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Coupled Linear Feedback and Sliding Model Control for a Serially Connected Euler-Bernoulli Beam

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Abstract

A system of serially connected string and Euler-Bernoulli beam with coupled linear feedback control and sliding model control is studied in the present paper. The system is formulated by partial differential equations with the boundary conditions. The eigenvalues and eigenfunctions of the system operator are discussed in the appropriate Hilbert spaces. A sliding model control is applied to the serially connected beam and it has been shown that the actual sliding mode of the system can be approximated by ideal sliding modes in any accuracy under certain conditions. In this paper, a significant semigroup property of restriction of the system operator is derived.

Keywords: *Serially Connected Euler-Bernoulli Beam, Feedback Control, Sliding Model Control, Semigroup of Linear Operators.*

1 Introduction

The vibration and control of serially connected strings and Euler-Bernoulli beams with linear feedback controls at joins have been studied extensively in the last two decades (see, e.g., [2-4,7,10,12,13,16,17]). In addition to the analysis of the distribution of eigenvalues, one also needs to establish the so-called spectrum-determined growth condition in order to conclude exponential stability for these infinite-dimensional systems form spectral analysis. In the case of serially connected strings, the first results on exponential stability were obtained in [12] for a 2-connected strings with linear feedbacks at the middle of the span. The stability of N-connected strings under joints feedback was studied in [13].

In this paper, we consider the following serially connected beams with linear feedback control

$$y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \quad L_{j-1} < x < L_j, \quad j = 1, 2, \cdots, n. \tag{1}$$

The boundary condition are

$$\begin{cases} y(0) = y_{xx}(0) = 0, \\ y_x(L_n) = y_{xxx}(L_n) = 0. \end{cases} \tag{2}$$

The linear feedback control at the joint points $L_j, j = 1, \cdots, n-1$, take the form

$$\begin{cases} y(L_j^-, t) = y(L_j^+, t), \\ y_{xx}(L_j^-, t) = y_{xx}(L_j^+, t) \\ y_{tx}(L_j^+, t) - y_{tx}(L_j^-, t) = (-1)^j r_j y_t(L_j^-, t) + p_j^2 y_{xx}(L_j^-, t), \\ y_{xxx}(L_j^+, t) - y_{xxx}(L_j^-, t) = -q_j^2 y_t(L_j^-, t) + (-1)^j s_j y_{xx}(L_j^-, t), \end{cases} \tag{3}$$

Where $0 = L_0 < L_1 < \cdots < L_n$ and

$$\begin{array}{l} p_j^2 \geq 0, \quad q_j^2 \geq 0, \quad p_j^2 + q_j^2 > 0, \quad r_j, s_j \in \mathbb{R} \\ p_j^2 \alpha^2 + q_j^2 \beta^2 + (r_j - s_j)\alpha\beta \geq 0, \quad \forall \alpha, \beta \in \mathbb{R} \end{array} \tag{4}$$

Let us defines the energy of system (1.1)-(1.4)as

$$E(t) = \frac{1}{2} \sum_{j=1}^{n} \int_{L_{j-1}}^{L_j} [y_t^2(x,t) + y_{xx}^2(x,t)] dx.$$

Then a simple computation shows that $\dot{E}(t) \leq 0$ and hence the system is dissipative.

Without loss of generality, we may assume that n is odd. For $j = 1, 2, \cdots, n$, we set

$$\begin{cases} u_j(x,t) = \frac{1}{2}\big[y_t(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t) \\ \qquad + \frac{(-1)^{j+1}}{l_j^2} y_{xx}(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t)\big], \\ v_j(x,t) = \frac{1}{2}[y_t(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t) \\ \qquad - \frac{(-1)^{j+1}}{l_j^2} y_{xx}(L_j + \frac{(-1)^j - 1}{2} l_j + (-1)^{j+1} l_j x, t)], \end{cases} \tag{5}$$

where $l_j = L_j - L_{j-1}, j = 1, 2, \cdots, n$, $0 \le x \le 1$.Then system (1.1)-(1.4) can be transformed into the form of

