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# On Some Subclasses of Analytic Functions With Negative Coefficients using a Convolution Approach

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**Abstract**

*In this work, we introduce and investigate new subclasses*

$$E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma) \quad and \quad \tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$$

*of analytic functions by making use of the familiar convolution structure of analytic functions whose Taylor-Maclaurin coefficients from the second onwards are all negative. In particular, we derive the coefficient inequalities and some other interesting properties for functions belonging to these subclasses. Our results generalize some earlier known results.*

**Keywords:** *Analytic functions, coefficient estimates, distortion inequalities, Littlewood subordination theorem, integral means.*

# 1 Introduction and Definitions

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$.
Also, let $S$ be the subclass of $A$, consisting of analytic and univalent functions in $U$.
We denote by

$$S^*(\alpha) \;\; and \;\; K(\alpha) \qquad (0 \le \alpha < 1)$$

the class of starlike functions of order $\alpha$ in $U$ and the class of convex functions of order $\alpha$ $in$ $U$, respectively.
Further denote by $E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta)$, the class of functions $f(z) \in A$ which are analytic in $U$ and satisfy therein the condition

$$\frac{D^m(f*\Phi)(z)}{D^n(f*\Psi)(z)} - \beta\left|\frac{D^m(f*\Phi)(z)}{D^n(f*\Psi)(z)} - 1\right| \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha \quad (z \in U), \tag{2}$$

where $*$ and $\prec$ denotes convolution(or Hadamard product) and subordination respectively, $(f * \Psi)(z) \neq 0$, $A$ and $B$ are arbitrarily fixed numbers such that $-1 \le B < A \le 1$ and $-1 \le B < 0$ and $0 \le \alpha < 1, \beta \le 0$ and $m \ge n$ $(m, n \in N_0)$.
The class $E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta)$ was introduced and studied by Srivastava et al.[3] and the class is due to the class $E_{m,n}(\Phi, \Psi; A, B, \alpha)$ which was earlier introduced by Eker and Seker [14].
Note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad (n \in N_0 = N \cup \{0\}), \tag{3}$$

where $D^n$ is the usual Salagean operator (see [1]) and

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k, \qquad \Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \tag{4}$$

which are also analytic in $U$ with $\lambda_k \ge 0$, $\mu_k \ge 0$ $and$ $\lambda_k \ge \mu_k$.

**Definition 1.1.** *(Hadamard Product(or Convolution)) If $f(z)$ and $g(z)$ are analytic in $U$, where $g(z)$ is given by*

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{5}$$

*then, their Hadamard product(or convolution), $f * g$ is the function*

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \qquad (6)$$

*The function $f * g$ is also analytic in $U$.*

**Definition 1.2.** *(Subordination Principle) Let $f(z)$ and $g(z)$ be analytic in the unit disk $U$. Then $g(z)$ is said to be subordinate to $f(z)$ in $U$ and we write*

$$g(z) \prec f(z), \quad z \in U,$$

*if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$, $|w(z)| < 1$ such that*

$$g(z) = f(w(z)), \quad z \in U. \qquad (7)$$

*In particular, if the function $f(z)$ is univalent in $U$, then $g(z)$ is subordinate to $f(z)$ if*

$$g(0) = f(0), \quad g(u) \subset f(u).$$

*(See for details Duren[10])*

In view of (1), (4) and definition 1.1 we note that

$$(f * \Phi)(z) = z + \sum_{k=2}^{\infty} a_k \lambda_k z^k, \quad (f * \Psi)(z) = z + \sum_{k=2}^{\infty} a_k \mu_k z^k, \qquad (8)$$

such that by using Binomial expansion on (8) we have

$$(f*\Phi)^{\gamma}(z) = z^{\gamma} + \sum_{k=2}^{\infty} a_k(\gamma)\lambda_k(\gamma) z^{k+\gamma-1}, \quad (f*\Psi)(z) = z^{\gamma} + \sum_{k=2}^{\infty} a_k(\gamma)\mu_k(\gamma) z^{k+\gamma-1}, \qquad (9)$$

Now, let

$$h(z) = (f * \Phi)^{\gamma}(z) = z^{\gamma} + \sum_{k=2}^{\infty} a_k(\gamma)\lambda_k(\gamma) z^{k+\gamma-1}$$

and

$$q(z) = (f * \Psi)(z) = z^{\gamma} + \sum_{k=2}^{\infty} a_k(\gamma)\mu_k(\gamma) z^{k+\gamma-1}$$

$\gamma \geq 0$.

