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Spatial and Descriptive Isometries in Proximity Spaces

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Abstract

The focus of this article is on two forms of isometries and homomorphisms in proximal relator spaces. A practical outcome of this study is the detection of descriptively near, disjoint sets in proximity spaces with application in the study of proximal algebraic structures.

Keywords: *Spatial, Descriptive, Isometry, Homomorphism, Proximal Algebraic Structure, Proximity space, Relator.*

1 Introduction

An *algebraic structure* is a set equipped with one or more binary operations. A *proximal algebraic structure* is an algebraic structure in a proximity space. This article introduces two forms of isometries and homomorphisms in proximity spaces. Proximity spaces were explored by Efremovič during the first part of 1930s and later formally introduced [2] and elaborated by Smirnov [13, 14]. The introduction of descriptive forms of isometry and homomorphism stems from recent work on near sets [11, 6, 12, 9], near groups [4] and proximal relator spaces [10].

2 Preliminaries

X denotes a metric topological space endowed with 1 or more proximity relations. 2^X denotes the collection of all subsets of a nonempty set X. Subsets $A, B \in 2^X$ are near (denoted by $A\ \delta\ B$), provided $A \cap B \neq \emptyset$. That is, nonempty sets are near, provided the sets have at least one point in common. The *closure* of a subset $A \in 2^X$ (denoted by $\mathrm{cl}(A)$) is the usual Kuratowski closure of a set defined by

$$\mathrm{cl}(A) = \{x \in X : D(x, A) = 0\}, \text{ where}$$
$$D(x, A) = inf\,\{d(x, a) : a \in A\}.$$

i.e., $\mathrm{cl}(A)$ is the set of all points x in X that are close to A ($D(x, A)$ is the Hausdorff distance [3, §22, p. 128] between x and the set A and $d(x, a) = |x - a|$ (standard distance)). A *discrete* proximity relation is defined by

$$\delta = \left\{(A, B) \in 2^X \times 2^X : \mathrm{cl}(A)\ \cap\ \mathrm{cl}(B) \neq \emptyset\right\}.$$

The following proximity space axioms are given by Ju.M. Smirnov [13] based on what V. Efremovič introduced during the first half of the 1930s [2]. Let $A, B \in 2^X$.

EF.1 If the set A is close to B, then B is close to A.
EF.2 $A \cup B$ is close to C, if and only if, at least one of the sets A or B is close to C.
EF.3 Two points are close, if and only if, they are the same point.
EF.4 All sets are far from the empty set $\emptyset$.
EF.5 For any two sets A and B which are far from each other, there exists C and D, $C \cup D = X$, such that A is far from C and B is far from D (*Efremovič axiom*).

In a proximity space X, the closure of A in X coincides with the intersection of all closed sets that contain A.

Theorem 1. [13] *The closure of any set A in the proximity space X is the set of points $x \in X$ that are close to A.*

2.1 Descriptive EF-Proximity Space

Descriptively near sets were introduced as a means of solving classification and pattern recognition problems arising from disjoint sets that resemble each other [8, 7]. Recently, the connections between near sets in EF-spaces and near sets in descriptive EF-proximity spaces have been explored in [12, 9].

Let X be a metric topological space containing non-abstract points and let $\Phi = \{\phi_1, \dots, \phi_n\}$ a set of probe functions that represent features of each $x \in X$. In a discrete space, a non-abstract point has a location and features that can be measured [5, §3]. A *probe function* $\phi : X \to \mathbb{R}$ represents a feature of a sample point in X. Let $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ denote a feature vector for x, which provides a description of each $x \in X$. To obtain a descriptive proximity relation (denoted by δ_Φ), one first chooses a set of probe functions. Let $A, B \in 2^X$ and $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. That is,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\},$$
$$\mathcal{Q}(B) = \{\Phi(b) : b \in B\}.$$

The expression $A\ \delta_\Phi\ B$ reads A *is descriptively near* B. Similarly, $A\ \underline{\delta}_\Phi\ B$ reads A *is descriptively far from* B. The descriptive proximity of A and B is defined by

$$A\ \delta_\Phi\ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset.$$

The *descriptive intersection* $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(A) \text{ and } \Phi(x) \in \mathcal{Q}(B)\}.$$