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = K\frac{\partial^2}{\partial x^2}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \\ [A,B][u_x(0),v_x(0),u(0),v(0)]^T = 0, \\ [E,F][u_x(1),v_x(1),u(1),v(1)]^T = 0, \end{cases} \tag{6}$$

where

$u(x,t) = [u_1(x,t), u_2(x,t), ..., u_n(x,t)]^T, v(x,t) = [v_1(x,t), v_2(x,t), ..., v_n(x,t)]^T$

and

$(2n \times 2n)$-matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & P_{21} & 0 & \dots & 0 & 0 & P_{22} & 0 & \dots & 0 \\ 0 & 0 & P_{41} & \dots & 0 & 0 & 0 & P_{42} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P_{(n-1)1} & 0 & 0 & 0 & \dots & P_{(n-1)2} \end{bmatrix}, \tag{6a}$$

$$B = \begin{bmatrix} P_{n1} & 0 & 0 & \dots & 0 & P_{n2} & 0 & 0 & \dots & 0 \\ 0 & \tilde{P}_{21} & 0 & \dots & 0 & 0 & \tilde{P}_{22} & 0 & \dots & 0 \\ 0 & 0 & \tilde{P}_{41} & \dots & 0 & 0 & 0 & \tilde{P}_{42} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{P}_{(n-1)1} & 0 & 0 & 0 & \dots & \tilde{P}_{(n-1)2} \end{bmatrix}, \tag{6b}$$

$$E = \begin{bmatrix} P_{11} & 0 & \dots & 0 & 0 & P_{12} & 0 & \dots & 0 & 0 \\ 0 & P_{31} & \dots & 0 & 0 & 0 & P_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_{(n-2)1} & 0 & 0 & 0 & \dots & P_{(n-2)2} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{6c}$$

$$F = \begin{bmatrix} \tilde{P}_{11} & 0 & \dots & 0 & 0 & \tilde{P}_{12} & 0 & \dots & 0 & 0 \\ 0 & \tilde{P}_{31} & \dots & 0 & 0 & 0 & \tilde{P}_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \tilde{P}_{(n-2)1} & 0 & 0 & 0 & \dots & \tilde{P}_{(n-2)2} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{6d}$$

where for $j = 1, 2, \cdots, n$,

$$P_{n1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, P_{n2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$P_j1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{l_j} & \frac{1}{l_j+1} \\ \frac{-1}{l_j} & \frac{1}{l_j+1} \end{bmatrix}, P_j2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{l_j} & \frac{1}{l_j+1} \\ \frac{1}{l_j} & \frac{1}{-l_j+1} \end{bmatrix},$$

$$\tilde{P}_j1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ p_j^2 - r_j & 0 \\ q_j^2 + s_j & 0 \end{bmatrix}, \tilde{P}_j2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ -p_j^2 - r_j & 0 \\ q_j^2 - s_j & 0 \end{bmatrix}.$$

Now we confine ourselves to system (1.1)-(1.4) with A, B, E, F specified by (1.6). Divide by $\rho\omega_1$ both sides of those equations which contain nonzero factors ρ in the system $\tilde{M}C = 0$; then we have becomes

$$\tilde{M}C = 0, \tag{7}$$

where

$$\tilde{M} = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \end{bmatrix} \tag{8}$$

and for $1 \leq k \leq 4$.

$$M_k = \begin{bmatrix} Q_0k & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & Q_2k & R_2k & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & Q_{n-k} & R_(n-1)k \\ Q_{1k^{\omega_k \rho l_1}} & R_{1k^{\omega_k \rho l_1}} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q_{(n-2)k^{\omega_k \rho l_{n-2}}} & R_{(n-2)k^{\omega_k \rho l_{n-1}}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & Q_n 1^{\omega_k \rho l_n} \end{bmatrix}, \tag{9}$$

with

$$\begin{gathered} Q_{01} = \begin{bmatrix} 1-i & 1+i \end{bmatrix}^T, \quad Q_{02} = \begin{bmatrix} 1+i & 1-i \end{bmatrix}^T, \\ Q_{03} = Q_{01}, \quad Q_{04} = Q_{02}, \quad Q_{n1} = Q_{01}, \\ Q_{n2} = Q_{02} \cdot i, \quad Q_{n3} = -Q_{n1}, \quad Q_{n4} = -Q_{n2}. \end{gathered} \tag{10}$$

