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Approximate Analytical and Numerical Solutions to Fractional KPP-like Equations

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Abstract

In this work some fractional KPP-like equations with initial condition are solved by Adomian decomposition method for two cases time-fractional order and time-space fractional order respectively. The fractional derivative are described in Caputo sense. The obtained solutions are presented in the form of convergent series then the numerical solutions are plotted and discussed in detail.

Keywords: *Adomian Decomposition Method; Fractional calculus; Newell-Whitehead equation.*

1 Introduction

Fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Consequently, considerable attention has been given to solve this kind of equations. Unfortunately, most of them do not have exact solutions. Recently, several numerical methods have been introduced for this purpose, one of them are the famous Adomian decomposition method (ADM). This method introduced in 1980 by George Adomian [1], is one of the most frequently used for computing solutions of a large class of linear and nonlinear ordinary and partial differential equations. In this method the solution is considered as the sum of an infinite series, rapidly converging

to an accurate solution without any need for linearization or discretization.
In outline of this work is as follows. We begin by giving some preliminary Definitions on fractional calculus, then we recall briefly the basic principle of the Adomian decomposition method. In section 4 the ADM is applied to the fractional Newell-Whitehead equation to obtain the exact solutions of it some concluding remarks are also given.

2 Preliminaries and Notations

In this section, we describe some necessary tools of the fractional calculus theory (fractional integration and differentiation) required for the reminder of this work. We stress that there are many books and papers that develop fractional calculus and various related Definitions, we refer the interested reader to [5] and the references therein.
Throughout this paper, the derivatives are considered in the Caputo sens, taking the advantage that such Definition allows traditional initial and boundary conditions to be included in the formulation of the considered problem.

Definition 2.1 *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exits a real number $\lambda > \mu$ such that $f(x) = x^\lambda g(x)$, where $g(x) \in C[0, \infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$ for $m \in I\!N$.*

Definition 2.2 *The Riemann-Liouville fractional integral operator of order α of a real function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0 \quad \textit{and} \quad J^0 f(x) = f(x). \quad (1)$$

The operator J^α has some proprieties, for $\alpha, \beta \geq 0$, $\gamma, \mu \geq -1$:

- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
- $J^\alpha x^\xi = \frac{\Gamma(\xi+1)}{\Gamma(\alpha+\xi+1)} x^{\alpha+\xi}$.

Next we define the Caputo fractional derivatives D^α of a function $f(x)$ of any real number α such that $m - 1 < \alpha \leq m$, $m \in I\!N$, for $x > 0$ and $f \in C_{-1}^m$ in the terms of J^α as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt \quad (2)$$

and has the following proprieties for $m - 1 < \alpha \leq m$, $m \in I\!N$, $\mu \geq -1$ and $f \in C_\mu^m$

- $D^\alpha J^\alpha f(x) = f(x)$,
- $J^\alpha D^\alpha = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}$, for $x > 0$

In this paper, the Caputo derivative is taken as the following

Definition 2.3 *For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as*

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-s)^{m-\alpha-1}\frac{\partial^m u(x,s)}{\partial s^m}ds, & for \quad m-1 < \alpha \leq m \\ \dfrac{\partial^m u(x,t)}{\partial t^m} & for \quad \alpha = m \in \mathbb{N}, \end{cases} \tag{3}$$

and the Caputo space-fractional derivative operator of order $\beta > 0$ is defined as

$$D_x^\beta u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^x (x-\tau)^{m-\beta-1}\frac{\partial^m u(\tau,t)}{\partial \tau^m}d\tau, & for \quad m-1 < \beta \leq m \\ \dfrac{\partial^m u(x,t)}{\partial x^m} & for \quad \alpha = m \in \mathbb{N}, \end{cases} \tag{4}$$

