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# **He's Variational Iteration Method for Solving Differential Equations of the Fifth Order**

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## **Abstract**

*In this work, a variational iteration method, which is a well-known method for solving functional equations, has been employed to solve differential equations of the fifth order. Some special cases of differential equations of the fifth order are solved as examples to illustrate the capability and reliability of the method. The results reveal that the method is very effective.*

**Keywords:** *Variational iteration method; differential equations of the fifth order*

**2000 MSC No:** 47G20

## **1 Introduction**

With the rapid development of linear and nonlinear science, many different methods were proposed to solve differential equations [1,2], such as the homotopy perturbation method (HPM) [3–6] and the variational iteration method (VIM) [7–9]. In this paper, we aim to apply the variational iteration method proposed by Ji-Huan He [10–12] to differential equations of the fifth order.

Consider the following differential equations of the fifth order

$$\begin{aligned} y^5(x) + f(x)y(x) &= g(x), \quad x \in [a, b], \\ y(a) = a_0, \quad y'(a) &= a_1, \quad y''(a) = a_2, \quad y'''(a) = a_3, \quad y^{(4)}(a) = a_4. \end{aligned} \quad (1)$$

Where  $a_1, a_2, a_3, a_4$ , are given real constants while the functions  $f(x)$  and  $g(x)$  are continuous on  $[a, b]$ .

To illustrate the method, consider the following general functional equation

$$Lu(t) + N(t) = g(t), \quad (2)$$

Where  $L$  is a linear operator,  $N$  is a non-linear operator and  $g(t)$  is a known analytical function. According to the variational iteration method, we can construct the following correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi) \{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\} d\xi, \quad (3)$$

Where  $\lambda$  is a general Lagrange multiplier which can be identified optimally via variational theory,  $u_0$  is an initial approximation with possible unknowns, and  $\tilde{u}_n$  is considered as restricted variation, i.e.,  $\delta\tilde{u}_n = 0$  [13].

## 2 numerical examples

In this section, we present examples of differential equations of the fifth order and results will be compared with the exact solutions.

**Example1.** Consider the following differential equations of the fifth order:

$$\begin{aligned} y^{(5)}(x) + xy(x) &= 5(x-1)\sin x + 5(x-x^2-5)\cos x, \quad 0 \leq x \leq 1, \\ y(0) = 5, \quad y'(0) &= -5, \quad y''(0) = -5, \quad y'''(0) = 15, \quad y^{(4)}(0) = 5. \end{aligned} \quad (4)$$

The analytical solution of the above problem is given by,

$$y(x) = 5(1-x)\cos x. \quad (6)$$

In the view of the variational iteration method, we construct a correction functional in the following form:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi) \left\{ y_n^{(5)}(\xi) + \xi \tilde{y}_n(\xi) - 5(\xi-1) \sin \xi - 5(\xi - \xi^2 - 5) \cos \xi \right\} d\xi, \quad (7)$$

To find the optimal  $\lambda(s)$ , calculation variation with respect to  $y_n$ , we have the following stationary conditions:

$$\begin{aligned} \delta y_n : \lambda^{(5)}(\xi) &= 0, \\ \delta y_n^{(4)} : \lambda(\xi) \Big|_{\xi=x} &= 0, \\ \delta y_n^{(3)} : \lambda'(\xi) \Big|_{\xi=x} &= 0, \\ &\vdots \\ \delta y_n : 1 - \lambda^{(4)}(\xi) \Big|_{\xi=x} &= 0. \end{aligned} \quad (8)$$

The Lagrange multiplier, therefore can identified as  $\lambda = \frac{-(x-\xi)^{(4)}}{4!}$ .

Substituting the identified multiplier into Eq. (7), we have the following iteration formula:

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(x-\xi)^{(4)}}{4!} \left\{ y_n^{(5)}(\xi) + \xi \tilde{y}_n(\xi) - 5(\xi-1) \sin \xi - 5(\xi - \xi^2 - 5) \cos \xi \right\} d\xi, \quad (9)$$

Starting with the initial approximation  $y_0 = 5 - 5x - \frac{5}{2}x^2 + \frac{5}{2}x^3 + \frac{5}{24}x^4$  in Eq. (9) successive approximations  $y_i(x)$ 's will be achieved. The plot of exact solution Eq. (4), the 4th order of approximate solution obtained using the VIM and absolute error between the exact and numerical solutions of this example are shown in Fig. 1.