For $j = 1, 3, \cdots, n-2$, $l = 2, 4, \cdots, n-1$,

$$Q_{j1} = \left[\begin{array}{cccc} 1-i+\frac{(1+i)p_j^2-(1-i)r_j}{\rho\omega_1}, & -(1+i)+\frac{(1-i)q_j^2+(1+i)s_j}{\rho\omega_1}, & 1-i, & 1+i \end{array}\right]^T, \tag{11a}$$

$$Q_{j2} = \left[\begin{array}{cccc} -(1-i)+\frac{(1-i)p_j^2-(1+i)r_j}{\rho\omega_1}, & -(1+i)+\frac{(1+i)q_j^2+(1-i)s_j}{\rho\omega_1}, & 1+i, & 1-i \end{array}\right]^T, \tag{11b}$$

$$Q_{j3} = \left[\begin{array}{cccc} -(1-i)+\frac{(1+i)p_j^2-(1-i)r_j}{\rho\omega_1}, & 1+i+\frac{(1-i)q_j^2+(1+i)s_j}{\rho\omega_1}, & 1-i, & 1+i \end{array}\right]^T, \tag{11c}$$

$$Q_{j4} = \left[\begin{array}{cccc} 1-i+\frac{(1-i)p_j^2-(1+i)r_j}{\rho\omega_1}, & 1+i+\frac{(1+i)q_j^2+(1-i)s_j}{\rho\omega_1}, & 1+i, & 1-i \end{array}\right]^T, \tag{11d}$$

$$Q_{l1} = \left[\begin{array}{cccc} 1+i+\frac{(1-i)p_l^2-(1+i)r_l}{\rho\omega_1}, & -(1-i)+\frac{(1+i)q_l^2+(1-i)s_l}{\rho\omega_1}, & 1+i, & 1-i \end{array}\right]^T, \tag{12a}$$

$$Q_{l2} = \left[\begin{array}{cccc} 1+i+\frac{(1+i)p_l^2-(1-i)r_l}{\rho\omega_1}, & 1-i+\frac{(1-i)q_l^2+(1+i)s_l}{\rho\omega_1}, & 1-i, & 1+i \end{array}\right]^T, \tag{12b}$$

$$Q_{l3} = \left[\begin{array}{cccc} -(1+i)+\frac{(1-i)p_l^2-(1+i)r_l}{\rho\omega_1}, & 1-i+\frac{(1+i)q_l^2+(1-i)s_l}{\rho\omega_1}, & 1+i, & 1-i \end{array}\right]^T, \tag{12d}$$

$$Q_{l4} = \left[\begin{array}{cccc} -(1+i)+\frac{(1+i)p_l^2-(1-i)r_l}{\rho\omega_1}, & -(1-i)+\frac{(1-i)q_l^2+(1+i)s_l}{\rho\omega_1}, & 1-i, & 1+i \end{array}\right]^T; \tag{12d}$$

$$\begin{aligned} R_{j1} &= \left[\begin{array}{cccc} 1+i, & 1-i, & -(1+i), & 1-i \end{array}\right]^T \\ R_{j2} &= \left[\begin{array}{cccc} 1+i, & -(1-i), & -(1-i), & 1+i \end{array}\right]^T \\ R_{j3} &= \left[\begin{array}{cccc} -(1+i), & -(1-i), & -(1+i), & 1-i \end{array}\right]^T \\ R_{j4} &= \left[\begin{array}{cccc} -(1+i), & 1-i, & -(1-i), & 1+i \end{array}\right]^T \end{aligned} \tag{13}$$

$$\begin{aligned} R_{l1} &= \left[\begin{array}{cccc} 1-i, & 1+i, & -(1-i), & 1+i \end{array}\right]^T \\ R_{l2} &= \left[\begin{array}{cccc} -(1-i), & 1+i, & -(1+i), & 1-i \end{array}\right]^T \\ R_{l3} &= \left[\begin{array}{cccc} -(1-i), & -(1+i), & -(1-i), & 1+i \end{array}\right]^T \\ R_{l4} &= \left[\begin{array}{cccc} 1-i, & -(1+i), & -(1+i), & 1-i \end{array}\right]^T \end{aligned} \tag{14}$$

