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Boehmians and Elzaki Transform

S.K.Q. Al-Omari

Department of Applied Sciences, Faculty of Engineering Technology,
Al-Balqa Applied University, Amman 11134, Jordan
E-mail: s.k.q.alomari@fet.edu.jo

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Abstract

A motivation of the classical Sumudu transform " Elzaki transform " was presented as a closely related transform to the Laplace transform. In the present work, we extend the cited transform to a Schwartz space of distributions of compact support and retain its classical properties. The Elzaki transform is extended to the context of Boehmian spaces and, further, shown to be well defined and linear mapping in the banach space of Lebesgue integrable Boehmians. Certain theorem is also proved in some detail.

Keywords: *Distribution Space, Sumudu Transform, Boehmian Spaces, Elzaki Transform.*

1 Introduction

Integral transforms play an important role in many fields of science. In the literature, integral transforms are widely used in mathematical physics, optics, engineering mathematics and, few others. Among these transforms which were extensively used and applied on theory and applications are : the Mellin, Hankel, Laplace and Sumudu transforms, to name, but a few. The Sumudu transform is defined on a set A of functions

$$A=\left\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{t}{\tau_j}},\text{if } t\in(-1)^j\times[0,\infty]\right\} \tag{1.1}$$

by the formula [14,15,16]

$$F(\zeta)=Sf(\zeta)=_0^\infty f(\zeta t)\,e^{-t}dt,t\in(-\tau_1,\tau_2). \tag{1.2}$$

The Sumudu transform has a strong relationship with other integral transforms. In particular, the relationship between the Sumudu transform and Laplace transform have been established by kilicman 2010. Recently, a motivation of the Sumudu transform, namely, Elzaki transform, is given by Elzaki [17, 18, 19]. The Elzaki transform of a function $f(t)$ over the set (1.1) of functions of exponential order, is given by [17, 18]

$$Ef(\zeta) = \zeta_0^{\infty} f(t)\, e^{\frac{-t}{\zeta}} dt, \zeta \in (-\tau_1, \tau_2). \tag{1.3}$$

In the above citations, the transform (1.3) is noted to facilitate the process of solving order and partial differential equations where examples are solved.

Let f be a function of exponential order. Let Lf and Ef be the Laplace and Elzaki transforms of f, respectively, then

$$Ef(\zeta) = \zeta Lf\left(\frac{1}{\zeta}\right). \tag{1.4}$$

and hence

$$Lf\left(\frac{1}{\zeta}\right) = \zeta E\left(\frac{1}{\zeta}\right). \tag{1.5}$$

Following, are considered as general properties of Elzaki transform.

(1) *If a and b are non-negative real numbers then*

$$E(af(t) + bg(t))(\zeta) = aEf(\zeta) + bEg(\zeta).$$

(2) $\lim_{t\to 0} f(t) = \lim_{\zeta\to 0} Ef(\zeta) = f(0)$.

The convolution product between two L^1 functions f and g is defined by

$$f * g(x) =_0^{\infty} f(t)\, g(x-t)\, dt \tag{1.6}$$

then

$$E(f*g)(\zeta) = \frac{f(\zeta)\, g(\zeta)}{\zeta}, \text{ see } [19].$$

2 The Elzaki Transform of Distributions

Let $\varepsilon(R_+)$ be the space of smooth functions of arbitrary support on R_+ and $\acute{\varepsilon}(R_+)$ be its strong dual of distributions of compact support. Denote by $D(R_+)$, the subspace of $\varepsilon(R_+)$ of test functions of compact support then its dual space $\acute{D}(R_+)$ consists of Schwartz distributions. Certainly, $D \subset \varepsilon$ and hence $\varepsilon \subset \acute{\varepsilon} \subset \acute{D}$. The kernel function $\zeta e^{\frac{-t}{\zeta}}$ of Elzaki transform is clearly a member of $\varepsilon(R_+)$. Hence, we define the generalized Elzaki transform $\hat{E}$ on $\acute{\varepsilon}(R_+)$ by the equation

$$\hat{E}f(\zeta) = \left\langle f(t), \zeta e^{\frac{-t}{\zeta}} \right\rangle. \tag{2.1}$$

for every distribution $f \in \acute{\varepsilon}(R_+)$.

Theorem 2.1. *$\hat{E}$ is a well-defined mapping in the space $\varepsilon(R_+)$.*

Proof. of this theorem is immediate, since $\zeta e^{\frac{-t}{\zeta}} \in \varepsilon(R_+)$.