Then a function $f \in A$ is said to be in the class $E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ if and only if

$$\frac{D^m h(z)}{D^n q(z)} - \beta \left| \frac{D^m h(z)}{D^n q(z)} - 1 \right| \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha \quad (z \in U), \qquad (10)$$

where $\prec$ denotes subordination as earlier defined, $q(z) \neq 0$, $A$ *and* $B$ are arbitrarily fixed numbers such $-1 \leq B < A \leq 1$ *and* $-1 \leq B < 0$ and $0 \leq \alpha < 1, \beta \leq 0$ *and* $m \geq n$ $(m, n \in N_0)$.
In other words, $f \in E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ if and only if there exists an analytic function $\omega(z)$ satisfying

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \qquad (z \in U)$$

such that

$$\frac{D^m h(z)}{D^n q(z)} - \beta\left|\frac{D^m h(z)}{D^n q(z)} - 1\right| \prec (1-\alpha)\frac{1 + A\omega(z)}{1 + B\omega(z)} + \alpha \quad (z \in U), \tag{11}$$

The condition (11) is equivalent to the following inequality:

$$\left|\frac{\frac{D^m h(z)}{D^n q(z)} - \beta|\frac{D^m h(z)}{D^n q(z)} - 1| - 1}{(A-B)(1-\alpha) - B\left(\frac{D^m h(z)}{D^n q(z)} - \beta|\frac{D^m h(z)}{D^n q(z)} - 1| - 1\right)}\right| < 1 \qquad (z \in U). \tag{12}$$

Let $\tau$ denote the subclass of $A$ whose Taylor-Maclaurin expansion about $z = 0$ can be expressed in the following form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{13}$$

We shall denote $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ the subclass of functions in $E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ that has their non-zero Taylor-Maclaurin coefficients, from the second term onwards, all negative.
Thus we can write

$$\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma) = E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma) \cap \tau. \tag{14}$$

It is easy to check that various known or new subclasses of $\tau$ referred to above can be represented in terms of $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ for suitable choices of the function $\Phi$ *and* $\Psi$ (and the parameters $m, n, A, B, \beta$ *and* $\gamma$).
For example, we have the following relationship with known classes of functions:

$$\tilde{E}_{0,0}(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 1, -1, \alpha, 0, 1) = S^*(\alpha)$$

and

$$\tilde{E}_{0,0}(\frac{z + z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 1, -1, \alpha, 0, 1) = k(\alpha)$$

which were studied by Silverman[7];

$$\tilde{E}_{0,0}(\frac{z}{(1-z)^2}, z; 1, -1, \alpha, 0, 1) = P^*(\alpha),$$

which was studied by Bhoosnurmath and swamy[13] and Gupta and Jain[15],

$$\tilde{E}_{0,0}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; 1, -1, \alpha, 0, 1) = R(\alpha),$$

which was studiedby Silverman and Silivia[6].

# 2 Main Results

We begin by proving the following results.

