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Cantor Theorem and Application in Some Fixed Point Theorems in a Generalized Metric Space

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Abstract

Some useful fixed point Theorems are derived by applying Cantor like Theorem as proved in complete Generalized metric spaces.

Keywords: *Cantor Theorem, Generalized metric spaces, fixed point Theorem*

1 Introduction

In 2000 A. Branciary[1] had initiated the study of Generalized metric spaces(g.m.s.). A g.m.s.(X, d) is one where $X \neq \phi$, and $d : X \times X \to R^+$ (non-negative reals) is a function to satisfy:

(i) $d(x, y) = 0$ if and only if $x = y$ in X

(ii) $d(x, y) = d(y, x)$ for $x, y \in X$

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and for all distinct members u, v as distinct from x and y.

While a metric space is treated as a g.m.s. Branciary has shown in [1] that there is a g.m.s. that is not a metric space. Since this initiation theory of g.m.s., especially in fixed point theory, a rapid stride has taken place, primarily through works of researchers like Lahiri and Das[3] who were also responsible for introducing Generalized vector Metric Space where they have proved analogue of Banach Contraction Principle in a complete metric space. B.E.Rhoades[4] and Azam and Arshad [2] have also contributed in this area by proving some useful fixed point Theorems. These researchers have employed Picards Iterative scheme in proving main Theorems. In this paper we have been able to invite an alternative route to achieve a fixed point of an operator that is not necessarily continuous over a g.m.s. To that end we have proved a Cantor like theorem in a g.m.s. and with its aid we deal with various types of operators acting on g.m.s. including mixed type contractive operator and Ciric-type contractive operator. Our findings shall include known and important results as available to date.

Definition 1.1 *A sequence $\{x_n\}$ is said to be a Cauchy sequence in a g.m.s. (X, d) if*

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0$$

Definition 1.2 *A g.m.s. (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.*

Let $x \in X$. For $r > 0$ let $B_r(x) = \{y \in X | d(x, y) < r\}$ be an open ball centered at x with radius r.

Theorem 1.3 *The family $\{B_r(x)\}$ together with empty set contributes a base for a topology τ_d in X.*

Branciari observed that in a g.m.s. (X, d) the topology τ_d is Hausdroff. See [1]. We assume that (X, d) is free from isolated points. Now we define $\rho : X \times X \to R^+$ by the following rule: $\rho(x, y) = \int_0^{d(x,y)} \varphi dt$ where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable function which is summable and non-negative such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi dt > 0$. Then by routine checkup we find (X, ρ) is a g.m.s. such that $\tau_d \subset \tau_\rho$.

Theorem 1.4 *If $\{x_n\}$ is a ρ-convergent in X, then it is ρ-Cauchy.*

Proof. Suppose $\rho - \lim\limits_{n\to\infty} x_n = u \in X$ i.e., $\lim\limits_{n\to\infty} \int_0^{d(x_n,u)} \varphi dt = 0$
For $\varepsilon > 0$ take v in X distinct from x_n and u so that $d(u, v) < \frac{\varepsilon}{4}$.
For large n we have $| \int_0^{d(x_n,u)} \varphi dt| < \frac{\varepsilon}{4}$. Now $d(x_m, x_n) \leq d(x_m, u) + d(u, v) + d(v, x_n)$
Since d is coordinate-wise continuous (See Branciary[1]) we have for large n

$d(v, x_n) \leq d(u, v) + \frac{\varepsilon}{4}$.
Thus $d(x_m, x_n) \leq d(x_m, u) + d(u, v) + d(v, x_n) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$ for large m and n.
So $\int_0^{d(x_m,x_n)} \varphi dt \leq \int_0^{\varepsilon} \varphi dt$. As $\varepsilon > 0$ is arbitrary, we have $\lim\limits_{m,n\to\infty} \int_0^{d(x_m,x_n)} \varphi dt = 0$, so $\{x_n\}$ is $\rho - Cauchy$ in X.

