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The Longest Path Problem in Fuzzy Project Networks: A Case Study

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Abstract

In This article a real-life international construction and building project network problem is presented, the problem of identifying the critical path of the project within fuzzy parameters (the longest path) is formulated by applying a linear programming approach which incorporates the concept of α-cuts into two (primal and dual) models. In order to determine the solution a code was developed to solve 22 linear programming models that had been constructed at different α-cuts during formulation. Yager's ranking method is applied to compare all paths, and determining the time of the longest one.

Keywords: *Construction & Building, Critical Path, Fuzzy CPM, Linear Programming, Project Management.*

1 Introduction

Scheduling the activities in project management is becoming increasingly important to obtain competitive priorities such as on-time delivery [7]. By using project management, managers are able to obtain a graphical display of project activities (tasks), an estimate of how long the project will take [3]. Critical Path

Method (CPM) is employed in a wide range of engineering and management applications [2,8]. However, there are many cases where the activity times may not be presented in a precise manner. To deal quantitatively with imprecise data, the Program Evaluation and Review Technique (PERT) [7,11] based on the probability theory can be employed which has detailed critiques illustrated in [10]. An alternative way to deal with imprecise data is to employ the concept of fuzziness; the main advantages of methodologies based on fuzzy theory are that they do not require prior predictable regularities or posterior frequency distributions [13]. The problems of computing the intervals of possible values of the latest starting times and floats of activities with imprecise durations represented by fuzzy or interval numbers have attracted intensively attentions and many solution methods have been proposed (e.g.,[6,9,14]). Most of them are straightforward extensions of deterministic CPM. However, many issues after had been aroused considering the drawbacks and failures specially when estimating the degree of criticality of each path as well as they did not completely conserve all the fuzziness of activity times, thus some useful insights and valuable information may be lost [5,6].
Some authors proposed Linear Programming (LP) approaches to overcome these drawbacks mentioned like in [2,4], a lot of modifications done since the first LP approach proposed. In [2] the author combined the duality concept to the α-cuts concept into two LP models to conserve fuzziness and overcome previous drawbacks mentioned. In construction and building projects the processes of activities' scheduling, identifying the critical path, determining the whole project's duration are very important and critical tasks. These tasks must be done in a very precise manner to avoid extra costs and time especially when many independent construction companies are involved in the project. In this paper a real-life project of constructing mall is introduced, the project is formulated into two LP models; the approach found in [2] is applied to solve the longest path problem. This paper is structured as following: section 2 is made for the linear programming approach, the case study is presented in section 3, solution and illustration are introduced in section 4, and finally section 5 is made for conclusion.

2 Linear Programming Approach

Consider a project network S = $\{V,A,t\}$ consisting of a finite set V of nodes (events) and a set A of arcs with crisp activity times, which are determined by a function $t : A \rightarrow \mathrm{R}^+$ and attached to the arcs. Denote t_{ij} as the time period of activity (i, j) belongs to A. An alternative way to determine the total duration and find critical paths is by using LP techniques. The basic idea of the LP formulation that it assumes that a unit flow enters the project network at the start node and leaves at the finish node. Let x_{ij} be the decision variable denoting the amount of flow in activity (i, j) belongs to A. Since only one unit of flow could be in any arc at any one time, the variable x_{ij} must assume binary values (0 or 1) only as shown in details in [2,7]. If any of the activity duration time t_{ij} is fuzzy, the total duration time D becomes fuzzy as well. The conventional CPM problem is then modified into the CPM problem with fuzzy parameters. Consequently, it cannot be maximized directly [1]. Consider a project network $S_f = \{V,A,\check{T}\}$ with fuzzy activity times. V and A are the same as in the crisp case except that the activity times are approximately known and defined by function $\check{T} : A \rightarrow \mathrm{FN}(R^+)$, where $\mathrm{FN}(R^+)$ is the set of non-negative fuzzy numbers. Consequently, the fuzzy CPM problem has the following form in model (1):

$$
\begin{aligned}
\widetilde{D} = \max \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{T}_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{1j} = 1, \\
& \sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki}, \quad i = 2, \ldots, n-1, \\
& \sum_{k=1}^{n} x_{kn} = 1, \\
& x_{ij} = 0 \text{ or } 1, \quad (i,j) \in A.
\end{aligned} \tag{1}
$$

