1/r}\,.$$

We first consider P. As $\big||x-y|-z\big|\le|x-y-z|$, we have

$$P\le\int_0^\infty\int_0^\infty\int_0^\infty \frac{f^p(x)\,y^{b-1}z^{c-1}}{x^{(a-1)(p-1)}\,\big||x-y|-z\big|^{\lambda-1}}\,dx\,dy\,dz$$

$$=\int_0^\infty x^{b+c+1-\lambda+(1-a)(p-1)}f^p(x)\,dx\int_0^\infty \frac{\left(\frac{y}{x}\right)^{b-1}\frac{1}{x}}{\left|1-\frac{y}{x}\right|^{\lambda-c-1}}\,dy\int_0^\infty \frac{\left(\frac{z}{|x-y|}\right)^{c-1}\frac{1}{|x-y|}}{\left|1-\frac{z}{|x-y|}\right|^{\lambda-1}}\,dz$$

$$=\int_0^\infty x^{b+c+1-\lambda+(1-a)(p-1)}f^p(x)dx\int_0^\infty \frac{t^{b-1}}{|1-t|^{\lambda-c-1}}\,dt\;\int_0^\infty \frac{t^{c-1}}{|1-t|^{\lambda-1}}\,dt$$

$$=\left[B(c,2-\lambda)+B(\lambda-c-1,2-\lambda)\right]\left[B(b,2+c-\lambda)+B(\lambda-b-c-1,2+c-\lambda)\right]$$

$$\times\int_0^\infty x^{b+c+1-\lambda+(1-a)(p-1)} f^p(x)dx,$$

by an application of the lemma.

$$Q = \int_0^\infty\int_0^\infty\int_0^\infty \frac{g^q(y)\, z^{c-1} x^{a-1}}{y^{(b-1)(q-1)}\,|x-y-z|^{\lambda-1}}\, dx\, dy\, dz$$

$$= \int_0^\infty y^{a+c+1-\lambda+(1-b)(q-1)} g^q(y)\, dy \int_0^\infty \frac{\left(\frac{z}{y}\right)^{c-1}\frac{1}{y}}{\left(1+\frac{z}{y}\right)^{\lambda-a-1}}\, dz \int_0^\infty \frac{\left(\frac{x}{y+z}\right)^{a-1}\frac{1}{y+z}}{\left|1-\frac{x}{y+z}\right|^{\lambda-1}}\, dx$$

$$= B(c,\lambda-a-c-1)\left[B(a,2-\lambda)+B(\lambda-a-1,2-)\right]\times$$

$$\int_0^\infty y^{a+c+1-\lambda+(1-b)\left(\frac{q}{p}+\frac{q}{r}\right)} g^q(y)dy.$$

Finally

$$R = \int_0^\infty\int_0^\infty\int_0^\infty \frac{h^r(z)\, x^{a-1} y^{b-1}}{z^{(c-1)(r-1)}\,|x-y-z|^{\lambda-1}}\, dxdydz$$

$$\le \int_0^\infty\int_0^\infty\int_0^\infty \frac{h^r(z)\, x^{a-1} y^{b-1}}{z^{(c-1)(r-1)}\,\big||x-z|-y\big|^{\lambda-1}}\, dxdydz$$

$$= \int_0^\infty z^{a+b+1-\lambda+(1-c)(r-1)} h^r(z)\, dz \int_0^\infty \frac{\left(\frac{x}{z}\right)^{a-1}\frac{1}{z}}{\left|1-\frac{x}{z}\right|^{\lambda-b-1}}\, dx \int_0^\infty \frac{\left(\frac{y}{|x-z|}\right)^{b-1}\frac{1}{|x-z|}}{\left|1-\frac{y}{|x-z|}\right|^{\lambda-1}}\, dy$$

$$= \left[B(b,2-\lambda)+B(\lambda-b-1,2-\lambda)\right]\left[B(a,2+b-\lambda)+B(\lambda-a-b-1,2+b-\lambda)\right]$$

$$\times\int_0^\infty z^{a+b+1-\lambda+(1-c)\left(\frac{r}{p}+\frac{r}{q}\right)} h^r(z)dz.$$

This completes the proof of the theorem.

**Theorem 2.** *Let* $f,g,h\geq 0,\ p,q,r>1,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,\ \frac{3}{2}<\lambda<\frac{1}{\gamma},\ \gamma\in\{p,q,r\},$

$$0<\int_0^\infty|1-t|^{\lambda p}\frac{f^p(t)}{t}dt<\infty,\ 0<\int_0^\infty|1-t|^{\lambda q}\frac{g^q(t)}{t}dt<\infty,\ \int_0^\infty|1-t|^{\lambda r}\frac{h^r(t)}{t}dt<\infty$$

.