That is, $x \in A \cup B$ is in $A \underset{\Phi}{\cap} B$, provided $\Phi(x) = \Phi(a) = \Phi(b)$ for some $a \in A, b \in B$. Observe that A and B can be disjoint and yet $A \underset{\Phi}{\cap} B$ can be nonempty. The descriptive proximity relation δ_Φ is defined by

$$\delta_\Phi = \left\{(A, B) \in 2^X \times 2^X : \mathrm{cl}(A) \underset{\Phi}{\cap} \mathrm{cl}(B) \neq \emptyset\right\}.$$

Whenever sets A and B have no points with matching descriptions, the sets are *descriptively far* from each other (denoted by $A\ \underline{\delta}_\Phi\ B$), where

$$\underline{\delta}_\Phi = 2^X \times 2^X \setminus \delta_\Phi.$$

The binary relation δ_Φ is a *descriptive EF-proximity*, provided the following axioms are satisfied for $A, B, C, D \in 2^X$.

dEF.1 If the set A is descriptively close to B, then B is descriptively close to A.
dEF.2 $A \cup B$ is descriptively close to C, if and only if, at least one of the sets A or B is descriptively close to C.
dEF.3 Two points $x, y \in X$ are descriptively close, if and only if, the description of x matches the description of y.

dEF.4 All nonempty sets are descriptively far from the empty set $\emptyset$.

dEF.5 For any two sets A and B which are descriptively far from each other, there exists C and D, $C \cup D = X$, such that A is descriptively far from C and B is descriptively far from D (*Descriptive Efremovič axiom*).

A *relator* is a nonvoid family of relations $\mathcal{R}$ on a nonempty set X. The pair $(X, \mathcal{R})$ (also denoted $X(\mathcal{R})$) is called a relator space [16]. Relator spaces are natural generalisations of ordered sets and uniform spaces [15]. With the introduction of a family of proximity relations $\mathcal{R}_\delta$ on X, we obtain a proximal relator space $(X, \mathcal{R}_\delta)$. For simplicity, we consider only two proximity relations, namely, the Efremovič proximity δ [2] and the descriptive proximity δ_Φ in defining the *proximal relator* $\mathcal{R}_{\delta_\Phi}$ on a metric topological space. The pair $(X, \mathcal{R}_{\delta_\Phi})$ is called a descriptive proximal relator space (briefly, *proximal relator space*) [10]. With the introduction of $(X, \mathcal{R}_{\delta_\Phi})$, the traditional closure of a subset provides a basis the descriptive closure of a subset.

In a proximal relator space X, the *descriptive closure of a set* A (denoted by $\mathrm{cl}_\Phi(A)$) is defined by

$$\mathrm{cl}_\Phi(A) = \{x \in X : \Phi(x) \in \mathcal{Q}(\mathrm{cl}(A))\}.$$

Theorem 2. [11] *The descriptive closure of any set A in the proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ is the set of points $x \in X$ that are descriptively close to A.*

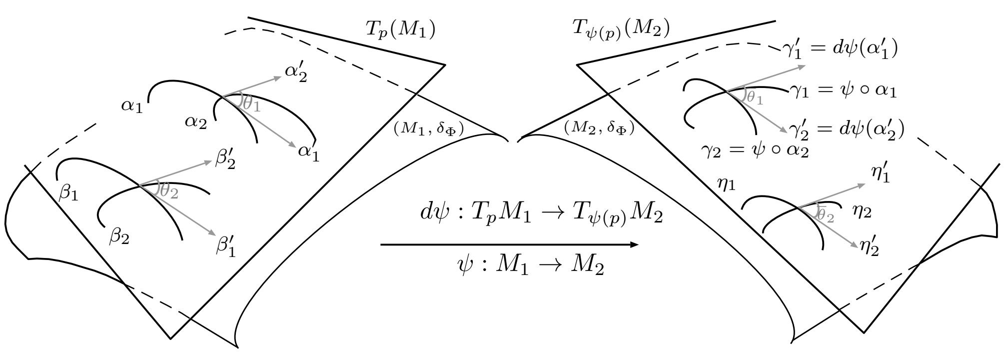

Figure 1: Descriptive isometry

3 Spatial and Descriptive Isometries and Homomorphisms in Proximity Spaces

Let $(X, \mathcal{R}_{\delta_\Phi})$, $(Y, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces and $A \subseteq X$, $B \subseteq Y$. A mapping $g_\Phi : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(A)$ is a *descriptive isometry*, provided $g_\Phi(\Phi(a)) =$