2 Spectral Analysis and Semigroup Generation

In this section, we derive the characteristic equation satisfied by eigenvalues of system (1.1-1.4). To begin with, we put system (1.1-1.4) into the framework of evolutionary equations in an underlying Hilbert space $\mathcal{H}$. Take $\mathcal{H} = (L^2(0,1))^{2n}$ and define $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} = K\frac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix}, \tag{15}$$

where

$$D(\mathcal{A}) = \left\{ [u,v]^T \in (H^2(0,1))^{2n} \;\middle|\; \begin{matrix} [A,B][u_x(0),v_x(0),u(0),v(0)]^T = 0, \\ [E,F][u_x(1),v_x(1),u(1),v(1)]^T = 0 \end{matrix} \right\}$$

and $H^2(0,1)$ denotes the usual Sobolev space. With this setting, system (1.1-1.4) can be considered as an abstract equation in $\mathcal{H}$:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{16}$$

Obviously, $\mathcal{A}$ is densely defined in $\mathcal{H}$. Next, we consider the eigenvalue problem for $\mathcal{A}$. For any given $\Phi = [f,g]^T \in \mathcal{H}$, solve the following equation:

$$(\lambda - \mathcal{A})\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{17}$$

i.e.,

$$\begin{cases} \dfrac{\partial^2}{\partial x^2}\begin{bmatrix} u \\ v \end{bmatrix} = \lambda K^{-1}\begin{bmatrix} u \\ v \end{bmatrix} - K^{-1}\Phi, \\ [A,B][u_x(0),v_x(0),u(0),v(0)]^T = 0, \\ [E,F][u_x(1),v_x(1),u(1),v(1)]^T = 0, \end{cases} \tag{18}$$

which can be further written as a first-order ordinary differential equation of the following form:

$$\begin{cases} \dfrac{\partial}{\partial x}\begin{bmatrix} u_x \\ v_x \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0_{2n} & \lambda K^{-1} \\ I_{2n} & 0_{2n} \end{bmatrix}\begin{bmatrix} u_x \\ v_x \\ u \\ v \end{bmatrix} - \begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix}, \\ [A,B][u_x(0),v_x(0),u(0),v(0)]^T = 0, \\ [E,F][u_x(1),v_x(1),u(1),v(1)]^T = 0, \end{cases} \tag{19}$$

where I_{2n} denotes the $2n \times 2n$ identity matrix. Set

$$K_\lambda = \begin{bmatrix} 0_{2n} & \lambda K^{-1} \\ I_{2n} & 0_{2n} \end{bmatrix} \tag{20}$$

Then the solution to the governing equation of 19 is

$$\begin{bmatrix} u_x(x) \\ v_x(x) \\ u(x) \\ v(x) \end{bmatrix} = e^{K_\lambda x} \begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} - \int_0^x e^{K_\lambda (x-s)} \begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix} ds. \tag{21}$$

In order for 21 to satisfy 19, the last two boundary conditions should be fulfilled, i.e.,

$$\begin{cases} [A,B][u_x(0), v_x(0), u(0), v(0)]^T = 0, \\ [E,F]e^{K_\lambda}[u_x(0), v_x(0), u(0), v(0)]^T = \displaystyle\int_0^1 [E,F]e^{K_\lambda(1-s)}[K^{-1}\Phi, 0]^T ds, \end{cases} \tag{22}$$

Define

$$H(\lambda) = \begin{bmatrix} [A,B] \\ [E,F]e^{K_\lambda} \end{bmatrix}. \tag{23}$$