3 The Adomian Decomposition Method

The principles of the Adomian decomposition method and its applicability for various kinds of differential equations can be found in [1][6] and related references.
We consider the general class of time-space fractional KPP equations of the form

$$D_t^\alpha u = D_x^\beta u + \phi(u) \qquad t > 0 \quad 0 < \alpha, \beta \leq 1 \tag{5}$$

with the initial condition

$$u(x,0) = f(x) \tag{6}$$

Where ϕ is a nonlinear function of u, differentiable for $0 \leq u \leq 1$, $\phi(0) = 0$, $\phi(u) > 0$ for $0 < u < 1$, $\phi(0) = 0$ and $\phi'(0) > \phi'(u)$ for $0 < u < 1$; of which the Fisher and the Newell-Whitehead equations are special cases. We recall that such equation has the same origin as the Zeldovich equation.
The Adomian method is based on applying the Riemann-Liouville integral operator J^α on both sides of the eq.(5) which yields

$$u(x,t) = u(x,0) + J^\alpha[\psi(u) + \phi(u)], \tag{7}$$

where $\psi(u) = D_x^{\beta} u$. The method assumes a series solution for $u(x,t)$ given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{8}$$

where th nonlinear term $\phi(u)$ is decomposed as the following

$$\phi(u) = \sum_{n=0}^{\infty} A_n(u_0, ..., u_n), \tag{9}$$

where A_n are called Adomian polynomials, which can be calculated for all forms of $\phi(u)$ throughout the general formula given by Adomian [1]:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\phi(\sum_{k=0}^{n}\lambda^k u_k)\right]_{\lambda=0}. \tag{10}$$

Substitution of (8) and (9) in (7) leads to

$$u(x,t) = u(x,0) + \sum_{n=0}^{\infty} J^{\alpha} A_n + \sum_{n=0}^{\infty} J^{\alpha}\psi(u_n). \tag{11}$$

From the above equation, the terms of $u_n(x,t)$ follows immediately

$$\begin{aligned}
u_0(x,t) &= u(x,0) = f(x),\\
u_1(x,t) &= J^{\alpha} A_0 + J^{\alpha}\psi(u_0),\\
u_2(x,t) &= J^{\alpha} A_1 + J^{\alpha}\psi(u_1)\\
&.\\
&.\\
&.\\
u_{n+1}(x,t) &= J^{\alpha} A_n + J^{\alpha}\psi(u_n)
\end{aligned} \tag{12}$$

Remark 3.1 The convergence of the series (8) has been investigated both theoretically and numerically in [2].

4 Applications and Discussions

In order to illustrate the efficiency of the method, the following two examples will be discussed. First we will consider a time-fractional Newell-Whithead equation, while the second deals with the same equation of both space and time fractional derivative.The obtained results are calculated using the symbolic calculus software Maple 13.

4.1 Exemple 1.

Consider the time-fractional Newell-Whitehead equation

$$D_t^\alpha u = u_{xx} + u - u^3, \tag{13}$$

with the initial condition

$$u(x,0) = u_0(x,t) = \frac{\sinh\frac{x}{\sqrt{2}}}{1+\cosh\frac{x}{\sqrt{2}}}. \tag{14}$$

The exact solution for eq.(13), for the case $\alpha = 1$, is given by

$$u(x,t) = \frac{e^{\frac{x}{\sqrt{2}}} - e^{-\frac{x}{\sqrt{2}}}}{e^{\frac{x}{\sqrt{2}}} + e^{-\frac{x}{\sqrt{2}}} + 2e^{-\frac{3}{2}t}}, \tag{15}$$

Equation (13) when $\alpha = 1$, called also *amplitude equation*, arises after carrying out a suitable normalization in the study of thermal convection of a fluid heated from below. Considering the perturbation from a stationary state, the equation describes the evolution of the amplitude of the vertical velocity if this varies slowly [3].
According to the previous section, we substitute the initial condition (14) into (12) and usig eq. (10) to get the Adomian polynomials, yields the following

$$\begin{aligned} u_0 &= \frac{\sinh\frac{x}{\sqrt{2}}}{1+\cosh\frac{x}{\sqrt{2}}}, \\ u_{n+1} &= J^\alpha(A_n + u_n + (u_n)_{xx}). \end{aligned} \tag{16}$$