$g_\Phi(\Phi(a'))$ when $\Phi(a) = \Phi(a')$, $a, a' \in A$. For a pseudometric d defined on X and Y, $g_\Phi : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(A)$ is a *descriptive isometry*, provided

$$d(g_\Phi(\Phi(a)), g_\Phi(\Phi(a'))) = 0 \text{ when } d(\Phi(a), \Phi(a')) = 0,$$

for $a, a' \in A$ [11]. Since a descriptive isometry is defined relative to matching descriptions, such an isometry can be defined without reference to a pseudometric.

Example 1. *In Fig. 1, let M_1, M_2 be manifolds endowed with a descriptive proximity relation δ_Φ, where Φ contains a probe function that represents the angles between two curves on manifolds and let $T_p(M_1), T_{\psi(p)}(M_2)$ be tangent spaces. Let $\psi : M_1 \longrightarrow M_2$ be a conformal map that for all $p \in M_1$ and all $v_1, v_2 \in T_p(M_1)$, we have $\langle d\psi_p(v_1), d\psi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$. The geometric meaning of this map is that the angles (but not necessarily the lengths) are preserved by conformal maps.*

In Fig. 1 , let we consider the pairs of curves (α_1, α_2),$(\beta_1, \beta_2) \in M_1$ and $(\gamma_1 = \psi \circ \alpha_1, \gamma_2 = \psi \circ \alpha_2)$,$(\eta_1 = \psi \circ \beta_1, \eta_2 = \psi \circ \beta_2) \in M_2$. Then

$$\cos\theta_1 = \frac{\langle \alpha_1', \alpha_2' \rangle}{|\alpha_1'|\,|\alpha_2'|}, \quad \cos\theta_2 = \frac{\langle \beta_1', \beta_2' \rangle}{|\beta_1'|\,|\beta_2'|}\ ,\ 0 < \theta_1, \theta_2 < \pi.$$

Observe that

$$\cos\bar{\theta}_1 = \frac{\langle \gamma_1', \gamma_2' \rangle}{|\gamma_1'|\,|\gamma_2'|} = \frac{\langle d\psi(\alpha_1'), d\psi(\alpha_2')\rangle}{|d\psi(\alpha_1')|\,|d\psi(\alpha_2')|} = \frac{\lambda^2\langle \alpha_1', \alpha_2' \rangle}{\lambda^2|\alpha_1'|\,|\alpha_2'|} = \cos\theta_1$$

and

$$\cos\bar{\theta}_2 = \frac{\langle \eta_1', \eta_2' \rangle}{|\eta_1'|\,|\eta_2'|} = \frac{\langle d\psi(\beta_1'), d\psi(\beta_2')\rangle}{|d\psi(\beta_1')|\,|d\psi(\beta_2')|} = \frac{\lambda^2\langle \beta_1', \beta_2' \rangle}{\lambda^2|\beta_1'|\,|\beta_2'|} = \cos\theta_2.$$

Hence, conformal map ψ is provided such that

$$\Phi((\psi(\alpha_1), \psi(\alpha_2))) = \Phi((\psi(\beta_1), \psi(\beta_2))),$$

when $\Phi((\alpha_1, \alpha_2)) = \Phi((\beta_1, \beta_2))$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in M_1$. Hence ψ is a descriptive isometry, but ψ is not an ordinary isometry. ■

Lemma 1. *Kuratowski closure of a set A is a subset of the descriptive closure of A in a pseudometric proximal relator space.*

Proof. Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space. Assume $A \subset X$ and that Φ is a set of probe functions the represent features of points in X. Let $a \in A$. Consequently, $\Phi(a) \in \mathcal{Q}(A)$, since $a \in \mathrm{cl}(A)$. Assume $x \in X$ and $x \notin \mathrm{cl}(A)$ such that $\Phi(x) = \Phi(a)$ for some $a \in A$. Hence, $\mathrm{cl}(A) \subseteq \mathrm{cl}_\Phi(A)$. □