Then for

$$h(\lambda) = \det H(\lambda) \neq 0, \tag{24}$$

it has

$$R(\lambda, \mathcal{A})\Phi = [0_{2n}, I_{2n}]e^{K_\lambda x} \begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} - \int_0^x [0_{2n}, I_{2n}]e^{K_\lambda(x-s)} \begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix} ds, \tag{25}$$

where

$$\begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} = H^{-1}(\lambda) \begin{bmatrix} 0 \\ \displaystyle\int_0^1 [E,F]e^{K_\lambda(1-s)} \begin{bmatrix} K^{-1}\Phi \\ 0 \end{bmatrix} ds \end{bmatrix}. \tag{26}$$

Therefore, in this case, $\lambda \in \rho(\mathcal{A})$ and $R(\lambda, \mathcal{A})$ is compact.

On the order hand, if $h(\lambda) = 0$, for any $4n \times 1$ nonzero column vector $Z = (u_x(0), v_x(0), u(0), v(0))^T$ satisfying $H(\lambda)Z = 0$, by setting $\Phi = 0$ in 21, we have

$$\begin{bmatrix} u_x(0) \\ v_x(0) \\ u(0) \\ v(0) \end{bmatrix} = e^{K_\lambda x} Z \neq 0$$

and hence $(u_x(0), v_x(0), u(0), v(0))^T \neq 0$. Therefore,

$$\begin{bmatrix} u \\ v \end{bmatrix} = [0_{2n}, I_{2n}] \begin{bmatrix} u_x \\ v_x \\ u \\ v \end{bmatrix} = [0_{2n}, I_{2n}] e^{K_\lambda x} Z \neq 0 \tag{27}$$

and satisfies

$$\mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}$$

In other words, $\lambda \in \sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.

To sum up, we have obtained the following Theorem.

Theorem 1: *Let $h(\lambda) = detH(\lambda)$ be defined by 24. Then $h(\lambda)$ is an entire function of λ, and the following statements hold:*

1. *$\lambda \in \sigma(\mathcal{A})$ if and only if $h(\lambda) = 0$, i.e.,*

   $$\sigma(\mathcal{A}) = \{\lambda | h(\lambda) = 0\}. \tag{28}$$

   The eigenvalues are symmetric with respect to the real axis.

2. *For each $\lambda \in \sigma(mathcalA)$, the corresponding eigenfunction $[u, v]^T$ is given by 27, where Z is any nonzero solution of the algebraic equation $H(\lambda)Z = 0$.*

3. *$\mathcal{A}$ is a densely defined discrete operator in $\mathcal{H}$, i.e., $\mathcal{A}$ is densely defined in $\mathcal{H}$ and $R(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ is compact for any $\lambda \in \rho(\mathcal{A})$.*

4. *$\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup in $\mathcal{H}$.*

3 Sliding Model Control

Let us establish a sliding model control for the system (15)

$$\begin{cases} \dfrac{\partial z}{\partial t} = \mathcal{A}z + Bw(z, t) \\ z(0) = z_0 \end{cases} \tag{1}$$

where B is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$, $w(z, t)$ is the control of the system (1) that is not continuous on the manifold $S = Cz = 0$, and C is a bounded linear operator with $S = S(z) = Cz \in R^n$.

Now, we consider the δ-neighborhood of sliding mode $S = Cz = 0$, where $\delta > 0$ is an arbitrary given positive number. Using a continuous control $\tilde{w}(z,t)$ to replace $w(z,t)$ in the system 1 yields

$$\begin{cases} \dot{z} = \mathcal{A}z + B\tilde{w}(z,t) \\ z(0) = z_0 \end{cases} \tag{2}$$

where $\dot{z} = \partial z/\partial t$, and the solution of (2) belongs to the boundary layer $\|S(z)\| \leq \delta$