Unfortunately, most of the existing techniques provide only crisp solutions. If the obtained objective value is a crisp value, then some helpful information for project management may be lost. To identify the critical paths of the project for the fuzzy CPM problem the solutions must conserve the fuzziness of the fuzzy CPM problem. In [2] an approach is developed to derive the membership function of the fuzzy total duration time analysis based on a combination of the concept of α-cut, two-level mathematical programming. This combination resulted in the following two (primal and dual) linear programming models (maximization and minimization) as shown below, which can be solved by the simplex method, interior-point algorithms, or other network analysis methods [2,7]. First, the primal problem (upper maximization model) presented in model (2):

$$
\begin{aligned}
D_\alpha^{U} = \max \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} (T_{ij})_\alpha^{U} x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{1j} = 1, \\
& \sum_{j=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{ki}, \quad i = 2, \ldots, n-1, \\
& \sum_{k=1}^{n} x_{kn} = 1 \\
& x_{ij} \geqslant 0, \quad (i,j) \in A.
\end{aligned} \tag{2}
$$

Second, the dual form (lower minimization model) listed in model (3):

$$D_\alpha^{\mathrm{L}} = \min \quad y_n - y_1$$
$$\text{s.t.} \quad y_j \geqslant y_i + (T_{ij})_\alpha^{\mathrm{L}}, \quad (i,j) \in A, \tag{3}$$
$$y_i, y_j \text{ unrestricted in sign} \quad \forall (i,j) \in A.$$

Where D_α^U and D_α^L are the upper and lower bounds of the α-cut of the membership function. Note that all α-cuts form a nested structure that is for two possibility levels α_1 and α_2 such that $0 < \alpha_2 < \alpha_1 < 1$. Therefore; the feasible region defined by α_2 is smaller than the one defined by α_1 thus $D_{\alpha_1}^L \geq D_{\alpha_2}^L$, $D_{\alpha_1}^U \leq D_{\alpha_2}^U$, for further details refer to [2].

3 Case Study

Recently, many foreign multinational companies are willing to invest in construction and building projects in Egypt, many factors help this type of projects to be developed and expanded over time in Egypt. *Makro* is a German company that works in hyper supermarkets, specialized in mass trade, selling food stuffs like: meats, fishes, vegetables, fruits, sugar, macaroni, rice, and many other commodities. It has many branches in several countries like Turkey, Germany, France, Italy and more than 30 other countries. Two years ago, a feasibility study was done by the company to enter the Egyptian market by investing 4.5 milliards L.E., and building 45 branches all over Egypt in the next five years; the estimated budgeted cost for each mall is 100 millions L.E., the first branch is planned to be built in Al-Salam City, the mall consists of one floor store of 15000 m^2 steel structure building, a 30000 m^2 parking area, and 5000 m^2 backyard for trucks maneuvering. The company chose many contractors to execute this project: the Egyptian DETAC CO. for concrete and finishes works, The Saudi AL-ZAMIL CO. for steel structure and external cladding, the Italian ECO CO. for air conditioning and fire fighting, the German SIEMENS CO. for electrical works, the Italian VERCOS CO. for internal panels for cold area, the Polish RINOHL CO. for floorings, the Turkish 3K CO. for the internal huge racks. The company chose the Egyptian consultant M.A. CONSULTANTS to be the project manager for the whole project.

3.1 Problem definition

As being the responsible of the co-ordination between the companies working in the project and scheduling the whole project's activities, the Egyptian consultant should be so precise in scheduling the times of the activities and the whole project duration. The consultant should have interactive discussions, agreements, and decisions with the executive companies to optimize both the time and the cost of the project, any deviation in the assessment of the activities' times will lead to extra cost and time. The activities' duration times in the project are not deterministic and imprecise so the concept of fuzziness is employed to deal with the vague activity times. The Egyptian consultant scheduled the project into 30 activities and represented their times by fuzzy sets after asking the experts, interacting with the companies to build the membership functions used. As shown in Table 1. the 30 activities are listed; their fuzzy operation times are illustrated by L-L fuzzy number type [15] where L,U, ls, and rs are for the lower, upper, left spread and right spread respectively.