Using the above to get the following first four terms of the decomposition series of $u(x,t)$

$$\begin{aligned} u_0(x,t) &= u(x,0) := f(x), \\ u_{n+1}(x,t) &= J^\alpha(A_n + u_n + u_{nxx}) = f_{n+1}(x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{17}$$

where the Adomian polynomials are defined

$$\begin{aligned} A_0 &= -u_0^3, \\ A_1 &= -u_0^2 u_1, \\ A_2 &= -3u_0u_1^2 - 3u_0^2u_2, \\ A_3 &= -u_1^3 - 6u_0u_1u_2 - 3u_0^2u_3, \end{aligned} \tag{18}$$

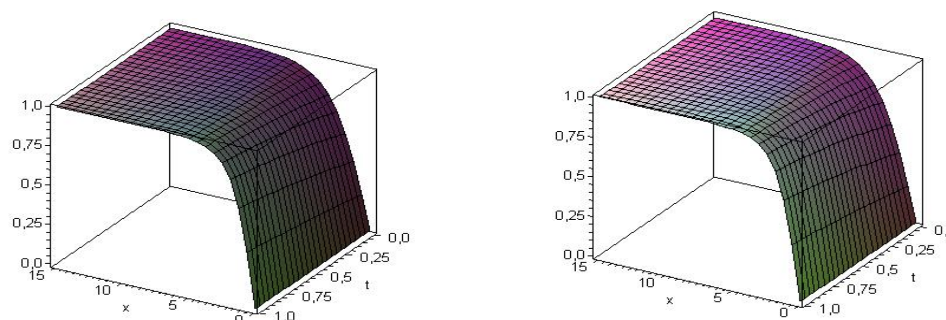
and the functions $(f_k)_{k=0..4}$ are given by

$$\begin{aligned}
f_0 &= u_0,\\
f_1 &= -f_0^3 + f_0 + f_0'',\\
f_2 &= -f_0^2 f_1 + f_1 + f_1'',\\
f_3 &= -3 f_0 f_1^2 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} - 3 f_0^2 f_2 + f_2 + f_2'',\\
f_4 &= -f_1^3 \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)^3} - 6 f_0 f_1 f_2 \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} - 3 f_0^2 f_3 + f_3 + f_3'''.
\end{aligned} \tag{19}$$

Then the solution in series form is given by

$$u(x,t) = f(x) + f_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + f_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3 \frac{t^{3\alpha}}{3\Gamma(\alpha+1)} + \ldots \tag{20}$$

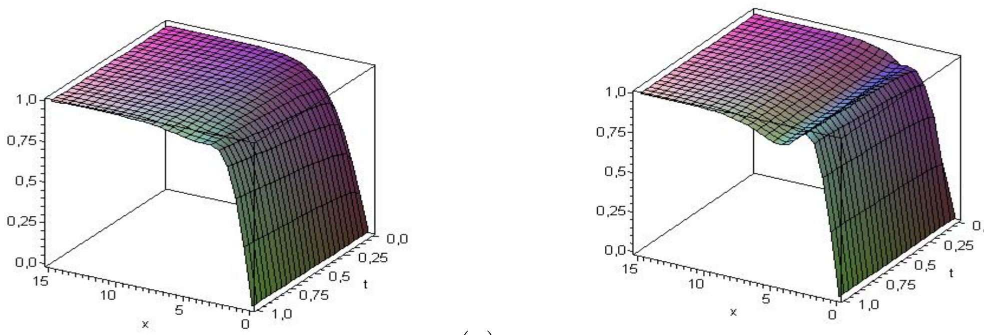
We stress that only four terms of the decomposition series were used for the approximate solution (20).



(a) (b)

Figure 1: The numerical solutions of $u(x,t)$: (a) by ADM (20) and (b) the exact solutions given in (15).

Figure.1 shows the evolution result for the Newell-Whitehead equation when $\alpha = 1$: (a) corresponds to the solution obtained by ADM (20) and (b) corresponds to the exact solution given in (15). It is easy to see that the two solutions look almost identical. Figure.2 (a) and (b) depict the evolution solution of the cases of $\alpha = 0.5$ and $\alpha = 0.05$ respectively. It is to be noted that as the time-fractional derivative parameter α decreases, the evolution solution $u(x,t)$ bifurcates for small values of $|x|$.