Theorem 1.5 *Let (X, ρ) is complete. If $\{F_n\}$ is a sequence of non-empty ρ-closed sets in X such that $F_1 \supset F_2 \supset \ldots$ with $\rho - Diam(F_n) \to 0$ as $n \to \infty$ then $\bigcap_{i=1}^{\infty} F_i$ is a singleton.*

Proof. Take $x_n \in F_n (n = 1, 2, \ldots)$. Now $x_m \in F_n$ if $m > n$. Therefore $\lim\limits_{m,n} \rho(x_m, x_n) \leq \rho - Diam(F_n) \to 0$ as $n \to \infty$. Similar is the case when $n > m$.
Thus $\{x_n\}$ is $\rho - Cauchy$ in X which is $\rho - Complete$. Take $u \in X$ such that $\rho - \lim\limits_{n\to\infty} x_n = u$.
or, $\lim\limits_{j\to\infty} \int_0^{d(x_{n+j},u)} \varphi dt = 0$
So, u is a ρ-limit point of F_n and therefore $u \in F_n$.
This is true for $n = 1, 2, \ldots$ and hence $u \in \bigcap_{i=1}^{\infty} F_i$. Assuming $v \in \bigcap_{i=1}^{\infty} F_i$, we have $\rho(u, v) \leq \rho - Diam(F_n) \to 0$ as $n \to \infty$, and hence $u = v$.
Proof is now complete.

Theorem 1.6 *Suppose (X, ρ) is complete and $f : (X, \rho) \to (X, \rho)$ is an operator satisfying $\int_0^{d(f(x),f(y))} \varphi dt \leq \alpha \int_0^{d(x,f(x))} \varphi dt + \beta \int_0^{d(y,f(y))} \varphi dt + \gamma \int_0^{d(x,y)} \varphi dt$ where $0 \leq \alpha, \beta, \gamma$ and $\sum \alpha < 1$; then f has a unique fixed point in X.*

The proof of Theorem (1.6) rests upon the following lemma.

Lemma 1.7 *Suppose (X, ρ) is complete and $f : (X, \rho) \to (X, \rho)$ satisfy $\int_0^{d(f(x),f(y))} \varphi dt \leq \alpha \int_0^{d(x,f(x))} \varphi dt + \beta \int_0^{d(y,f(y))} \varphi dt + \gamma \int_0^{d(x,y)} \varphi dt$ where $0 \leq \alpha, \beta, \gamma$ and $\sum \alpha < 1$. Then $G_\lambda = \{x \in X : \int_0^{d(x,f(x))} \varphi dt \leq \lambda, \lambda \in R^+\}$ is a non-empty ρ-closed,ρ-bounded set in X such that $f(G_\lambda) \subset G_\lambda$.*

Proof. Take $x = x_0 \in X$, and put $x_n = f(x_{n-1}), n = 1, 2, \ldots$.
Then $\rho(x_2, x_1) = \rho(f(x_1), f(x_0)) = \int_0^{d(f(x_0),f(x_1))} \varphi dt$
$\leq \alpha \int_0^{d(x_1,f(x_1))} \varphi dt + \beta \int_0^{d(x_0,f(x_0))} \varphi dt + \gamma \int_0^{d(x_1,x_0)} \varphi dt$
$= \alpha \int_0^{d(x_1,x_2)} \varphi dt + \beta \int_0^{d(x_0,x_1)} \varphi dt + \gamma \int_0^{d(x_1,x_0)} \varphi dt$
or, $(1 - \alpha)\rho(x_2, x_1) \leq (\beta + \gamma)\rho(x_1, x_0)$
or, $\rho(x_2, x_1) \leq \frac{\beta+\gamma}{1-\alpha} \rho(x_1, x_0)$
And by induction, $\rho(x_{n+1}, x_n) \leq (\frac{\beta+\gamma}{1-\alpha})^n \rho(x_1, x_0)$ which can be made arbitrarily small with inrease of n as $\frac{\beta+\gamma}{1-\alpha} < 1$. Hence $x_n \in G_\lambda$ for large n, i.e., $G_\lambda \neq \phi$.
Let $\{x_n\} \subset G_\lambda$ with $\rho - \lim\limits_{n\to\infty} x_n = u \in X$. Then

$$\rho(u, f(u)) \leq \rho(u, x_n) + \rho(x_n, f(x_n)) + \rho(f(x_n), f(u)) \tag{1}$$

write $\rho(f(x_n), f(u)) = \int_0^{d(f(x_n),f(u))} \varphi dt \leq \alpha \int_0^{d(x_n,f(x_n))} \varphi dt + \beta \int_0^{d(u,f(u))} \varphi dt + \gamma \int_0^{d(x_n,u)} \varphi dt$
This gives from (1)

$$\rho(u, f(u)) = \frac{1+\gamma}{1-\beta}\rho(u, x_n) + \frac{1+\alpha}{1-\beta}\lambda \to \frac{1+\alpha}{1-\beta}\lambda, n \to \infty \tag{2}$$

