Gen. Math. Notes, Vol. 1, No. 2, December 2010, pp. 159-165
ISSN 2219-7184; Copyright ©ICSRS Publication, 2010
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A Note on the Generalized Shift Map

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(Received 18.11.2010, Accepted 25.11.2010)

Abstract

In this short note we have discussed generalized shift map in the symbol space Σ_2. Some stronger chaotic properties have been proved. Some special properties are discussed in a different section. In the last section we have also provided few examples.

Keywords: *Symbolic dynamics, Shift map, Generalized shift map, Strong sensitive dependence on initial conditions, Periodic points.*

2010 MSC No: 37B10, 37D45, 74H62.

1 Introduction

A dynamical system is sometimes defined as a pair (X, f) consisting of a set X together with a continuous map f from X into itself. Chaotic dynamical systems constitute a special class of dynamical systems. Symbolic dynamics is also an example of chaotic dynamical systems. In particular, there are several works on symbolic dynamics such as [1, 2, 3, 4, 5, 8, 9, 11, 13]. Of particular interest is the space Σ_2 which has been considered in a large number of works. Devaney [6] have given vivid description of the space Σ_2. By symbolic dynamical system we mean here the space of sequences $\Sigma_2 = \{\alpha : \alpha = (\alpha_0\alpha_1................), \alpha_i = 0$ or $1\}$ along with the shift map defined on it. It is known that Σ_2 is a compact metric space by the metric

$$d(s,t)=\sum_{i=0}^{\infty}\frac{|s_i-t_i|}{2^{i+1}},\ \text{where } s=(s_0s_1..........)\text{ and } t=(t_0t_1.........)\text{ are any two}$$

points of Σ_2.

The present authors extended the idea of the shift map into the generalized shift map in [1] and proved that it is chaotic both in the sense of Devaney [6] and Li-Yorke [10]. It is also chaotic in the sense of Du [5, 7].

In this short note we have proved some stronger chaotic properties of the generalized shift map. A comparative study of dynamics of the shift map and the generalized shift map is given. Lastly, we have given some examples.

We now give some definitions and lemmas which are required for this note.

Definition 1.1 (Shift map [6]) The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(\alpha_0\alpha_1..........) = (\alpha_1\alpha_2..........)$, where $\alpha = (\alpha_0\alpha_1................)$ is any point of Σ_2.

Definition 1.2 (Generalized shift map [1]) The generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ is defined by $\sigma_n(s) = (s_ns_{n+1}s_{n+2}.........)$, where, $s = (s_0s_1.....s_n.....)$ is any element of Σ_2. For $n = 1$, the generalized shift map reduces to the shift map and $n \geq 1$ is a finite positive integer.

Definition 1.3 (Sensitive dependence on initial conditions [6]) Let (S, ρ) be a compact metric space. A continuous map $f : S \to S$ is said to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in S$ and any neighborhood $N(x)$ of x there exist $y \in N(x)$ and $n \geq 0$ such that $\rho(f^n(x), f^n(y)) > \delta$.

In the following we give a stronger version of the above definition.

Definition 1.4 (Strong sensitive dependence on initial conditions [4]) Let (S, ρ) be a compact metric space. A continuous map $f : S \to S$ has strong sensitive dependence on initial conditions if for any $x \in S$ and any non empty open set U of S (not necessarily an open neighborhood of x), there exist $y \in U$ and $n \geq 0$ such that $\rho(f^n(x), f^n(y))$ is maximum in S.

It is obvious that if a map has strong sensitive dependence on initial conditions, it has also sensitive dependence on initial conditions. At the end of this paper we give an example to establish that the converse is not necessarily true.

Definition 1.5 (Totally transitive [12]) Let (X, ρ) be a compact metric space. A continuous map $f : X \to X$ is called totally transitive if f^n is topologically transitive for all $n \geq 1$.

Definition 1.6 (Transitive point [6]) In the symbol space Σ_2 there are points whose orbit comes arbitrarily close to any given sequence of Σ_2, that is, the point with dense orbit. Those points are called transitive points.

Definition 1.7 (Fixed point [6]) Let $f : I \to I$ be a continuous map. If a point $a \in I$ be such that $f(a) = a$, then a is called a fixed point of f.

Definition 1.8 (Periodic point [6]) Let $f : I \to I$ be a continuous map. The point $x \in I$ is called a periodic point of least period n if $f^n(x) = x$ and $f^m(x) \neq x$, for all $m < n$ where m and n are positive integers.

We also require the following lemma.