Table 1: Makro construction project (time in days)

Activity Item	Activity Description	Precedence Item	Fuzzy Operation Times			
			L	U	ls	rs
P1	Concrete works foundation	-	28	30	3	5
P2	Insulation works	P1	4	4	1	1
P3	Parking area + Roads + Landscape	P2	29	30	4	5
P4	Back filling works	P3	7	12	4	3
P5	Sub-base	P4	6	6	1	4
P6	Steel structure erection	P5	30	35	4	5
P7	Under ground drainage system	P5	10	10	3	3
P8	Water tank – civil works	-	21	21	6	4
P9	Steel structure testing	P6	3	4	1	1
P10	Roofing works	P6	10	12	1	3
P11	Water tank - finishing	P8	7	8	1	2
P12	HVAC works – 1st fix	P9	14	12	2	2
P13	Fire fighting works 1st fix	P9	9	11	2	1
P14	Electrical system works – 1st fix	P12,P13	6	7	1	3
P15	Flooring	P14	9	11	2	1
P16	HVAC work – 2nd fix	P9	14	12	2	2
P17	Fire fighting works – 2nd fix	P9	9	11	2	1
P18	Cladding works	P9	24	25	9	5
P19	Electrical system works – 2nd fix	P16,17	6	7	1	3
P20	Water tank - MEP	P11	11	12	2	2
P21	Finishing works	P15	18	18	3	2
P22	HVAC works – 3rd	P9	14	12	2	2
P23	Fire fighting work – 3rd fix	P9	9	11	2	1
P24	Electrical system works –3rd fix	P22,P23	6	7	1	3
P25	Plumbing works – 1st fix	P14	6	6	1	2
P26	Plumbing works – 2nd fix	P19	6	6	1	2
P27	Plumbing works – 3rd fix	P24	6	6	1	2
P28	Water tank testing	P20	2	2	1	1
P29	Testing and commissioning	P28	2	2	1	1
P30	Snag list and Initial handling	P29	7	7	2	2

Many LP approaches tried to solve the problem of finding the critical path of the project (the longest path) within these fuzzy parameters, in next subsection the problem is formulated into two LP models, applying the approach found in [2] to the problem. The project is plotted using the Activity On Arc (AOA) method as shown in Fig.1, dotted arcs are for dummy activities.