Theorem 3. *Let $(X, \mathcal{R}_{\delta_\Phi}, d_X)$, $(Y, \mathcal{R}_{\delta_\Phi}, d_Y)$ be pseudometric proximal relator spaces, $A \subseteq X$ and $f : X \longrightarrow Y$ be an isometry. Then $cl(f(A)) \subseteq cl_\Phi(f(A))$.*

Proof. Let $y \in cl\,(f\,(A))$, $y = f\,(x)$, $x \in A$. Then $d_X\,(x, A) = 0$. Since f is an isometry $d_X\,(x, A) = d_Y\,(f\,(x)\,, f\,(A)) = 0$, then $d\,(\Phi\,(f\,(x))\,, \Phi\,(f\,(A))) = 0$. Consequently $\Phi\,(f\,(x)) \in \mathcal{Q}\,(f\,(A))$. Hence $y = f\,(x) \in cl_\Phi\,(f\,(A))$ and $cl\,(f\,(A)) \subseteq cl_\Phi\,(f\,(A))$. □

The following result for a descriptive isometry on a proximal relator space X into a proximal relator space Y, is obtained without using a metric.

Theorem 4. *Let $(X, \mathcal{R}_{\delta_\Phi})$, $(Y, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces, $A \subseteq X$, $B \subseteq Y$, $g_\Phi : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(B)$ be a descriptively isometry. Then $cl\,(g_\Phi\,(\mathcal{Q}(A))) \subseteq cl_\Phi\,(g_\Phi\,(\mathcal{Q}(A)))$.*

Proof. Let $y \in cl\,(g_\Phi\,(\mathcal{Q}(A)))$, then $y = g_\Phi\,(x)$, $x \in A$, $\Phi\,(y) \in \mathcal{Q}(B)$, $\Phi\,(x) \in \mathcal{Q}(A)$. Then $\Phi\,(g_\Phi\,(\Phi(x))) \in \mathcal{Q}(g_\Phi\,(\mathcal{Q}(A)))$, $\Phi\,(x) \in \mathcal{Q}(A)$.
Consequently, $y = g_\Phi\,(\Phi(x)) \in cl_\Phi\,(g_\Phi\,(\mathcal{Q}(A)))$. Hence, $cl\,(g_\Phi\,(\mathcal{Q}(A))) \subseteq cl_\Phi\,(g_\Phi\,(\mathcal{Q}(A)))$.
□

Theorem 5. *Let $(X, \delta)\,, (Y, \delta)$ be EF-proximity spaces, $A_1, A_2 \subseteq X$ and $f : X \longrightarrow Y$ be an isometry, then*

$$\delta\,(A_1, A_2) = 0 \Rightarrow \delta\,(f\,(A_1)\,, f\,(A_2)) = 0.$$

Theorem 6. *Let $(X, \mathcal{R}_{\delta_\Phi})$, $(Y, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces, $A_1, A_2 \subseteq X$ and $f : X \longrightarrow Y$ be an isometry, then*

$$\delta_\Phi\,(A_1, A_2) = 0 \Rightarrow \delta_\Phi\,(f\,(A_1)\,, f\,(A_2)) = 0.$$

Theorem 7. *Let (X, δ_Φ), (Y, δ_Φ) be proximal relator spaces, $A_1, A_2 \subseteq X$, $B \subseteq Y$, $g_\Phi : \mathcal{Q}(X) \longrightarrow \mathcal{Q}(Y)$ be a descriptive isometry. Then $\delta_\Phi\,(A_1, A_2) = 0 \Rightarrow \delta_\Phi\,(g_\Phi\,(\mathcal{Q}(A_1))\,, g_\Phi\,(\mathcal{Q}(A_2))) = 0$.*