Lemma 2.1 [**6**] *Let $s, t \in \Sigma_2$ and $s_i = t_i$, for $i = 0, 1, ..., m$. Then $d(s,t) < \frac{1}{2^m}$ and conversely if $d(s,t) < \frac{1}{2^m}$ then $s_i = t_i$, for $i = 0, 1, ..., m$.*

2 The Main Results

Theorem 2.1 *Let $\sigma_n : \Sigma_2 \to \Sigma_2$ be the generalized shift map. Then for any point x of Σ_2 and any open neighborhood U of x, there exist two non empty subsets K and L of U, which satisfy the following conditions:*

i) both K and L are countable,

ii) $K \cap L = \phi$ and

iii) $d(\sigma_n^{n_j}(k), \sigma_n^{n_j}(x)) = 1$, for all $k \in K$ and $d(\sigma_n^{m_j}(l), \sigma_n^{m_j}(x)) = 0$, for all $l \in L$, where n_j's and m_j's are different for different points of K and L and depend on the minimum distance of x from the boundary U.

Proof. Let $x = (x_0x_1......)$ be any point of Σ_2 and U be any open neighborhood of x such that minimum distance of x from the boundary of U is $\varepsilon > 0$. We now choose $p \geq 5$ as an integer such that $\frac{1}{2^{np}} < \varepsilon$, for all $n \geq 1$. We now consider the two sets $K = \{k_i : k_i = (x_0x_1........x_{2ni-1}x'_{2ni}x'_{2ni+1}x'_{2ni+2}........), i \geq p\}$ and $L = \{l_j : l_j = (x_0x_1......x_{2nj-1}x'_{2nj}.......x'_{2nj+2n-1}x_{2nj+2n}x_{2nj+2n+1}........), j \geq p\}$.

Now by our construction we see that all k_i's of K agree with x at least up to x_{np}. Hence by Lemma 2.1 we get that $d(x, k_i) < \frac{1}{2^{np}}$, for all $k_i \in K$, that is, $d(x, k_i) < \varepsilon$, for all $k_i \in K$. So $k_i \in U$, for all $i \geq p$ and we get that K is a non empty subset of U. Similarly, we can show that L is a non empty subset of U. Again by our construction we see that both K and L are countable. This proves i).

We now observe the two sets K and L and see that, after the $(2ni+2n)$-th (or $(2nj + 2n)$-th) term all terms of k_i (or l_i) are mutually complementary terms for all i (or j). Hence $l_j \neq k_i$, for all i and j, that is, $K \cap L = \phi$. This proves ii).

$$\begin{aligned}
\text{Now, } d(\sigma_n^{2i}(k_i), \sigma_n^{2i}(x)) &= d((x'_{2ni}x'_{2ni+1}................), (x_{2ni}x_{2ni+1}................)) \\
&= \tfrac{1}{2} + \tfrac{1}{2^2} + ... \\
&= 1, \text{ for all } k_i \in K, \text{ and} \\
d(\sigma_n^{2j+2}(l_j), \sigma_n^{2j+2}(x)) &= d((x_{2nj+2n}x_{2nj+2n+1}........), (x_{2nj+2n}x_{2nj+2n+1}.......)) \\
&= \tfrac{0}{2} + \tfrac{0}{2^2} + ... \\
&= 0, \text{ for all } l_j \in L.
\end{aligned}$$

Also by our constructions of K and L we see that $i \geq p$ and $j \geq p$, for K and L respectively and p is depending on ε, where ε is the minimum distance of x from the boundary of U. So we conclude that n_j's and m_j's are depending on the minimum distance of x from the boundary of U. This proves iii).

Hence the theorem is proved.

Theorem 2.2 *The generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions.*

Proof. Let $x = (x_0x_1.......)$ be any point of Σ_2 and U be any non empty open set of Σ_2. Hence we can take an open ball V with radius $\varepsilon > 0$ and center at $\alpha = (\alpha_0\alpha_1.......)$, such that $V \subset U$. Let $p > 0$ be an integer, such that $\frac{1}{2^{np-1}} < \varepsilon$. We now consider the point $y = (\alpha_0\alpha_1............\alpha_{np-1}x'_{np}x'_{np+1}.............)$. Then the point y agrees with α up to α_{np-1} and after that all terms of y are the complementary terms of the point x starting with x'_{np}.

By the application of Lemma 2.1 above we get that $d(\alpha, y) < \frac{1}{2^{np-1}} < \varepsilon$. Hence $y \in V$, that is, $y \in U$ also.

Again we get
$$\begin{aligned} d(\sigma_n^p(x), \sigma_n^p(y)) &= d((x_{np}x_{np+1}...........), (x'_{np}x'_{np+1}...........)) \\ &= \frac{1}{2} + \frac{1}{2^2} + ... \\ &= 1, \text{ (this is maximum)} \end{aligned}$$
that is, $y \in U$ such that $d(\sigma_n^p(x)\sigma_n^p(y)) = 1$, where U is an arbitrary open set of Σ_2.

Hence the generalized shift map $\sigma : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions.