Theorem 2.2. *$\hat{E}$ is infinitely smooth and*

$$\frac{d^k}{d\zeta^k}\hat{E}f(\zeta) = \left\langle f(t), \frac{d^k}{d\zeta^k}\left(\zeta e^{\frac{-t}{\zeta}}\right)\right\rangle.$$

for every $f \in \acute{\varepsilon}(R_+)$.

This theorem can be proved by an argument similar to that used in [9, Theorem 2.9.1.], detailed proof thus avoided. Next, is a theorem describing linearity of the distributional Elzaki transform.

Theorem 2.3. *$\hat{E}$ is linear.*

Let $f, g \in \acute{\varepsilon}(R_+)$. We define the generalized convolution between f and g by

$$\langle f * g(x), \psi(x)\rangle = \langle f(x), \langle g(t), \psi(x+t)\rangle\rangle. \tag{2.2}$$

for every $\psi \in \varepsilon(R_+)$. Hence using (2.1) and (2.2) together with simple calculations yields

$$\hat{E}(f * g)(\zeta) = \frac{\hat{E}f(\zeta)\,\hat{E}g(\zeta)}{\zeta}.$$

Theorem 2.4. *Let $f \in \acute{\varepsilon}(R_+)$ and $g(t) = \begin{cases} f(t-\tau), & t \geq \tau \\ 0, & t < \tau \end{cases}$ then*

$$\hat{E}g(\zeta) = e^{\frac{-\tau}{\zeta}}\hat{E}f(\zeta).$$

Proof. It is clear that $g \in \acute{\varepsilon}(R_+)$. The translation property of distributions through τ [9, p.26], implies

$$\hat{E}g(\zeta) = \left\langle f(t-\tau), \zeta e^{-t/\zeta}\right\rangle = e^{-\tau/\zeta}\hat{E}f(\zeta).$$

Hence, the theorem.

Theorem 2.5. *Let $f \in \acute{\varepsilon}(R_+)$ then the following holds*

(1) $\hat{E}(tf(t))(\zeta) = \zeta^2\frac{d}{d\zeta}\hat{E}f(\zeta) - \zeta\hat{E}f(\zeta)$.

(2) $\hat{E}(t^2 f(t))(\zeta) = \zeta^4\frac{d^2}{d\zeta^2}\hat{E}f(\zeta)$.

Proof. Considering properties of Elzaki transform (2.1) and Theorem 2.2 , we get

$$\frac{d}{d\zeta}\hat{E}(\zeta)=\frac{d}{d\zeta}\left\langle f(t),\zeta e^{-t/\zeta}\right\rangle=\left\langle f(t),\frac{d}{d\zeta}\left(\zeta e^{-t/\zeta}\right)\right\rangle$$

Differentiating inside the inner product yields

$$\frac{d}{d\zeta}\hat{E}(\zeta)=\left\langle f(t),\frac{t}{\zeta}e^{-t/\zeta}+e^{-t/\zeta}\right\rangle$$

Properties of distributions imply

$$\frac{d}{d\zeta}\hat{E}(\zeta)=\left\langle tf(t),\frac{1}{\zeta}e^{-t/\zeta}\right\rangle+\left\langle f(t),e^{-t/\zeta}\right\rangle$$

Multiplying both sides by ζ^2 and rearranging complete the proof of the first part of the theorem. Proof of the second part is similar. Hence we avoid the same. This proof is therefore completed.

Theorem 2.6. (Shifting Theorem) . *Let* $f\in\acute{\varepsilon}(R_+)$ *then*

$$\hat{E}\left(e^{at}f(t)\right)(\zeta)=\tfrac{1}{1-a\zeta}\hat{E}\left(\tfrac{\zeta}{1-a\zeta}\right).$$

The proof is straightforward .

3 Boehmians

Let G be a linear space and S be a subspace of G. We assume that to each pair of elements $f\in G$ and $\phi\in S$, is assigned the product $f*g$ such that the following conditions are satisfied:

(1) $\phi,\psi\in S\Rightarrow\phi*\psi\in S$ and $\phi*\psi=\psi*\phi$.

(2) $f\in G$, $\phi,\psi\in S\Rightarrow(f*\phi)*\psi=f*(\phi*\psi)$.

(3)If $f,g\in G,\phi\in S$ and $\lambda\in R$, then $(f+g)*\phi=f*\phi+g*\phi$ and $\lambda(f*\phi)=(\lambda f)*\phi$. Let Δ be a family of sequences from S, such that

(1)*If* $f,g\in G,(\gamma_n)\in\Delta$ *and* $f*\gamma_n=g*\gamma_n\ (n=1,2,...)$, *then* $f=g$.