Now $0 \leq \alpha, \beta, \gamma < 1$ and $\alpha + \beta + \gamma < 1$ give $\frac{\alpha+\gamma}{1-\beta} < 1$.
Therefore $\sup\limits_{\gamma}\{\frac{\alpha+\gamma}{1-\beta}\} \leq 1$
or, $\frac{\alpha+1}{1-\beta} \leq 1$
Passing on $n \to \infty$ in (2) we have $\rho(u, f(u)) \leq \lambda$ and hence $u \in G_\lambda$. So G_λ is $\rho - closed$. Finally take $x, y \in G_\lambda$; so we have $\rho(x, f(x)) \leq \lambda$ and $\rho(y, f(y)) \leq \lambda$;so $\rho(x, y) \leq \rho(x, f(x))+\rho(f(y), f(x))+\rho(y, f(y)) \leq 2\lambda+\rho(f(y), f(x))$, where $\rho(f(y), f(x)) \leq \alpha\rho(x, f(x)) + \beta\rho(y, f(y)) + \gamma\rho(y, x) \leq (\alpha + \beta)\lambda + \gamma\rho(x, y)$.
So $\rho(x, y) \leq 2\lambda + (\alpha + \beta)\lambda + \gamma\rho(x, y)$
And $\rho(x, y) \leq \frac{\alpha+\beta+2}{1-\gamma}\lambda$
Therefore $\sup\limits_{x,y\in G_\lambda} \rho(x, y) \leq \frac{\alpha+\beta+2}{1-\gamma}\lambda$
or, $\rho - Diam(G_\lambda) < \infty$ and G_λ is $\rho - bounded$.
Finally, taking $x \in G_\lambda$, we have
$\rho(f(x), f(f(x))) \leq \alpha\rho(x, f(x)) + \beta\rho(f(x), f(f(x))) + \gamma\rho(x, f(x))$
or, $\rho(f(x), f(f(x))) \leq \frac{\alpha+\gamma}{1-\beta}\rho(x, f(x)) \leq \frac{\alpha+\gamma}{1-\beta}\lambda \leq \lambda$ since $\alpha + \beta + \gamma < 1$.
Therefore $f(x) \in G_\lambda$ and $f(G_\lambda) \subset G_\lambda$. Proof of Lemma (1.7) is now complete.

Proof of Theorem (1.6) Take $\lambda = \frac{1}{n}$ and $G_n = \{x \in X : \rho(x, f(x)) \leq \frac{1}{n}\}$. Then G_n is a decreasing chain of non-empty $\rho - bounded$ and $\rho - closed$ sets such that $f : G_n \to G_n$ such that $\rho - Diam(G_n) \leq \frac{\alpha+\beta+2}{1-\gamma}\frac{1}{n}$ (See Lemma (1.7))$\to 0$ as $n \to \infty$. Hence Theorem (1.5) applies to show that $\bigcap\limits_{i=1}^{\infty} G_i$ is a singleton= $\{v\}$ for some $v \in X$. Clearly $f(v) = v$, and uniqueness of v is also clear

We close the paper by adding another application of Theorem (1.5) to prove a fixed point Theorem in a g.m.s. where operators involved form a class so large that includes several contraction type of operators as known to date.