(a) (b)

Figure 2: The numerical solutions of $u(x,t)$: (a) $\alpha = 0.50$ (b) $\alpha = 0.05$.

Remark 4.1 The case of a Newell-Whitehead equation with space-fractional derivative

$$\frac{\partial u}{\partial t} = \frac{\partial^\beta u}{\partial x^\beta} + u - u^3 \qquad \text{with} \quad \beta \in (0,1]. \tag{21}$$

can be handled similarly.

4.2 Exemple 2.

In this example we consider the Newell-whitehead equation both time and space fractional derivative, namely

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\beta u}{\partial x^\beta} + u - u^3 \qquad \text{with} \quad 0 < \alpha \leq 1 \quad \text{and} \quad 1 < \beta \leq 2, \tag{22}$$

with a simple initial condition $u(x,0) = x^2$. In analogous way, the ADM analysis gives

$$u_0 = u(x,0), \tag{23}$$

and

$$u_{n+1} = J^\alpha(A_n + u_n + D_x^\beta(u_n)). \tag{24}$$

With the aid of the above recursive relationship equations and the Adomian polynomials, the first three terms of $u(x,t)$ follow immediately upon setting

$$\begin{aligned} &u_0(x,t) = u(x,0) := f(x), \\ &u_{n+1}(x,t) = J^\alpha(A_n + u_n + D_x^\beta u_n) = f_{n+1}(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{25}$$

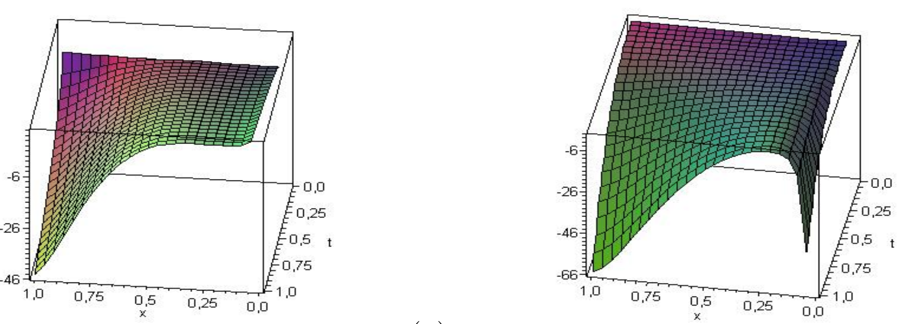
where the functions $(f_k)_{k=0..3}$ are given by

$$
\begin{aligned}
f_0 &= u_0,\\
f_1 &= f_0^2 + u_0 + D_x^\beta f_0,\\
f_2 &= -f_0^2 f_1 + f_1 + D_x^\beta f_1,\\
f_3 &= -3 f_0 f_1^2 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} - 3 f_0^2 f_2 + f_2 + D_x^\beta f_2.
\end{aligned}
\tag{26}
$$

The solution in series form reads

$$
u(x,t) = f(x) + f_1 \frac{t^\alpha}{\Gamma(\alpha+1)} + f_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3 \frac{t^{3\alpha}}{3\Gamma(\alpha+1)} + \ldots \tag{27}
$$

For this example, only three terms of the decomposition series were used for the approximate solution. Figure.3 shows the evolution result for the Newell-Whithead equation with time-space fractional derivatives: (a) $\alpha = 0.20$, $\beta = 1.2$ and (b) $\alpha = 0.75$, $\beta = 1.75$. As it can be seen the profile of $u(x,t)$ for small value of α (resp. β) is quite different from one of large values of α (resp. β).



(a) (b)

Figure 3: The numerical solutions of $u(x,t)$: (a)$\alpha = 1$ et$\beta = 1.2$, (b): $\alpha = 0.75$ et $\beta = 1.75$.

5 Conclusion

In this paper, the Adomian decomposition method was applied for solving time- and space- fractional Newell-Whitehead equation with initial conditions. The analytical results have been given in terms of a power series with easily computed terms. The fractional derivative was defined in the Caputo sense. The results show that the solution strongly depends on the fractional derivation order parameter.

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