(2) $(\gamma_n),(\tau_n)\in\Delta\Rightarrow(\gamma_n*\tau_n)\in\Delta$.

then each elements of Δ will be called *delta sequence.*

Consider the class A of pairs of sequences defined by

$$A=\left\{((f_n),(\gamma_n)):(f_n)\subseteq G^N,(\gamma_n)\in\Delta\right\},$$

for each $n\in\mathbf{N}$. Then, an element $((f_n),(\gamma_n))\in A$ is called a *quotient of sequences*, denoted by $\frac{f_n}{\gamma_n}$if

$$f_i*\gamma_j=f_j*\gamma_i,\forall i,j\in\mathbf{N}.$$

Two quotients of sequences $\frac{f_n}{\gamma_n}$ and $\frac{g_n}{\tau_n}$ are said to be *equivalent,* $\frac{f_n}{\gamma_n} \sim \frac{g_n}{\tau_n}$, if

$$f_i * \gamma_j = g_j * \tau_i, \forall i, j \in \mathbf{N}.$$

The relation $\sim$ is an equivalent relation on A and hence, splits A into equivalence classes. The equivalence class containing $\frac{f_n}{\gamma_n}$ is denoted by $\left[\frac{f_n}{\gamma_n}\right]$.These equivalence classes are called *Boehmians* and the *space of all Boehmians* is denoted by H.

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way

$$\left[\frac{f_n}{\gamma_n}\right] + \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n*\tau_n+g_n*\gamma_n}{\gamma_n*\tau_n}\right]$$

and

$$a\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{af_n}{\gamma_n}\right], a \in C.$$

The operation $*$ and the differentiation are defined by

$$\left[\frac{f_n}{\gamma_n}\right] * \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n*g_n}{\gamma_n*\tau_n}\right]$$

and

$$D^\alpha\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{D^\alpha f_n}{\gamma_n}\right].$$

Many a time, G is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $*$ are given by:

(1) *If* $f_n \to f$ *as* $n \to \infty$ *in* G *and,* $\phi \in S$ *is any fixed element, then*

$$f_n * \phi \to f * \phi \text{ in } G \ (\text{as } n \to \infty).$$

(2) *If* $f_n \to f$ *as* $n \to \infty$ *in* G *and* $(\gamma_n) \in \Delta$, *then*

$$f_n * \gamma_n \to f \text{ in } G \ (\text{as } n \to \infty).$$

The operation $*$ can be extended to $H \times S$ by the following definition.

Definition 3.1. *If* $\left[\frac{f_n}{\gamma_n}\right] \in H$ *and* $\phi \in S$ *,then* $\left[\frac{f_n}{\gamma_n}\right] * \phi = \left[\frac{f_n*\phi}{\gamma_n}\right]$.

In H, two types of convergence, $\delta-$convergence and $\Delta-$convergence, are defined as follows:

Definition 3.2. *A sequence of Boehmians* (β_n) *in* H *is said to be* $\delta-$*convergent to a Boehmian* β *in* H, *denoted by* $\beta_n \xrightarrow{\delta} \beta$, *if there exists a delta sequence* (γ_n) *such that*

$$(\beta_n * \gamma_n), (\beta * \gamma_n) \in G, \forall k, n \in \mathbf{N},$$

and

$$(\beta_n * \gamma_k) \to (\beta * \gamma_k) \ \text{ as } n \to \infty, \text{in } G, \ \text{for every } k \in \mathbf{N}.$$

The following lemma is equivalent for the statement of $\delta-$*convergence*

Lemma 3.3. *$\beta_n \overset{\delta}{\to} \beta$ $(n \to \infty)$ in H if and only if there is $f_{n,k}, f_k \in G$ and $\gamma_k \in \Delta$ such that $\beta_n = \left[\frac{f_{n,k}}{\gamma_k}\right], \beta = \left[\frac{f_k}{\gamma_k}\right]$ and for each $k \in \mathbf{N}$,*

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } G.$$

Definition 3.4. *A sequence of Boehmians (β_n) in H is said to be $\Delta-$convergent to a Boehmian β in H, denoted by $\beta_n \overset{\Delta}{\to} \beta$, if there exists a $(\gamma_n) \in \Delta$ such that $(\beta_n - \beta) * \gamma_n \in G, \forall n \in \mathbf{N}$, and $(\beta_n - \beta) * \gamma_n \to 0$ as $n \to \infty$ in G.* see [1-5,78,10,11,13]

4 The Elzaki Transform of Boehmians

Let $G = L^1(R_+)$ and $S = D(R_+)$. Let Δ be the collection of sequences (γ_n) from $D(R_+)$ such that

(1)$_{R_+} \gamma_n(t)\, dt = 1$.