Theorem 1.8 *Let (X, ρ) be complete and $f : X \to X$ satisfy $\int_0^{d(f(x),f(y))} \varphi dt \leq \psi[\max\{\int_0^{d(x,y)} \varphi dt, \int_0^{d(f(x),x)} \varphi dt, \int_0^{d(y,f(y))} \varphi dt\}]$ where $\varphi : R^+ \to R^+$ is summable (Lebesgue) and non-negative such that for each $\varepsilon > 0, \int_0^{\varepsilon} \varphi dt > 0$; and $\psi : R^+ \to R^+$ is upper semi-continuous with $\psi(t) \neq t$ as $t > 0$ such that $0 < \sup\limits_{t>0} \frac{t}{t-\psi(t)} < 1$. Then f has unique fixed point in X.*

The proof of Theorem (1.8) rests on the following lemma:

Lemma 1.9 *Under the hypothesis of Theorem* (1.8) *if* $\alpha_n = \rho(x_n, x_{n+1})$ *where* $x_n = f^n(x)$ *and* $x \in X$ *and* $\rho(u,v) = \int_0^{d(u,v)} \varphi dt, u, v \in X$ *then* $\lim_{n\to\infty} \alpha_n = 0$.

Proof. Suppose $\alpha_n > 0$ for all n. Then
$\alpha_n = \rho(x_n, x_{n+1}) = \int_0^{d(f(x_{n-1}),f(x_n))} \varphi dt$
$\leq \psi[\max\{\int_0^{d(x_{n-1},x_n)} \varphi dt, \int_0^{d(f(x_{n-1}),x_{n-1})} \varphi dt, \int_0^{d(x_n,f(x_n))} \varphi dt\}]$
$= \psi[\max\{\rho(x_n, x_{n-1}), \rho(x_n, x_{n-1}), \rho(x_n, x_{n+1})\}]$
$= \psi[\max\{\rho(x_n, x_{n-1}), \rho(x_n, x_{n+1})\}]$
If max. value$= \rho(x_n, x_{n+1})$, then one has $\alpha_n \leq \psi(\alpha_n) < \alpha_n$ which is untenable. Hence max. value$= \rho(x_n, x_{n-1})$. So we have $\alpha_n \leq \psi(\alpha_{n-1}) < \alpha_{n-1}$. That means $\{\alpha_n\}$ is a decreasing sequence, and let $\lim_n \alpha_n = \alpha$. If $\alpha > 0$ we have $\psi(\alpha) < \alpha$. By u.s.c. property of ψ we get $\alpha = \lim_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \psi(\alpha_{n-1}) \leq \psi(\lim_{n\to\infty} \alpha_{n-1}) = \psi(\alpha) < \alpha$, which is a contradiction. Therefore $\alpha = 0$.

Proof of Theorem (1.8)**.** From $\sup_{t>0} \frac{t}{t-\psi(t)} < 1$ it follows that $\psi(t) < t$ for $t > 0$. For any natural number n, put $G_n = \{x \in X : \rho(x, f(x)) \leq \frac{1}{n}\}$. By Lemma (1.9) we may assume $G_n \neq \phi$ for all n. Now we verify that f maps G_n in G_n. Take $x \in G_n$.
The $\rho(f(x), f(f(x))) \leq \psi[\max\{\rho(f(x), x), \rho(f(x), x), \rho(f(x), f(f(x)))\}]$
$= \psi[\max\{\rho(f(x), x), \rho(f(x), f(f(x)))\}]$
If max. value$= \rho(f(x), f(f(x)))$, we get $\rho(f(x), f(f(x))) \leq \psi(\rho(f(x), f(f(x)))) < \rho(f(x), f(f(x)))$- a contradiction. Hence max. value$= \rho(f(x), x)$. So $\rho(f(x), f(f(x))) \leq \psi[\rho(x, f(x))] < \rho(x, f(x)) \leq \frac{1}{n}$. That means $f(x) \in G_n$ i.e., $f(G_n) \subset G_n$.
We now show that G_n is $\rho-$*closed* in X. Let $\{x_{n_k}\} \subset G_n$ with $\rho - \lim_k x_{n_k} = \xi \in X$. So $\rho(x_{n_k}, f(x_{n_k})) \leq \frac{1}{n}$ for all k. So

$$\rho(\xi, f(\xi)) \leq \rho(\xi, x_{n_k}) + \rho(x_{n_k}, f(x_{n_k})) + \rho(f(x_{n_k}), f(\xi)) \tag{3}$$