Let $\dot{S}(z) = c\dot{z} = 0$. Applying C to the first equation of (1) leads to the following the equivalent control:

$$w_{eq}(z,t) = -(CB)^{-1}C(\mathcal{A}z)$$

with assumption that $(CB)^{-1}$ exists. Substitute $w_{eq}(z,t)$ into 1 to find

$$\dot{z} = [I - B(CB)^{-1}C]\mathcal{A}z. \tag{3}$$

Denote $P = B(CB)^{-1}C$ and $\mathcal{A}_0 = (I-P)\mathcal{A}$, then 1 becomes

$$\dot{z} = \mathcal{A}_0 z, \qquad z(0) = z_0 \tag{4}$$

In the rest part of this paper, we are going to show that the actual sliding mode $Z(t)$ will approach uniformly to the ideal sliding mode $\overline{Z}(t)$ under certain conditions.

If $(CB)^{-1}$ is a compact operator and $P\mathcal{A} = \mathcal{A}P$, then $\mathcal{A}_0 = (I-P)\mathcal{A}$ generates a C_0-semigroup $T_2(t)$ in $\mathcal{H}$ and $T_2(t) = (I-p)T_1(t)$, where $T_1(t)$ is the C_0-semigroup generated by $\mathcal{A}$.

Since $(CB)^{-1}$ is a compact operator, B and C are bounded linear operators, we see from the definition of P that P is compact, and therefor the range of $I-P$ is a closed subspace of $\mathcal{H}$. Since $P^2 = P$ and $(1-P)^2 = I-P$, $I-P$ can be viewed as the identity operator on $(I-P)\mathcal{H}$. It can be easily seen that $T_2(t) = (I-P)T_1(t)$ is a C_0-semigroup in $(I-P)\mathcal{H}$.

Next, we shall prove that the infinitesimal generator of $T_2(t)$ is $(I-P)\mathcal{A}$ and $\mathcal{D}((I-P)\mathcal{A}) = (I-P)\mathcal{D}(\mathcal{A})$.

In fact, for every $x \in (I-P)\mathcal{D}(\mathcal{A})$, there is a $x_1 \in \mathcal{D}(\mathcal{A})$ such that $x = (I-P)x_1$. It should be noted that $T_1(t)$ and $I-P$ are commutative because $\mathcal{A}$ and P are commutative. We see that

$$\begin{aligned}\lim_{t\to 0^+}\frac{T_2(t)x-x}{t} &= \lim_{t\to 0^+}\frac{(I-P)T_1(t)(I-P)x_1-(I-P)x_1}{t}\\ &= \lim_{t\to 0^+}\frac{(I-P)^2T_1(t)x_1-(I-P)x_1}{t}\\ &= \lim_{t\to 0^+}\frac{(I-P)T_1(t)x_1-(I-P)x_1}{t}\\ &= (I-P)\lim_{t\to 0^+}\frac{T_1(t)x_1-x_1}{t}\\ &= (I-P)\mathcal{A}x_1.\end{aligned}$$

Let $\tilde{\mathcal{A}}$ be the infinitesimal generator of $T_2(t)$. Since the limit on the left exists, we can assert that $x\in\mathcal{D}(\tilde{\mathcal{A}})$ and $(I-P)\mathcal{D}(\mathcal{A})\subseteq\mathcal{D}(\tilde{\mathcal{A}})$.

On the other hand, for any $x\in\mathcal{D}(\tilde{\mathcal{A}})$, since $\mathcal{D}(\tilde{\mathcal{A}})\subseteq(I-P)\mathcal{H}$, there exists $\tilde{x}\in\mathcal{H}$, such that $x=(I-P)\tilde{x}$, and

$$\begin{aligned}\lim_{t\to 0^+}\frac{T_2(t)x-x}{t} &= \lim_{t\to 0^+}\frac{T_2(t)(I-P)\tilde{(x)}-(I-P)\tilde{(x)}}{t}\\ &= \lim_{t\to 0^+}\frac{(I-P)T_1(t)\tilde{x}-(I-P)\tilde{x}}{t}\\ &= (I-P)\lim_{t\to 0^+}\frac{T_1(t)\tilde{x}-\tilde{x}}{t}\\ &= (I-P)\mathcal{A}\tilde{x}.\end{aligned}$$