Proof. Let $\delta_\Phi\,(A_1, A_2) = 0$. Then $\mathcal{Q}(A_1) \cap \mathcal{Q}(A_2) \neq \emptyset$, i.e., $\Phi\,(a_1) = \Phi\,(a_2)$, $a_1 \in A_1$, $a_2 \in A_2$. Since g_Φ is a descriptive isometry $\Phi\,(g_\Phi\,(\Phi(a_1)) = \Phi\,(g_\Phi\,(\Phi(a_2)))$. Hence,

$$\mathcal{Q}(g_\Phi\,(\mathcal{Q}(A_1))) \cap \mathcal{Q}(g_\Phi\,(\mathcal{Q}(A_2))) \neq \emptyset,$$

i.e., $\delta_\Phi\,(g_\Phi\,(\mathcal{Q}(A_1))\,, g_\Phi\,(\mathcal{Q}(A_2))) = 0$. □

4 Descriptive Homomorphism

A *binary operation* on a set S is a mapping of $S \times S$ into S, where $S \times S$ is the set of all ordered pairs of elements of S. A *groupoid* (denoted $S(\circ)$) is a non-empty set S equipped with a binary operation $\circ$ on S. Let $A\,(\circ)$ and $B\,(\bullet)$ be groupoids. A mapping h from A *into* B is a *homomorphism*, if $h(x \circ y) = h(x) \bullet y(y)$ for all $x, y \in A$ [1, §1.3, p. 9]. A one-to-one homomorphism h from A into B is called an *isomorphism* on A to B.

Let $(X, \mathcal{R}_{\delta_\Phi})$, $(Y, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces and consider the groupoids $\mathcal{Q}(A)(\circ_1)$, $\mathcal{Q}(B)(\circ_2)$, where $A \subset X, B \subset Y$.

A mapping

$$h_\Phi : \mathcal{Q}(B) \longrightarrow \mathcal{Q}(A)$$

is called a *descriptive homomorphism*, provided $h_\Phi(\Phi_B(b_1) \circ_2 \Phi_B(b_2)) = h_\Phi(\Phi_B(b_1)) \circ_1 h_\Phi(\Phi_B(b_2))$ for all $\Phi_B(b_1), \Phi_B(b_2) \in \mathcal{Q}(B)$.

A one-to-one descriptive homomorphism h_Φ is called a *descriptive monomorphism*, a descriptive homomorphism h_Φ of $\mathcal{Q}(B)$ onto $\mathcal{Q}(A)$ is called a *descriptive epimorphism* and one-to-one descriptive homomorphism h_Φ of $\mathcal{Q}(B)$ onto $\mathcal{Q}(A)$ is called a *descriptive isomorphism*.

Example 2. *Let $M_1 = M_2 = \mathbb{R}^2$ be manifolds, $(M_1, \mathcal{R}_{\delta_\Phi})$, $(M_2, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces, $A \subset M_1, B \subset M_2$ be sets of all 2-dimensional shapes and $\Phi = \{\varphi : \varphi$ is a area of shapes$\}$. Let us consider the rotation*

$$h : B \longrightarrow A, (x, y) \longmapsto (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

Observe that if area of b_1 matches area of b_2, then area of $h(b_1)$ matches area of $h(b_2)$. That is, rotation h is provided $\Phi(h(b_1)) = \Phi(h(b_2))$ when $\Phi(b_1) = \Phi(b_2)$, $b_1, b_2 \in B$. Hence h is a descriptive isometry. ■

Example 3. *Again, let $M_1 = M_2 = \mathbb{R}^2$ be manifolds, $(M_1, \mathcal{R}_{\delta_\Phi})$, $(M_2, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces, $A \subset M_1, B \subset M_2$ be sets of all 2-dimensional shapes and $\Phi = \{\varphi : \varphi$ is a area of shapes$\}$. Let $\mathcal{Q}(A)(\circ_1)$ and $\mathcal{Q}(B)(\circ_2)$ be groupoids, where*

$$\begin{aligned}
\circ_1 :& \mathcal{Q}(A) \times \mathcal{Q}(A) \longrightarrow \mathcal{Q}(A) : \\
& (\Phi(a_1), \Phi(a_2)) \longmapsto \min\{\Phi(a_1), \Phi(a_2)\}, \\
\circ_2 :& \mathcal{Q}(B) \times \mathcal{Q}(B) \longrightarrow \mathcal{Q}(B) : \\
& (\Phi(b_1), \Phi(b_2)) \longmapsto \min\{\Phi(b_1), \Phi(b_2)\}.
\end{aligned}$$

Let $h_\Phi : \mathcal{Q}(B) \longrightarrow \mathcal{Q}(A)$ be a map such that $h_\Phi(\Phi_B(b)) = \Phi_A(h(b))$, for all $b \in B$ and $\Phi_B(b) \in \mathcal{Q}(B)$.