3 Some Special Properties

In this section we discuss some basic differences of dynamics of the generalized shift map and the shift map. We also present a comparative study between the shift map and the generalized shift map.

We know that transitive points play a big role in any Devaney's chaotic system. For the shift map σ, if a point of Σ_2 which contains every finite sequence of 0's and 1's, the point is a transitive point. But there is a different situation for the generalized shift map σ_n. A point of Σ_2 which contains every finite sequence of 0's and 1's with a power n is a transitive point with respect to the generalized shift map. The following is an example. If we consider a point a of Σ_2 as given below,

$$a = (\overbrace{(0)^n(1)^n}^{1\ block}\ \overbrace{(00)^n(01)^n(10)^n(11)^n}^{2\ block}\ \overbrace{(000)^n(001)^n........}^{3\ block}\ \overbrace{(0000)^n........}^{4\ block}\ ....),$$

then obviously $a \in \Sigma_2$ is a transitive point with respect to the generalized shift map. But

$$b = (\overbrace{01}^{1\ block}\ \overbrace{00\ 01\ 10\ 11}^{2\ block}\ \overbrace{000\ 001.......}^{3\ block}\ \overbrace{0000..........}^{4\ block}\ ....),$$ is a transitive point

for σ. Hence we can say that all transitive points of the generalized shift map σ_n are also transitive points of the shift map σ, but not conversely. Note that, at this time $(\alpha_0\alpha_1\alpha_2)^n = (\alpha_0\alpha_1\alpha_2)(\alpha_0\alpha_1\alpha_2)................(\alpha_0\alpha_1\alpha_2)$ $n-$times.

We now discuss the periodic points of the generalized shift map. Throughout this paper periods mean prime periods. If $\sigma : \Sigma_2 \to \Sigma_2$ is the shift map then we all know that any repeating sequence of 0's and 1's is always a periodic point of σ. For example, $\beta = (\beta_0\beta_1...........\beta_{n-1}\beta_0\beta_1............\beta_{n-1}...........)$ is a periodic point of period n of σ, for all $n \geq 1$. But $\sigma_n(\beta) = \beta$, that is, β is a fixed point of σ_n. On the other hand if we consider the points $O = (0000...........)$ and $I = (1111............)$ of Σ_2. These are the only fixed points of σ. The above two points are fixed points of σ_n also, but there exist other fixed points of σ_n in Σ_2. For example, $x = (x_0x_1............x_{n-1}x_0x_1............x_{n-1}...........)$ is a fixed points of σ_n, where x_i's are not all 0 or 1 at the same time.

Hence we conclude that periodic points of σ and σ_n are not same in general.

4 Conclusions

In this note we have proved some stronger chaotic properties of the generalized shift map. Since the generalized shift map is chaotic in the sense of Devaney, it is topologically transitive on Σ_2. Hence we can say that the shift map is totally transitive on Σ_2. So a question arises whether all topologically transitive maps are totally transitive? The answer is no. In the following we give an example to establish this fact.

Example 4.1. Let $f(x)$ be a continuous map from $[0,1]$ onto itself defined by

$$f(x) = \begin{cases} 4x + \frac{1}{3},\ 0 \leq x \leq \frac{1}{6} \\ -4x + \frac{5}{3},\ \frac{1}{6} \leq x \leq \frac{1}{3} \\ -\frac{1}{2}x + \frac{1}{2},\ \frac{1}{3} \leq x \leq 1 \end{cases}$$

It can be easily proved that the function f is topologically transitive on [0,1]. On the other hand it is not totally transitive, since the subintervals $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$ are invariant under f^2, so f^2 is not topologically transitive on [0,1]. Hence $f(x)$ is not totally transitive on [0.1].

As noted earlier that if a continuous map has strong sensitive dependence on initial conditions then it has sensitive dependence on initial conditions, but the converse is not always true. The following example establishes this fact.

Example 4.2. Let $f : [-1, 1] \to [-1, 1]$ be a map defined by

$$f(x) = \begin{cases} \frac{3}{2}x + \frac{3}{2}, -1 \leq x \leq -\frac{1}{3} \\ \\ -3x, \quad -\frac{1}{3} \leq x \leq \ 0 \\ \\ \quad -x, \quad 0 \leq x \leq 1 \end{cases}$$

The function defined above is obviously a continuous map. Also it can be easily proved that the function has sensitive dependence on initial conditions. Note that maximum distance between any two points of [-1,1] is equal to 2. We now consider the point $-\frac{3}{5}$ and the open interval $U = (0, 1)$. Then there exists no point $y \in U$ such that $d(f^n(x), f^n(y)) = 2$, for any $n \geq 0$. Hence $f(x)$ does not have not strong sensitive dependence on initial conditions.

ACKNOWLEDGEMENTS. Indranil Bhaumik acknowledges his father Mr. Sadhan Chandra Bhaumik for his help in preparing the manuscript.

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