(2) $\|\gamma_n\|_{L^1} < B$ for all $(\gamma_n) \in \Delta$ where B is certain positive constant.

(3)$_{|x|>\epsilon} |\gamma_n(t)|\, dt \to 0$ as $n \to \infty, \epsilon > 0$.

The corresponding space of Boehmians $H(L^1, D, *, \Delta)$ is a convolution algebra with the operations

$$\left[\frac{f_n}{\gamma_n}\right] + \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n * \tau_n + g_n * \gamma_n}{\gamma_n * \tau_n}\right]. \tag{4.1}$$

and

$$a\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{a f_n}{\gamma_n}\right], a \in R. \tag{4.2}$$

The operation $*$ in $H(L^1, D, *, \Delta)$ can be defined by

$$\left[\frac{f_n}{\gamma_n}\right] * \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n * g_n}{\gamma_n \tau_n}\right]. \tag{4.3}$$

Differentiation in $H(L^1, D, *, \Delta)$ is defined by

$$D^k\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{D^k f_n}{\gamma_n}\right], k \in \mathbf{N}. \tag{4.4}$$

If $(\gamma_n) \in \Delta$ then, certainly, $E\gamma_n(\zeta) \to \zeta$ as uniformly $n \to \infty$, on compact subsets of R_+.

Lemma 4.1. *If* $f_n \in L^1$ *such that* $\left[\frac{f_n}{\gamma_n}\right] \in H\left(L^1, D, *, \Delta\right)$ *then*

$$Ef_n\left(\zeta\right) =_{R_+} \zeta e^{\frac{-t}{\zeta}} f_n\left(t\right) dt.$$

converges uniformly on each compact set of R_+.

Proof. Since $E\gamma_n \to \zeta$ as $n \to \infty$ on compact subsets of $R_+, E\gamma_n > 0$ for almost all $k \in \mathbf{N}$ and hence

$$\begin{aligned} Ef_n\left(\zeta\right) &= Ef_n\left(\zeta\right)\frac{E\gamma_k(\zeta)}{E\gamma_k(\zeta)} \\ &= \frac{\zeta E(f_n*\gamma_k)(\zeta)}{E\gamma_n(\zeta)} \\ &= \frac{\zeta E(f_k*\gamma_n)}{E\gamma_n(\zeta)} \\ &= \frac{Ef_k(\zeta)}{E\gamma_k(\zeta)} E\gamma_n\left(\zeta\right) \text{ on } K. \end{aligned}$$

where K is certain compact subset of R_+. Considering limit as $n \to \infty$ we get $Ef_n\left(\zeta\right) \to \frac{\zeta Ef_k(\zeta)}{E\gamma_k(\zeta)}$.

From Theorem 4.1 we define the Elzaki transform of $\beta \in H\left(L^1, D, *, \Delta\right)$, where $\beta = \left[\frac{f_n}{\gamma_n}\right]$, by the formula

$$\tilde{E}\beta = \lim_{n\to\infty} Ef_n,$$

on compact subsets of R_+. Now, we show the above definition is well defined. For, if $\beta_1 = \beta_2$ where, $\beta_1 = \left[\frac{f_n}{\gamma_n}\right], \beta_2 = \left[\frac{g_n}{\tau_n}\right]$ then $f_n * \tau_m = g_m * \gamma_n = g_n * \gamma_m$. Employing Elzaki transform on both sides yields

$$\frac{Ef_n\left(\zeta\right) E\tau_m\left(\zeta\right)}{\zeta} = \frac{Eg_n\left(\zeta\right) E\gamma_m\left(\zeta\right)}{\zeta}.$$

Hence, allowing $m \to \infty$ yidels

$$\lim_{n\to\infty} Ef_n\left(\zeta\right) = \lim_{n\to\infty} Eg_n\left(\zeta\right)$$

Hence

$$\tilde{E}\beta_1 = \tilde{E}\beta_2.$$

This completes the proof.