## 2.1 A Set of Coefficient Inequalities

**Theorem 2.1.** *If $f(z) \in A$ satisfies the following inequality:*

$$\begin{aligned}&\sum_{k=2}^{\infty}\{(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)]\\&+(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\}|a_k(\gamma)|\\&\leq(A-B)(1-\alpha)\gamma^m-(1-B)(1+\beta)(\gamma^m-\gamma^n)\end{aligned} \tag{15}$$

*where*

$$(\lambda_k(\gamma) \geq \mu_k(\gamma) \geq 0; 0 \leq \alpha < 1; \beta \geq 0; m \geq n \quad (m, n \in N_0))$$

*and $a_k(\gamma), \lambda_k(\gamma), \mu_k(\gamma)$ are the coefficients $a_k, \lambda_k, \mu_k$ depending on $\gamma$, then $f(z) \in E_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$.*

**Proof.** Let the condition (15) hold true. Then it suffice to show that

$$\left|\frac{\frac{D^m h(z)}{D^n q(z)}-\beta|\frac{D^m h(z)}{D^n q(z)}-1|-1}{(A-B)(1-\alpha)-B\left(\frac{D^m h(z)}{D^n q(z)}-\beta|\frac{D^m h(z)}{D^n q(z)}-1|-1\right)}\right|<1 \quad (z \in U). \tag{16}$$

Thus we have that

$$
\begin{aligned}
&\left|D^m h(z) - D^n q(z) - \beta e^{i\theta}|D^m h(z) - D^n q(z)|\right| \\
&- \left|(A-B)(1-\alpha)D^n q(z) - B[D^m h(z) - D^n q(z) - \beta e^{i\theta}|D^m h(z) - D^n q(z)|]\right| \\
&= |(\gamma^m - \gamma^n)z^\gamma + \sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]a_k(\gamma) z^{\gamma+K-1} \\
&- \beta e^{i\theta}|(\gamma^m - \gamma^n)z^\gamma + \sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]a_k(\gamma) z^{\gamma+K-1}|| \\
&- |(A-B)(1-\alpha)\gamma^m z^\gamma + (A-B)(1-\alpha)\sum_{k=2}^{\infty}(\gamma+k-1)^n a_k(\gamma)\mu_k(\gamma) z^{\gamma+k-1} \\
&- B\{(\gamma^m - \gamma^n)z^\gamma + \sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]a_k(\gamma) z^{\gamma+k-1} \\
&- \beta e^{i\theta}|(\gamma^m - \gamma^n z^\gamma) + \sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k - (\gamma+k-1)^n \mu_k(\gamma)]a_k(\gamma) z^{\gamma+k-1}|]| \\
&\leq (\gamma^m - \gamma^n) + \sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]|a_\gamma|\frac{|z|^r|z|^k}{|z|} \\
&+ \beta(\gamma^m - \gamma^n) + \beta\sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]|a_k(\gamma)|\frac{|z|^r|z|^k}{|z|} \\
&- (A-B)(1-\alpha)\gamma^m|z|^\gamma \\
&+ (A-B)(1-\alpha)\sum_{k=2}^{\infty}(\gamma+k-1)^n|a_k(\gamma)|\mu_k(\gamma)\frac{|z|^r|z|^k}{|z|} \\
&+ |B|(\gamma^m - \gamma^n)|z|^r + |B|\sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]|a_k(\gamma)|\frac{|z|^r|z|^k}{|z|} \\
&+ |B|\beta(\gamma^m - \gamma^n) + |B|\beta\sum_{k=2}^{\infty}[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]|a_k(\gamma)|\frac{|z|^r|z|^k}{|z|} \\
&\leq (1-B)(1+\beta)(\gamma^m - \gamma^n) - (A-B)(1-\beta)\gamma^m
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{k=2}^{\infty}\{(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) \\
&- (\gamma+k-1)^n \mu_k(\gamma)] + (A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\}|a_k(\gamma)| \leq 0,
\end{aligned}
$$

which implies that

$$\sum_{k=2}^{\infty}\{(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\}|a_k(\gamma)| \\ \leq (A-B)(1-\beta)\gamma^m+(1-B)(1+\beta)(\gamma^m-\gamma^n)$$

This completes the proof of Theorem 2.1
In theorem 2.2 below, it is shown that the condition (15) is also necessary for functions $f(z)$ of the form (13) to be in the class $\tilde{E}_{m,n}(\Phi,\Psi;A,B,\alpha,\beta,\gamma)$.