Observe that $h_\Phi(\Phi_B(b_1) \circ_2 \Phi_B(b_2)) = h_\Phi(\Phi_B(b_1)) \circ_1 h_\Phi(\Phi_B(b_2))$, for all $b_1, b_2 \in B$. Hence, h_Φ is a descriptive homomorphism. ■

5 Descriptive Epimorphism

Theorem 8. *A descriptive isomorphism is a descriptive epimorphism.*

Proof. Immediate from the definition of the definition of a descriptive epimorphism. □

Theorem 9. *The descriptive homomorphism in Example 3 is a descriptive epimorphism.*

Theorem 10. *Let X, Y be proximal relator spaces and let $A \subset X, B \subset Y$ be proximal groupoids $A(\circ), B(\bullet)$. If $h : \mathcal{Q}(A) \longrightarrow \mathcal{Q}(B)$ is a descriptive homomorphism such that every $\Phi(b) \in \mathcal{Q}(B)$ has a corresponding $\Phi(x) \circ \Phi(y) \in \mathcal{Q}(A)$ such that*

$$h(\Phi(x) \circ \Phi(y)) = h(\Phi(x)) \bullet h(\Phi(y)) = \Phi(b) \in \mathcal{Q}(B),$$

then h is a descriptive epimorphism.

Proof. Immediate from the definition of a descriptive epimorphism from an algebraic structure onto another algebraic structure. □

6 Object Description

Let $A(\bullet), \mathcal{Q}(A)(\circ)$ be ordinary groupoid and descriptive groupoid, respectively. Let $a \in A$. An object description Φ_A is defined by a mapping

$$A \longrightarrow \mathcal{Q}(A) : a \longmapsto \Phi(a).$$

The object description Φ_A of A into $\mathcal{Q}(A)$ is an *object description homomorphism*, provided

$$\Phi_A(x \bullet y) = \Phi_A(x) \circ \Phi_A(y) \text{ for all } x, y \in A.$$

$$\begin{tikzcd}
B_\Phi \arrow[r, "h"] \arrow[d, "\Phi_B"'] & A_\Phi \arrow[d, "\Phi_A"] \\
\mathcal{Q}(B) \arrow[r, "h_\Phi"'] & \mathcal{Q}(A)
\end{tikzcd}$$

Figure 2: Object Description Homomorphism Diagram

Let $h : B \longrightarrow A$ be a homomorphism and let $h_\Phi : \mathcal{Q}(B) \longrightarrow \mathcal{Q}(A)$ be a descriptive homomorphism such that

$$h_\Phi(\Phi_B(b)) = \Phi_A(h(b)).$$

See the arrow diagram in Fig. 2 for the object description homomorphism and descriptive homomorphism mappings gathered together. For all $b \in B$,

$$(h_\Phi \circ \Phi_B)(b) = h_\Phi(\Phi_B(b)) = \Phi_A(h(b)) = (\Phi_A \circ h)(b).$$

This leads to the following result.

Lemma 2. *$h_\Phi \circ \Phi_B = \Phi_A \circ h$.*

Theorem 11. *Let $(X, \mathcal{R}_{\delta_\Phi})$, $(Y, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces, $B(\cdot_2)$, $A(\cdot_1)$, $\mathcal{Q}(B)(\circ_2)$ and $\mathcal{Q}(A)(\circ_1)$ be groupoids and h be a homomorphism from $B(\cdot_2)$ to $A(\cdot_1)$. If there are a descriptive monomorphism h_Φ of $\mathcal{Q}(B)$ to $\mathcal{Q}(A)$ and an object description homomorphism Φ_A of A to $\mathcal{Q}(A)$, then there is an object description homomorphism Φ_B of B to $\mathcal{Q}(B)$.*