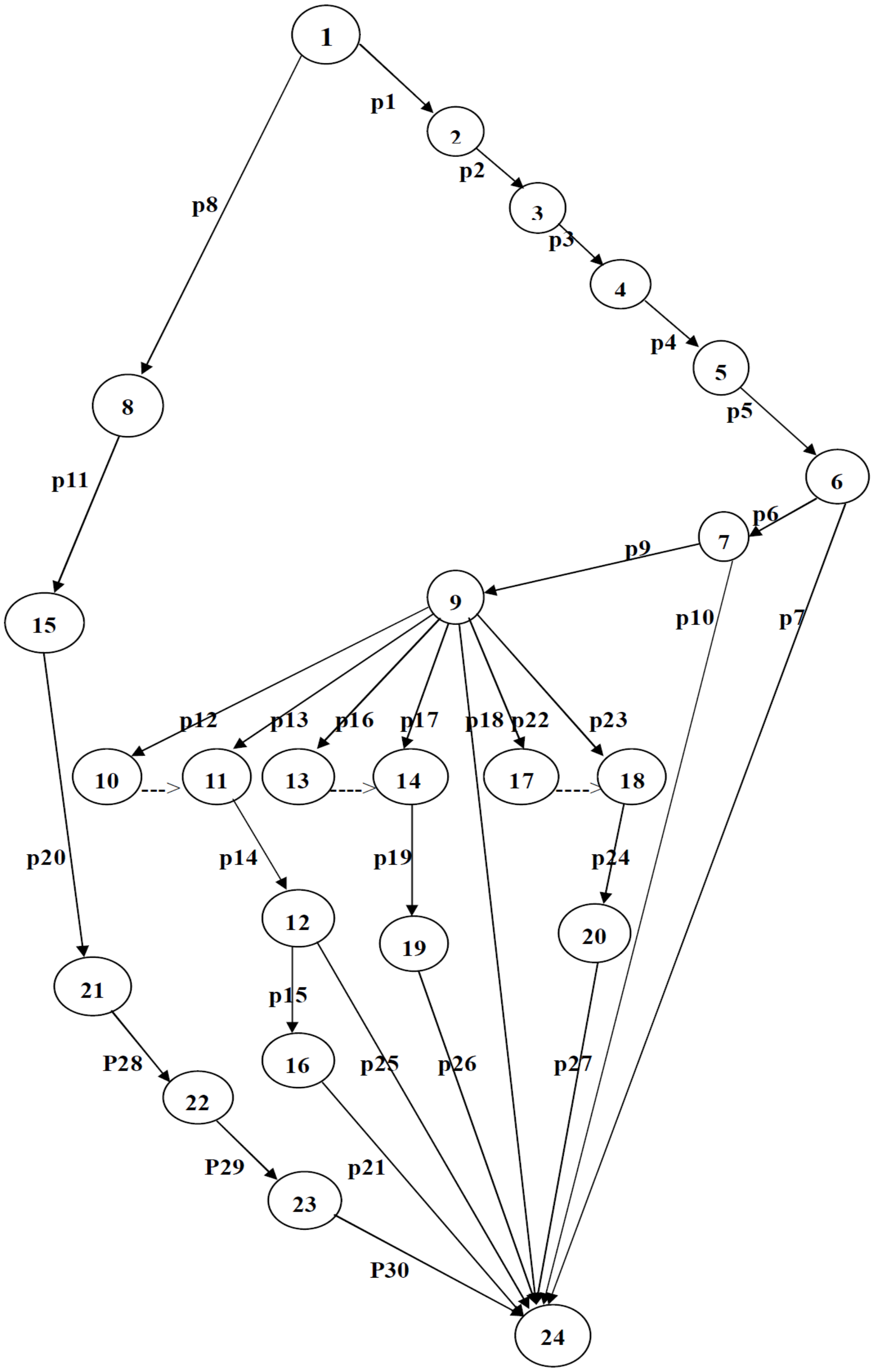

Fig.1 Macro project's network

3.2 Problem formulation

In addition to the α-cuts concept used in [2], the linear reference function method and fuzzy numbers of L-L type as shown in [14] were incorporated in the formulation process resulted in 11 different linear programming models to be solved for each model (maximization and minimization).

The upper (maximization) illustrated in model (4):

$$
\begin{aligned}
D_\alpha^U = \max\ & (35-5\alpha)P1+(5-\alpha)P2+(35-5\alpha)P3+(15-3\alpha)P4+(10-4\alpha)P5+(40-5\alpha)P6+ \\
& (13-3\alpha)P7+(25-4\alpha)P8+(5-\alpha)P9+(15-3\alpha)P10+(10-2\alpha)P11+(16-2\alpha)P12+ \\
& (12-\alpha)P13+(10-3\alpha)P14+(12-\alpha)P15+(16-2\alpha)P16+(12-\alpha)P17+(30-5\alpha)P18+ \\
& (10-3\alpha)P19+(14-2\alpha)P20+(20-2\alpha)P21+(16-2\alpha)P22+(12-\alpha)P23+(10-3\alpha)P24+ \\
& (8-2\alpha)P25+(8-2\alpha)P26+(8-2\alpha)P27+(3-\alpha)P28+(3-\alpha)P29+(9-2\alpha)P30 \\
\text{s.t.}\ & \\
& P1+P8=1 \\
& P1=P2 \\
& P2=P3 \\
& P3=P4 \\
& P4=P5 \\
& P5=P6+P7 \\
& P6=P9+P10 \\
& P8=P11 \\
& P11=P20 \\
& P20=P28 \\
& P28=P29 \\
& P29=P30 \\
& P9=P12+P13+P16+P17+P18+P22+P23 \qquad (4) \\
& P12+P13=P14 \\
& P14=P15+P25 \\
& P15=P21 \\
& P16+P17=P19 \\
& P19=P26 \\
& P22+P23=P24 \\
& P24=P27 \\
& P7+P10+P18+P21+P25+P26+P27+P30=1 \\
& \\
& P_i \geq 0, \quad i=1,2,3,\dots,30.
\end{aligned}
$$

The lower (minimization) presented in model (5):