**Theorem 2.2.** *Let $f(z)\in\tau$, then $f(z)\in\tilde{E}_{m,n}(\Phi,\Psi;A,B,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}\{(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\}|a_k(\gamma)| \\ \leq (A-B)(1-\alpha)\gamma^m-(1-B)(1+\beta)(\gamma^m-\gamma^n) \tag{17}$$

*where*

$$(\lambda_k(\gamma)\geq\mu_k(\gamma)\geq 0; 0\leq\alpha<1;\beta\geq 0; m\geq n \quad (m,n\in N_0))$$

*and $a_k(\gamma),\lambda_k(\gamma),\mu_k(\gamma)$ are the coefficients $a_k,\lambda_k,\mu_k$ depending on $\gamma$.*

**Proof.** Since $\tilde{E}_{m,n}(\Phi,\Psi;A,B,\alpha,\beta,\gamma)\subset E_{m,n}(\Phi,\Psi;A,B,\alpha,\beta,\gamma)$,
we only need to prove the *only if* part of Theorem 2.2. For functions $f(z)\in\tau$, we can write

$$\left|\frac{(D^mh(z)/D^nq(z))-(\beta|D^mh(z)/D^nq(z)-1|)-1}{(A-B)(1-\alpha)-B[(D^mh(z)/D^nq(z))-\beta|(D^mh(z)/D^nq(z)-1|)-1]}\right| \\ =\left|\frac{D^mh(z)-D^nq(z)-\beta e^{i\theta}|D^mh(z)-D^nq(z)|}{(A-B)(1-\alpha)D^nq(z)-B[D^mh(z)-D^nq(z)-\beta e^{i\theta}|D^mh(z)-D^nq(z)|}\right|=$$

$$\left|\frac{\begin{array}{c}(\gamma^m-\gamma^n)z^\gamma\\ +\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}\\ +\beta e^{i\theta}|(\gamma^m-\gamma^n)z^\gamma\\ +\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}|\end{array}}{\begin{array}{c}(A-B)(1-\alpha)z^\gamma\gamma^n+(A-B)(1-\alpha)\sum_{k=2}^{\infty}(\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}\\ +B\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}\\ +B\beta e^{i\theta}|\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}|\end{array}}\right|$$

$< 1.$

Since $Re(z) \leq |z| < 1 \quad (z \in U,)$ we thus find that

$$Re\left\{\frac{\begin{array}{c}(\gamma^m - \gamma^n)z^\gamma + \sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}\\ +\beta e^{i\theta}|(\gamma^m-\gamma^n)z^\gamma\\ +\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}|\end{array}}{\begin{array}{c}(A-B)(1-\alpha)z^\gamma\gamma^n + (A-B)(1-\alpha)\sum_{k=2}^{\infty}(\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}\\ +B\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}\\ +B\beta e^{i\theta}|\sum_{k=2}^{\infty}[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)]a_k(\gamma)z^{\gamma+k-1}|\end{array}}\right\}$$

$< 1.$

If we now choose $z$ to be real and let $z \to 1^-$, we have

$$\begin{aligned}&\sum_{k=2}^{\infty}\{(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)]\\ &+ (A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\}a_k(\gamma)\\ &\leq (A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m-\gamma^n)\end{aligned} \tag{18}$$

which is equivalent to equation (17).