Where $\rho(f(x_{n_k}), f(\xi)) \leq \psi[\max\{\rho(\xi, x_{n_k}), \rho(x_{n_k}, f(x_{n_k})), \rho(\xi, f(\xi))\}]$. Now two cases arise to consider.
Case 1. Let max. value$= \int_0^{d(x_{n_k})} \varphi dt$ which $\to 0$ as $k \to \infty$. And therefore $\int_0^{d(f(x_{n_k}),f(\xi))} \varphi dt \to 0$ as $k \to \infty$ and in consequence from (3) we have

$$\int_0^{d(\xi,f(\xi))} \varphi dt \leq \frac{1}{n} \tag{4}$$

Case 2. Let max. value$= \max\{\int_0^{d(f(x_{n_k}),x_{n_k})} \varphi dt, \int_0^{d(\xi,f(\xi))} \varphi dt\}$
$= \max\{\frac{1}{n}, \int_0^{d(\xi,f(\xi))} \varphi dt\}$ as $\int_0^{d(f(x_{n_k}),x_{n_k})} \varphi dt \leq \frac{1}{n}$;
If max. value$= \frac{1}{n}$; that means

$$\int_0^{d(\xi,f(\xi))} \varphi dt \leq \frac{1}{n} \tag{5}$$

Otherwise max. value= $\int_0^{d(\xi,f(\xi))}\varphi dt$,and from (3) we get
$\int_0^{d(\xi,f(\xi))}\varphi dt \leq \frac{1}{n}+\psi(\int_0^{d(\xi,f(\xi))}\varphi dt \leq \frac{1}{n})$
or, $\int_0^{d(\xi,f(\xi))}\varphi dt(1-\frac{\psi(\int_0^{d(\xi,f(\xi))}\varphi dt)}{\int_0^{d(\xi,f(\xi))}\varphi dt}) \leq \frac{1}{n}$
or, $\int_0^{d(\xi,f(\xi))}\varphi dt \leq \frac{1}{n}\frac{\int_0^{d(\xi,f(\xi))}\varphi dt}{\int_0^{d(\xi,f(\xi))}\varphi dt-\psi(\int_0^{d(\xi,f(\xi))}\varphi dt)} \leq \frac{1}{n}\sup_{t>0}\frac{t}{t-\psi(t)} < \frac{1}{n}$. Therefore

$$\int_0^{d(\xi,f(\xi))}\varphi dt \leq \frac{1}{n} \tag{6}$$

Combining (4), (5) and (6) we conclude that $\xi \in G_n$ and G_n is $\rho - closed$. Finally, to estimate $\rho - Diam(G_n)$, take $x, y \in G_n$. Then

$$\int_0^{d(x,y)}\varphi dt \leq \int_0^{d(x,f(x))}\varphi dt + \int_0^{d(f(x),f(y))}\varphi dt + \int_0^{d(f(y),y)}\varphi dt \leq \frac{1}{n}+\frac{1}{n}+\int_0^{d(f(x),f(y))}\varphi dt \tag{7}$$

$\int_0^{d(f(x),f(y))}\varphi dt \leq \psi[\max\{\int_0^{d(x,y)}\varphi dt, \int_0^{d(f(x),x)}\varphi dt, \int_0^{d(y,f(y))}\varphi dt\}]$
$\leq \psi[\max\{\int_0^{d(x,y)}\varphi dt, \frac{1}{n}\}]$
Case 1 arises due to

$$\int_0^{d(x,y)}\varphi dt \leq \frac{1}{n} \tag{8}$$

Case 2 arises due to

$$\int_0^{d(x,y)}\varphi dt > \frac{1}{n} \tag{9}$$

In case 2 we will have from (7), $\int_0^{d(x,y)}\varphi dt \leq \frac{2}{n}+\psi(\int_0^{d(x,y)}\varphi dt)$. As before we arrive at

$$\int_0^{d(x,y)}\varphi dt \leq \frac{2}{n}\sup_{t>0}\frac{t}{t-\psi(t)} < \frac{2}{n} \tag{10}$$

Hence upon combining (8) and (10) one concludes that $\rho - Diam(G_n) < \infty$ and $\rho - Diam(G_n) \to 0$ as $n \to \infty$. So we invite Theorem (1.5) to apply here for desired conclusion. The proof is now complete.

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