$$
\begin{aligned}
D_\alpha^L = \min \quad & y_{24} - y_1 \\
\text{s.t.} \quad & \\
& y_2 \geq y_1 + (25 + 3\alpha) \\
& y_3 \geq y_2 + (3 + \alpha) \\
& y_4 \geq y_3 + (25 + 4\alpha) \\
& y_5 \geq y_4 + (3 + 4\alpha) \\
& y_6 \geq y_5 + (5 + \alpha) \\
& y_7 \geq y_6 + (26 + 4\alpha) \\
& y_9 \geq y_7 + (2 + \alpha) \\
& y_{10} \geq y_9 + (10 + 2\alpha) \\
& y_{11} \geq y_9 + (7 + 2\alpha) \\
& y_{11} \geq y_{10} \\
& y_{12} \geq y_{11} + (5 + \alpha) \\
& y_{16} \geq y_{12} + (7 + 2\alpha) \\
& y_{13} \geq y_9 + (10 + 2\alpha) \\
& y_{14} \geq y_9 + (7 + 2\alpha) \qquad (5) \\
& y_{14} \geq y_{13} \\
& y_{19} \geq y_{14} + (5 + \alpha) \\
& y_{17} \geq y_9 + (10 + 2\alpha) \\
& y_{18} \geq y_9 + (7 + 2\alpha) \\
& y_{18} \geq y_{17} \\
& y_{20} \geq y_{18} + (5 + \alpha) \\
& y_8 \geq y_1 + (15 + 6\alpha) \\
& y_{15} \geq y_8 + (6 + \alpha) \\
& y_{21} \geq y_{15} + (9 + 2\alpha) \\
& y_{22} \geq y_{21} + (1 + \alpha) \\
& y_{23} \geq y_{22} + (1 + \alpha) \\
& y_{24} \geq y_{12} + (5 + \alpha) \\
& y_{24} \geq y_{16} + (15 + 3\alpha) \\
& y_{24} \geq y_{19} + (5 + \alpha) \\
& y_{24} \geq y_9 + (15 + 9\alpha) \\
& y_{24} \geq y_{20} + (5 + \alpha) \\
& y_{24} \geq y_7 + (9 + \alpha) \\
& y_{24} \geq y_6 + (7 + 3\alpha) \\
& y_{24} \geq y_{23} + (5 + 2\alpha) \\
& y_i \text{ unrestricted in sign}, \quad i = 1,2,3,\ldots,24.
\end{aligned}
$$

4 Solution and Illustration

Eleven α-cuts (0, 0.1, 0.2,…, 1) were used in the two models (4), and (5) resulting in 22 different linear programming models to be solved, MATLAB software was employed to develop a code to solve these linear models, building the membership functions, determining the lower and upper bounds for each model, and estimate the degree of criticality for each path using yager's ranking method [12]. Table 2. shows the solution of the 22 linear programs corresponding to all α-cuts for both models, it is obvious that the path P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21 is the only solution for all linear programs, also the path gives the highest degree of criticality in Table 3. which was set for comparing all paths found of the project after been ranked. The solution should lies between the values of 203 and 152 days (upper and lower), after ranking it yields to be 163 days.

Table 2: Critical path (solution) at each α-cut for upper and lower models

Alpha (α)	Lower bound (Minimization model)		Upper bound (Maximization model)	
	Critical path	Duration	Critical path	Duration
0	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	126	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	203
0.1	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	128.6	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	199.8
0.2	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	131.2	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	196.6
0.3	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	133.8	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	193.4
0.4	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	136.4	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	190.2
0.5	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	139	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	187
0.6	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	141.6	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	183.8
0.7	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	144.2	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	180.6
0.8	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	146.8	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	177.4
0.9	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	149.4	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	174.2
1	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	152	P1-P2-P3-P4-P5-P6-P8-P9-P12-P14-P15-P21	171

Table 3: Degree of criticality for all paths

No.	Path	Duration	Deg. Of criticality	Solution
1	P1-P2-P3-P4-P5-P7	89.25	0.547546	
2	P1-P2-P3-P4-P5-P6-P10	123.5	0.757669	
3	P1-P2-P3-P4-P5-P6-P9-P18	139	0.852761	
4	P1-P2-P3-P4-P5-P6-P9-P22-P24-P27	141.75	0.869632	
5	P1-P2-P3-P4-P5-P6-P9-P23-P24-P27	138.5	0.849693	
6	P1-P2-P3-P4-P5-P6-P9-P16-P19-P26	141.75	0.869632	
7	P1-P2-P3-P4-P5-P6-P9-P17-P19-P26	138.5	0.849693	
8	P1-P2-P3-P4-P5-P6-P9-P12-P14-P15-P21	163	1	✓
9	P1-P2-P3-P4-P5-P6-P9-P12-P14-P25	141.75	0.869632	
10	P1-P2-P3-P4-P5-P6-P9-P13-P14-P15-P21	148.25	0.909509	
11	P1-P2-P3-P4-P5-P6-P9-P13-P14-P25	138.5	0.849693	
12	P8-P11-P20-P28-P29-P30	101.5	0.622699	

5 Conclusion

A fuzzy CPM real-life construction problem is introduced, aggregating many international companies working together. We apply a LP approach which employs the α-cuts concept to two-linear programs which constitute so far a new approach in the field [2].The main idea behind using this new approach because of it superiority over the related previous work in conserving the fuzziness of the activities which had been the main drawback of forward and backward recursions proposals. A code was developed; also some modifications were made for this case study concerning the formulation.

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