**Remark 2.3.** *Taking different choices for the function $\Phi(z)$ and $\Psi(z)$ as stated in section 1 when $\gamma = 1$, Theorem 2.2 leads us to the necessary and sufficient conditions for a function $f$ to be in each of the following classes:*

$$S^*(\alpha), \quad K(\alpha), \quad P^*(\alpha) \quad and \quad R[\alpha].$$

**Remark 2.4.** *If we set $\gamma = 1$ in Theorem 2.1 and 2.2 above, we obtain the result given in [9].*

**Remark 2.5.** *If we set $m = n = \beta = 0$ and $\gamma = 1$ in Theorem 2.1 and 2.2 above, we obtain the results given in [9].*
*Moreover, for $m \geq n \quad (m, n \in N_0)$, $\beta = 0$ and $\gamma = 1$, our results will coincide with those presented in [14].*

## 2.2 Distortion Theories Involving Fractional Calculus

In this section, we shall prove several distortion theorems for functions belonging to the general class $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$. Each of these theorems would involve certain operators of fractional calculus (i.e, fractional integrals and fractional derivatives), which are defined as follows(see for details,[12, 11, 2, 5, 4]):

**Definition 2.6.** *The fractional integral of order $\delta$ is defined, for a function $f$, by*

$$D_z^{-\delta}f(z) = \frac{1}{\Gamma(\delta)}\int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}}d\xi \qquad (\delta > 0,) \tag{19}$$

*where $f$ is an analytic function in a simply-connected region of the complex $z$-plane containing the origin, and the multiplicity of $(z-\xi)^{\xi-1}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi>0$.*

**Definition 2.7.** *The fractional derivative of order $\delta$ is defined, for a function $f$, by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \qquad (0 \le \delta < 1,) \tag{20}$$

*where $f$ is constrained, and the multiplicity of $(z-\xi)^{\xi}$ is removed as in definition* 2.6

**Definition 2.8.** *Under the hypothesis of the definition* 2.7, *the fractional derivative of order $n+\delta$ is defined, for a function $f$, by*

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n}\{D_z^{\delta} f(z)\} \qquad (0 \le \delta < 1, \qquad n \in N_0).$$

By the virtue of Definitions $3, 4$, and $5$, we have

$$D_z^{-\delta} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} z^{k+\delta} \qquad (k \in N; \delta > 0)$$

and

$$D_z^{\delta} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} z^{k-\delta} \qquad (k \in N; 0 \le \delta < 1).$$

**Theorem 2.9.** *Let $f(z)$ defined by equation* (13) *be in the class $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$. Then,*

$$\left| D_z^{-\delta} f(z) \right| \ge \frac{|z|^{1+\delta}}{\Gamma(2+\delta)}$$

$$\left[1 - \left( \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\substack{(2+\delta\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_k(\gamma) - (\gamma+1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_k(\gamma)\})|z|}} \right) \right]$$

$$(\delta > 0; \quad z \in U)$$

*and*

$$\left| D_z^{-\delta} f(z) \right| \le \frac{|z|^{1+\delta}}{\Gamma(2+\delta)}$$

$$\left[1 + \left( \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\substack{(2+\delta\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_k(\gamma) - (\gamma+1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_k(\gamma)\})|z|}} \right) \right]$$

$$(\delta > 0; \quad z \in U).$$

*Each of these results is sharp.*

**Proof.** Let

$$F(z) = \Gamma(2+\delta) z^{-\delta} D_z^{\delta} f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+1+\delta)} a_k(\gamma) z^k$$
$$= z - \sum_{k=2}^{\infty} \Omega(k) a_k(\gamma) z^k$$

where $\Omega(k) = \dfrac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+1+\delta)} \qquad (k \in N\ \{1\}).$
since $\Omega(k)$ is a decreasing function of $k$, we can write

$$0 < \Omega(k) \leq \Omega(2) = \frac{2}{2+\delta}. \tag{21}$$

Furthermore, in view of Theorem 2.2, we have

$$\Big\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)]$$
$$+ (A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\Big\} \sum_{k=2}^{\infty} a_k(\gamma)$$
$$\leq \sum_{k=2}^{\infty} \{(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)]$$
$$+ (A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\} a_k(\gamma)$$
$$\leq (A-B)(1-\alpha)(\gamma)^m - (1-B)(1+\beta)(\gamma^m - \gamma^n),$$

which evidently yields

$$\sum_{k=2}^{\infty} a_k(\gamma) \leq \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - \gamma + 1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\end{array}} \tag{22}$$