Since the limit of the left hand side exists, and so the limit of the right hand side exists, and $\tilde{x}\in\mathcal{D}(\mathcal{A})$ which implies that $\mathcal{D}(\tilde{\mathcal{A}})\subseteq(I-P)\mathcal{D}(\mathcal{A})$. Thus, $\mathcal{D}(\tilde{\mathcal{A}})=(I-P)\mathcal{D}(\mathcal{A})$ and $\tilde{\mathcal{A}}$, the infinitesimal generator of $T_2(t)$, is $(I-P)\mathcal{A}$.

The proof of the lemma is complete.

Suppose that in the system 1,

1. $(CB)^{-1}$ exists and it is compact,
2. $P\mathcal{A}=\mathcal{A}P$, where $P=B(CB)^{-1}C$.

Then for any solution $z(t)$ of the system 4 satisfying $S(\overline{z}_0)=0$, $\overline{z}_0\in\mathcal{D}(\mathcal{A}_0)$ and $\|z_0-\overline{z}_0\|\leq\delta$, $z_0\in\mathcal{D}(\mathcal{A})$, we have

$$\lim_{\delta\to 0}\|z(t)-\overline{z}(t)\|=0$$

uniformly on $[0,T]$ for any positive number T.

We see from the Theorem 2 and Lemma 3 that $\mathcal{A}$ and $\mathcal{A}_0=(I-P)\mathcal{A}$ are infinitesimal generators of C_0-semigroups $T_1(t)$ and $T_2(t)$ respectively. It follows from theory of semigroup of linear operators that there are positive constants $M_1,\ M_2,\ \omega_1$ and ω_2 such that

$$\|T_1(t)\| \leq M_1 e^{\omega_1 t}, \quad \|T_2(t)\| \leq M_2 e^{\omega_2 t}. \quad (0 \leq t \leq T) \tag{5}$$

In the boundary layer $\|T_1(t)\| \leq \delta$, the equivalent control is

$$w_{eq}(z,t) = -(CB)^{-1}C\mathcal{A}z + (CB)^{-1}C\dot{z} \tag{6}$$

Substitute (6) into (1) to find

$$\dot{z} = (I - P)\mathcal{A}z + P\dot{z} \tag{7}$$

Hence, the solution of (7) can be expressed as follows:

$$z(t) = T_2(t)z_0 + \int_0^t T_2(t-s)P\dot{z}(s)ds, \tag{8}$$

and the solution of (4) can be written as

$$\overline{z}(t) = T_2(t)\overline{z}_0 \tag{9}$$

Substracting (9) from (8) yields

$$z(t) - \overline{z}(t) = T_2(t)(z_0 - \overline{z}_0) + \int_0^t T_2(t-s)P\dot{z}(s)ds \tag{10}$$

Since $P\mathcal{A} = \mathcal{A}P$, we see that $PT_1(t) = PT_1(t)$. It should be emphasized that $(I-P)P = 0$ and $T_2(t) = (I-P)T_1(t)$, and consequently,

$$\begin{aligned}\int_0^t T_2(t-s)P\dot{z}(s)ds &= \int_0^t (I-P)T_1(t-s)P\dot{z}(s)ds \\ &= \int_0^t T_1(t-s)(I-P)P\dot{z}(s)ds \\ &= 0\end{aligned}$$

It can be obtained from (10) and (5) that

$$\|z(t) - \overline{z}(t)\| \leq \|T_2(t)\|\|z_0 - \overline{z}_0\| \leq M_2 e^{\omega_2 T}\|z_0 - \overline{z}\|,$$

Since $\|z_0 - z_0\| \leq \delta$, we have

$$\|z(t) - \overline{z}(t)\| \leq M_2 e^{\omega_2 T}\delta.$$

Thus,

$$\lim_{\delta \to 0}\|z(t) - \overline{z}_0\| = 0.$$

The proof of the theorem is complete.

We see from the Theorem 3 that the actual sliding mode can be approximated by ideal sliding mode in any accuracy.

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