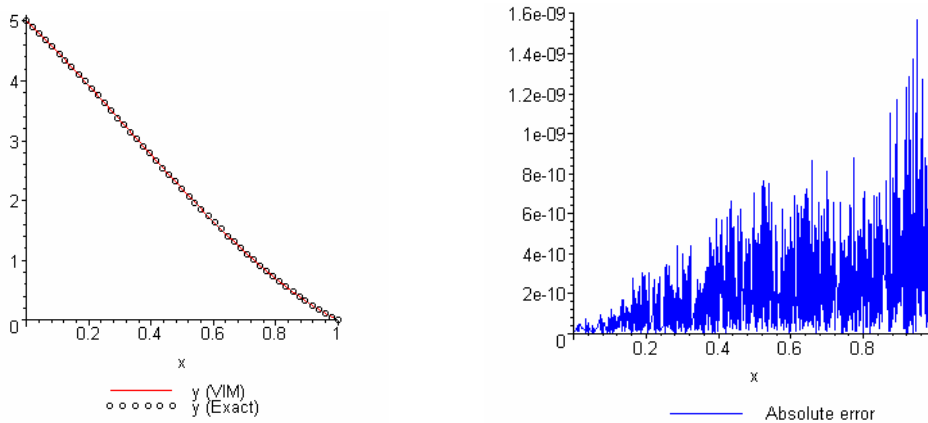


Fig. 1. The plots of approximate solution, exact solution and absolute error for Example 1.

**Example 2.** Now we consider the following general fifth order differential equations:

$$\begin{aligned}
 & y^{(5)}(x) + (x-2)y^{(4)}(x) + 2y^{(3)}(x) - (x^2 + 2x - 1)y''(x) + (2x^2 + 4x)y'(x) \\
 & - 2x^2y(x) = 4e^x \cos x - 2x^4 + 4x^3 + 6x^2 - 4x + 2, \quad 0 < x < 1, \\
 & y(0) = 0, y'(0) = 2, y''(0) = 6, y^{(3)}(0) = 4, y^{(4)}(0) = 0.
 \end{aligned}
 \tag{10}$$

The analytical solution of the above problem is given by,

$$y(x) = 2e^x \sin x + x^2.
 \tag{12}$$

For solving by VIM we obtain the recurrence relation

$$\begin{aligned}
 y_{n+1}(x) = y_n(x) - \int_0^x \frac{(x-\xi)^4}{4!} \{ & y^{(5)}(\xi) + (\xi-2)y^{(4)}(\xi) + 2y^{(3)}(\xi) \\
 & - (\xi^2 + 2\xi - 1)y''(\xi) + (2\xi^2 + 4\xi)y'(\xi) - 2\xi^2y(\xi) \\
 & - 4e^\xi \cos \xi + 2\xi^4 - 4\xi^3 - 6\xi^2 + 4\xi - 2 \} d\xi,
 \end{aligned}
 \tag{13}$$

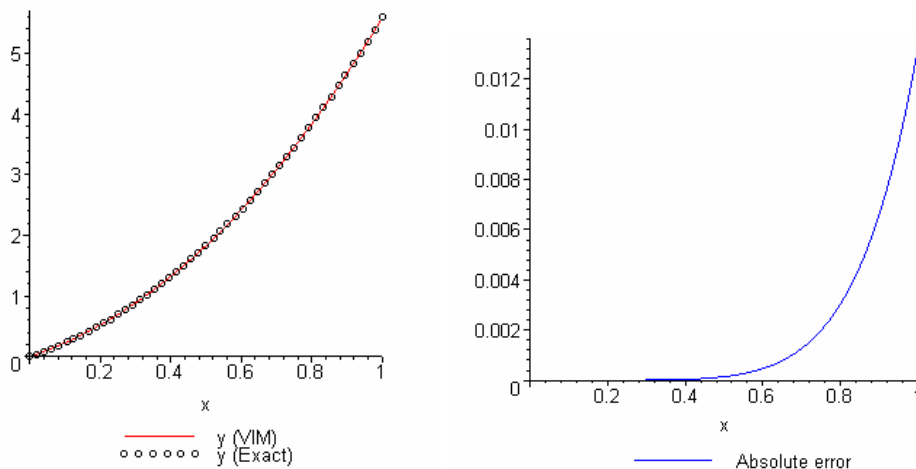


Fig. 2. The plots of approximate solution, exact solution and absolute error for Example 2.

Using the initial approximation  $y_0 = 2x + 3x^2 + \frac{2}{3}x^3$  in Eq. (13), approximations  $y_i(x)$ 's will be calculated, successively. The plot of exact solution Eq. (10), the 4th order of approximate solution obtained using the VIM and absolute error between the exact and numerical solutions of this example are shown in Fig. 2.

### 3 Conclusion

The variational iteration method is remarkably effective for solving differential equations of the fifth order. Four iterations are enough to obtain very highly accurate solution. The results show that the variational iteration method is a powerful mathematical tool for finding the numerical solutions of differential equations.

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