Therefore, by using equation (21)and (22) we can see that

$$|F(z)| \geq |z| - |\Omega(2)| \cdot |z|^2 \sum_{k=2}^{\infty} a_k(\gamma)$$

$$\geq |z| - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}\end{array}} |z|^2$$

and

$$|F(z)| \le |z| + |\Omega(2)| \cdot |z|^2 \sum_{k=2}^{\infty} a_k(\gamma)$$

$$\le |z| + \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \atop +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}} |z|^2$$

which prove Theorem 2.9

Finally, since the equalities are attained for the function $f(z)$ defined by

$$D_z^{-\delta} f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)}$$

$$\left(1 - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \atop +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}} z\right)$$

or equivalently, by

$$f(z) = z - \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \atop +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)} z^2,$$

Our proof of Theorem 2.9 is completed.

**Corollary 2.10.** *Under the hypothesis of Theorem* 2.9, $D_z^{-\delta} f(z)$ *is included in a disk with its center at the origin and radius* $r_1$ *given by*

$$r_1 = \frac{1}{\Gamma(2+\delta)} \left(1 - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \atop +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}}\right).$$

**Theorem 2.11.** *Let the function* $f(z)$ *defined by equation* (13) *be in the class* $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$. *Then,*

$$\left| D_z^{\delta} f(z) \right| \ge \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}$$

$$\left(1 - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{(2+\delta\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_k(\gamma) - (\gamma+1)^n \mu_k(\gamma)] \atop +(A-B)(1-\alpha)(\gamma+1)^n \mu_k(\gamma)\})} |z|\right)$$

$$(0 \le \delta < 0; \quad z \in U)$$

*and*

$$\left| D_z^\delta f(z) \right| \le \frac{|z|^{1-\delta}}{\Gamma(2-\delta)}$$

$$\left( 1 + \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(2+\delta\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_k(\gamma) - (\gamma+1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_k(\gamma)\})\end{array}} |z| \right)$$

$$(0 \le \delta < 0; \quad z \in U).$$

*Each of these results is sharp.*

**Proof.** Let

$$G(z) = \Gamma(2-\delta) z^\delta D_z^\delta f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} k a_k(\gamma) z^k = z - \sum_{k=2}^{\infty} \Lambda(k) k a_k(\gamma) z^k$$

where $\Lambda(k) = \dfrac{\Gamma(k)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} \qquad (k = \{2, 3, \cdots\})$.
Since $\Lambda(k)$ is a decreasing function of $k$, we can write

$$0 < \Lambda(k) \le \Lambda(2) = \frac{1}{2-\delta}. \tag{23}$$

Furthermore, in view of Theorem 2.2, we have

$$\begin{aligned}
&\Big\{ (1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\
&+ (A-B)(1-\alpha)(\gamma+1)^{n-1} \mu_2(\gamma) \Big\} \sum_{k=2}^{\infty} k a_k(\gamma) \\
&\le \sum_{k=2}^{\infty} \{(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\
&+ (A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\} a_k(\gamma) \\
&\le (A-B)(1-\alpha)(\gamma)^m - (1-B)(1+\beta)(\gamma^m - \gamma^n),
\end{aligned}$$

which evidently yields

$$\sum_{k=2}^{\infty} k a_k(\gamma) \le \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^{n-1} \mu_2(\gamma)\end{array}}. \tag{24}$$

Therefore, by using equation (23) and (24) we can see that

$$|G(z)| \geq |z| + |\Lambda(2)| \cdot |z|^2 \sum_{k=2}^{\infty} k a_k(\gamma)$$

$$\geq |z| - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}\end{array}} |z|^2$$

and

$$|G(z)| \leq |z| + |\Lambda(2)| \cdot |z|^2 \sum_{k=2}^{\infty} k a_k(\gamma)$$

$$\leq |z| + \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}\end{array}} |z|^2$$

which together yields the inequalities asserted by Theorem 2.11. Equalities are attained for the function $f(z)$ defined by

$$D_z^{\delta} f(z) = \frac{z^{1-\delta}}{\Gamma(2-\delta)}$$

$$\left(1 - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(2-\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^n \mu_2(\gamma)\}\end{array}} z\right)$$