*Then,*

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{f(x)g(y)h(z)}{|1-xyz|^\lambda}dxdydz\leq\quad 2B(\lambda/2,1-\lambda)K_p^{1/p}K_q^{1/q}K_r^{1/r\ \ 1/p}\times$$

$$\left(\int_0^\infty|1-t|^{\lambda p}\frac{f^p(t)}{t}dt\right)^{1/p}\left(\int_0^\infty|1-t|^{\lambda q}\frac{g^q(t)}{t}dt\right)^{1/q}\left(\int_0^\infty|1-t|^{\lambda r}\frac{h^r(t)}{t}dt\right)^{1/r},$$

*where*

$$K_\gamma = B(\gamma(1-\lambda/2),1-\lambda\gamma)+B(\gamma(3\lambda/2-1,1-\lambda\gamma)).$$

**Proof.** Applying the lemma, with $\lambda-1$ replaced by $\lambda$ ,we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{f(x)g(y)h(z)}{|1-xyz|^\lambda}dxdydz$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\frac{f(x)\,x^{\frac{\lambda/2-1}{p}}z^{\frac{\lambda/2-1}{p}}\left|\frac{1-x}{1-y}\right|^\lambda}{y^{(\lambda/2-1)(1-1/p)}|1-xyz|^{\lambda/p}}\ \frac{g(y)\,y^{\frac{\lambda/2-1}{q}}x^{\frac{\lambda/2-1}{q}}\left|\frac{1-y}{1-z}\right|^\lambda}{z^{(\lambda/2-1)(1-1/q)}|1-xyz|^{\lambda/q}}$$

$$\times\frac{h(z)z^{\frac{\lambda/2-1}{r}}y^{\frac{\lambda/2-1}{r}}\left|\frac{1-z}{1-x}\right|^\lambda}{x^{(\lambda/2-1)(1-1/r)}|1-xyz|^{\lambda/r}}\ dxdydz$$

$$\leq \left( \int_0^\infty \int_0^\infty \int_0^\infty \frac{f^p(x)\, x^{\lambda/2-1}\, z^{\lambda/2-1} |1-x|^{\lambda p}}{y^{(\lambda/2-1)(p-1)} |1-y|^{\lambda p} |1-xyz|^{\lambda}} dxdydz \right)^{1/p} \times$$

$$\left( \int_0^\infty \int_0^\infty \int_0^\infty \frac{g^q(y)\, y^{\lambda/2-1}\, x^{\lambda/2-1} |1-y|^{\lambda q}}{z^{(\lambda/2-1)(q-1)} |1-z|^{\lambda q} |1-xyz|^{\lambda}} dxdydz \right)^{1/q} \times$$

$$\left( \int_0^\infty \int_0^\infty \int_0^\infty \frac{h^r(z)\, z^{\lambda/2-1}\, y^{\lambda/2-1} |1-z|^{\lambda r}}{x^{(\lambda/2-1)(r-1)} |1-x|^{\lambda r} |1-xyz|^{\lambda}} dxdydz \right)^{1/r}$$

$$= L^{1/p} M^{1/q} N^{1/r}.$$

Observe that

$$L = \int_0^\infty |1-x|^{\lambda p} \frac{f^p(x)}{x} dx \int_0^\infty \frac{y^{p(1-\lambda/2)-1}}{|1-y|^{\lambda p}} dy \int_0^\infty \frac{(xyz)^{\lambda/2-1}\, xy}{|1-xyz|^{\lambda}} dz$$

$$= 2\, B(\lambda/2, 1-\lambda)\, [B(p(1-\lambda/2), 1-\lambda p) + B(3\lambda/2-1, 1-\lambda p)]$$

$$\times \int_0^\infty |1-x|^{\lambda p} \frac{f^p(x)}{x} dx$$

$$= 2B(\lambda/2, 1-\lambda)\, K_p \int_0^\infty |1-x|^{\lambda p} \frac{f^p(x)}{x} dx.$$

Similarly, we can show that

$$M = 2B(\lambda/2, 1-\lambda)\, K_q \int_0^\infty |1-y|^{\lambda q} \frac{g^q(y)}{y} dy,$$

$$N = 2B(\lambda/2, 1-\lambda)\, K_r \int_0^\infty |1-z|^{\lambda r} \frac{h^r(z)}{z} dz.$$

The proof is complete.

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