Proof. For all $b_1, b_2 \in B$ and $\Phi(b_1), \Phi(b_2) \in \mathcal{Q}(B)$,

$$\begin{aligned} h_\Phi(\Phi_B(b_1 \cdot_2 b_2)) &= \Phi_A(h(b_1 \cdot_2 b_2)) = \Phi_A(h(b_1) \cdot_1 h(b_2)) \\ &= \Phi_A(h(b_1)) \circ_1 \Phi_A(h(b_2)) \\ &= h_\Phi(\Phi_B(b_1)) \circ_1 h_\Phi(\Phi_B(b_2)) \\ &= h_\Phi(\Phi_B(b_1) \circ_2 \Phi_B(b_2)) \end{aligned}$$

Consequently $\Phi_B(b_1 \cdot_2 b_2) = \Phi_B(b_1) \circ_2 \Phi_B(b_2)$. Hence Φ_B is an object description homomorphism from B into $\mathcal{Q}(B)$. □

Theorem 12. *Let $(X, \mathcal{R}_{\delta_\Phi})$, $(Y, \mathcal{R}_{\delta_\Phi})$ be proximal relator spaces, $A \subset X$, $B \subset Y$ and let h_Φ be a descriptive homomorphism. Then h is a descriptive isometry from $\mathcal{Q}(B)$ to $\mathcal{Q}(A)$.*

Proof. Let $\Phi_B(b_1) = \Phi_B(b_2)$, $b_1, b_2 \in B$. Then $\Phi_A(h(\Phi(b_1))) = h_\Phi(\Phi_B(b_1)) = h_\Phi(\Phi_B(\Phi(b_2))) = \Phi_A(h(\Phi(b_2)))$. Hence h is a descriptive isometry. □

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References

[1] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, American Mathematical Society, Providence, R.I., 1964, xv+224pp.

[2] V.A. Efremovič, *The geometry of proximity I (in Russian)*, Mat. Sb. (N.S.) **31(73)** (1952), no. 1, 189–200.

[3] F. Hausdorff, *Grundzüge der mengenlehre*, Veit and Company, Leipzig, 1914, viii + 476 pp.

[4] E. İnan and M.A. Öztürk, *Near groups on nearness approximation spaces*, Hacettepe J. of Math. and Statistics **41** (2012), no. 4, 545–558.

[5] M.M. Kovár, *A new causal topology and why the universe is co-compact*, arXive:1112.0817[math-ph] (2011), 1–15.

[6] S.A. Naimpally and J.F. Peters, *Topology with applications. Topological spaces via near and far*, World Scientific, Sinapore, 2013.

[7] J.F. Peters, *Near sets. General theory about nearness of objects*, Applied Mathematical Sciences **1** (2007), no. 53, 2609–2029.

[8] ______, *Near sets. Special theory about nearness of objects*, Fundam. Inf. **75** (2007), no. 1-4, 407–433.

[9] ______, *Near sets: An introduction*, Math. in Comp. Sci. **7** (2013), no. 1, 3–9, DOI 10.1007/s11786-013-0149-6.

[10] ______, *Proximal relator spaces*, FILOMAT (2014), 1–5, *in press.*

[11] ______, *Topology of Digital Images. Visual Pattern Discovery in Proximity Spaces*, Intelligent Systems Reference Library, vol. 63, Springer, 2014, ISBN 978-3-642-53844-5, pp. 1-342.

[12] J.F. Peters and S.A. Naimpally, *Applications of near sets*, Notices of the Amer. Math. Soc. **59** (2012), no. 4, 536–542, DOI: `http://dx.doi.org/10.1090/noti817`.

[13] Ju. M. Smirnov, *On proximity spaces*, Math. Sb. (N.S.) **31** (1952), no. 73, 543–574, English translation: Amer. Math. Soc. Trans. Ser. 2, 38, 1964, 5-35.

[14] ______, *On proximity spaces in the sense of V.A. Efremovič*, Math. Sb. (N.S.) **84** (1952), 895–898, English translation: Amer. Math. Soc. Trans. Ser. 2, 38, 1964, 1-4.

[15] Á Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.

[16] ______, *An extension of Kelley's closed relation theorem to relator spaces*, FILOMAT **14** (2000), 49–71.