**Corollary 2.12.** *Under the hypothesis of Theorem* 2.11, $D_z^{\delta} f(z)$ *is included in a disk with its center at the origin and radius* $r_2$ *given by*

$$r_2 = \frac{1}{\Gamma(2-\delta)} \left(1 - \frac{2(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(2+\delta)\{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+1)^{n-1} \mu_2(\gamma)\}\end{array}}\right).$$

## 2.3 Extreme Points for the Functions in Class $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$

**Theorem 2.13.** *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_2(\gamma)\end{array}} z^k \tag{25}$$

$$(k = \{2, 3, \cdots\}).$$

*Then,* $f(z) \in \tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z),$$

*where* $\eta_k \geq 0$ *and* $\sum_{k=1}^{\infty} \eta_k = 1$.

**Proof.** Suppose that

$$f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z)$$

$$= z - \sum_{k=2}^{\infty} \eta_k \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\substack{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_2(\gamma)}} z^k.$$

Then, from Theorem 2.2, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \{(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\
&+ (A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\} \\
&\left( \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\substack{(1-B)(1+\beta)[(\gamma+1)^m \lambda_2(\gamma) - (\gamma+1)^n \mu_2(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_2(\gamma)}} \right) \eta_k \\
&= [(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)] \sum_{k=2}^{\infty} \eta_k \\
&= [(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)](1-\eta_1) \\
&\leq (A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n).
\end{aligned}$$

Thus in view of Theorem 2.2, we find that

$$f(z) \in \tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma).$$

Conversely, let us suppose that

$$\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma).$$

Then, since

$$a_k(\gamma) \leq \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\substack{(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)}}$$

$$(k = \{2, 3, \cdots\}),$$

we may set

$$\eta_k = \frac{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\end{array}}{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)} a_k(\gamma)$$

$$(k = \{2, 3, \cdots\}),$$

and $\eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$.
Thus, clearly, we have

$$f(z) = z - \sum_{k=2}^{\infty} \eta_k f_k(z).$$

This completes the proof of Theorem 2.13

**Corollary 2.14.** *The extreme points of the functions* $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$ *are given by*

$$f_1(z) = z$$

*and*

$$f_k(z) = z - \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma) - (\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\end{array}} z^k$$

$$(k = \{2, 3, \cdots\}).$$

## 2.4 Integral Means Inequalities for the Function Class $\tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$

In the year 1925, Littlewood [8] proved the following:

**Lemma 2.15.** *If the function $f(z)$ and $g(z)$ are analytic in $U$ with $g(z) \prec f(z)$, $\quad (z \in U)$, then for $p > 0$ and $z = re^{i\theta}, (0 < r < 1)$, we have*

$$\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |g(z)|^p d\theta.$$

We now make use of Lemma 2.15 to prove Theorem 2.16 below:

**Theorem 2.16.** *Let $f(z) \in \tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta, \gamma)$. Suppose also that $f_k(z)$ is defined by equation (25). If $\exists$ an analytic function $\omega(z)$ given by*

$$[\omega(z)]^{k-1} = \frac{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\end{array}}{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)} a_k(\gamma)$$

$$\times \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1},$$

*then, for* $z = r^{i\theta}$ *and* $0<r<1$,

$$\int_0^{2\pi} |f(r^{i\theta})|^p d\theta \le \int_0^{2\pi} |f_k(r^{i\theta})|^p d\theta \quad (p > 0).$$

**Proof.** We need to show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1} \right|^p d\theta \le$$

$$\int_0^{2\pi} \left| 1 - \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\end{array}} z^{k-1} \right|^p d\theta.$$

By applying Lemma 2.15 above, it would suffice to show that

$$1 - \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1}$$

$$< 1 - \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\end{array}} z^{k-1}$$

$(z \in U)$

By setting

$$1 - \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1}$$

$$= 1 - \frac{(A-B)(1-\alpha)\gamma^m - (1-B)(1+\beta)(\gamma^m - \gamma^n)}{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m \lambda_k(\gamma) - (\gamma+k-1)^n \mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n \mu_k(\gamma)\end{array}} [\omega(z)]^{k-1},$$

we find that

$$\begin{aligned}
&[\omega(z)]^{k-1} \\
&= \frac{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\end{array}}{(A-B)(1-\alpha)\gamma^m-(1-B)(1+\beta)(\gamma^m-\gamma^n)} \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1},
\end{aligned}$$

which readily yields $\omega(0) = 0$.
Furthermore by using equation (17), we obtain

$$\begin{aligned}
&|[\omega(z)]|^{k-1} \\
&\leq \left| \frac{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\end{array}}{(A-B)(1-\alpha)\gamma^m-(1-B)(1+\beta)(\gamma^m-\gamma^n)} \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1} \right| \\
&\leq \frac{\begin{array}{c}(1-B)(1+\beta)[(\gamma+k-1)^m\lambda_k(\gamma)-(\gamma+k-1)^n\mu_k(\gamma)] \\ +(A-B)(1-\alpha)(\gamma+k-1)^n\mu_k(\gamma)\end{array}}{(A-B)(1-\alpha)\gamma^m-(1-B)(1+\beta)(\gamma^m-\gamma^n)} \sum_{k=2}^{\infty} a_k(\gamma) z^{k-1} \\
&\leq |z|^{k-1} < 1.
\end{aligned}$$

This completes the proof of Theorem 2.16

# References

[1] G.S. Salagean, Subclasses of univalent functions in complex analysis, *Fifth Romanian-Finish Seminar, Part 1 (Bucharest, 1981)*, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, Heidelberg and New York, 1013(1983), 362-372.

[2] H.M. Srivastava and M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions withnegative coefficients, I and II, *J. Math. Anal. Appl.*, 192(1995), 673-688, and (1992), 1-13.

[3] H.M. Srivastava, S.S. Eker and B. Seker, Certain convolution approach for subclasses of analytic functions with negative coefficients, *Integral Transforms and Special Functions*, 20(9) (2009), 687-699.

[4] H.M. Srivastava and S. Owa, *Univalent Functions, Fractional Calculus, and their Applications*, Halsled Press, Ellis Horwood Limited, Chichester and John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1989).

[5] H.M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, Linera operators and certain subclasses of analytic functions, *Nagoya Math. J.*, 106(1987), 1-28.

[6] H. Silverman and E.M. Silvia, Prestarlike functions with negative coefficients, *Internat. J. Math. Math. Sc.*, 2(1979), 427-439.

[7] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, 51(1975), 109-116.

[8] J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Mtah. Soc.*, 2(1925), 481-519.

[9] O.P. Juneja, T.R. Reddy and M.L. Mogra, A convolution approach for analytic functions with negative coefficients, *Soochow J. Math.*, 11(1985), 69-81.

[10] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York Inc, (1983).

[11] S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, 39(1987), 1057-1077.

[12] S. Owa, On the distortion theorems: I, *Kyungpook Math. J.*, 18(1978), 53-59.

[13] S.S. Bhoosnurmath and S.R. Swamy, Analytic functions with negative coefficients, *Indian J. Pure Appl. Math.*, 12(1981), 738-742.

[14] S.S. Eker and B. Seker, A convolution approach for analytic functions with negative coefficients, *Proceedings of the International Symposium on Geometric Function Theory and Applications, GFTA 2007 Proceedings*, Istanbul, Turkey, August 20-24, (2007).

[15] V.P. Gupta and P.K. Jain, Certain classes of univalent functions with negative coefficients- II, *Bull. Austral. Math. Soc.*, 15(1976), 467-473.