Theorem 4.2. *Let* $x_1, x_2 \in H\left(L^1, D, *, \Delta\right)$ *and* $a \in C$ *then*

(1) $\tilde{E}\left(x_1 + x_2\right) = \tilde{E}x_1 + \tilde{E}x_2$.
(2) $\tilde{E}\left(ax_1\right) = a\tilde{E}x_1$.
(3) $\tilde{E}\left(x_1 * \gamma_n\right) = \tilde{E}\left(\gamma_n * x_1\right) = \frac{1}{\zeta}\tilde{E}x_1$.

$(4)\, \tilde{E}x_1 = 0 \Rightarrow x_1 = 0.$
$(5)\, x_n \overset{\Delta}{\to} x \in H\left(L^1, D, *, \Delta\right) \Rightarrow$

$$\tilde{E}x_n \overset{\Delta}{\to} \tilde{E}x \in H\left(L^1, D, *, \Delta\right) \ \textit{as } n \to \infty$$

on compact subsets.

Proof. of Parts $(1-2)$, and (4) follows from the corresponding properties of the classical Elzaki transform. Proof of Part(3): Let $x_1 \in H\left(L^1, D, *, \Delta\right)$ such that $x_1 = \left[\frac{f_n}{\gamma_n}\right]$ then $x_1 * \gamma_n = \left[\frac{f_n * \gamma_n}{\gamma_n}\right]$. Hence,

$$\tilde{E}\left(x_1 * \gamma_n\right) = \frac{1}{\zeta} \lim_{n\to\infty} E f_n\left(\zeta\right) = \frac{1}{\zeta} \tilde{E} x_1.$$

Finally, the proof of Part (5) have analysis similar to that employed in the proof of Part (f) of from [7,Theorem 2]. This completes the proof of the theorem.

References

[1] S.K.Q. Al-Omari, D. Loonker, P.K. Banerji and S.L. Kalla, Fourier sine(cosine) transform for ultradistributions and their extensions to tempered and ultra Boehmian spaces, *Integral Transforms Spec. Funct.*, 19(6) (2008), 453–462.

[2] S.K.Q. Al-Omari, The generalized stieltjes and fourier transforms of certain spaces of generalized functions, *Jord. J. Math. Stat.*, 2(2) (2009), 55-66.

[3] S.K.Q. Al-Omari, On the distributional mellin transformation and its extension to Boehmian spaces, *Int. J. Contemp. Math. Sciences*, 6(17) (2011), 801-810.

[4] S.K.Q. Al-Omari, A mellin transform for a space of Lebesgue integrable Boehmians, *Int. J. Contemp. Math. Sciences*, 6(32) (2011), 1597-1606.

[5] T.K. Boehme, The support of Mikusinski operators, *Tran. Amer. Math. Soc.*, 176(1973), 319-334.

[6] P.K. Banerji, S.K.Q. Al-Omari and L. Debnath, Tempered distributional Fourier sine(cosine)transform, *Integral Transforms Spec. Funct.*, 17(11) (2006),759-768.

[7] P. Mikusinski, Fourier transform for integrable Boehmians, *Rocky Mountain J. Math.*, 17(3) (1987), 577-582.

[8] P. Mikusinski, Tempered Boehmians and ultradistributions, *Proc. Amer. Math. Soc.*, 123(3) (1995), 813-817.

[9] R.S. Pathak, *Integral Transforms of Generalized Functions and Their Applications*, Gordon and Breach Science Publishers, Australia, Canada, India, Japan, (1997).

[10] R. Roopkumar, Mellin transform for Boehmians, *Bull. Institute of Math.*, Academica Sinica, 4(1) (2009), 75-96.

[11] R. Roopkumar, Stieltjes transform for Boehmians, *Integral Transf. Spl. Funct.*, 18(11) (2007), 845-853.

[12] A.H. Zemanian, *Generalized Integral Transformation*, Dover Publications, Inc., New York, (1987), First published by Interscience Publishers, New York, (1968).

[13] P. Mikusinski, Convergence of Boehmianes, *Japan. J. Math*, 9(1) (1983), 169-179.

[14] G.K. Watugala, Sumudu transform: A new integral transform to solve differential equations and control engineering problems, *Int. J. Math. Edu. Sci. Technol.*, 24(1) (1993), 35-43.

[15] S. Weerakoon, Application of Sumudu transform to partial differential equations, *Int. J. Math. Edu. Sci. Technol.*, 25(1994), 277-283.

[16] F.B.M. Belgasem, A.A. karaballi and S.L. Kalla, Analytical investigations of the Sumudu transform and applications to integral integral production equations, *Math. probl. Ing.*, 3-4(2003), 103-118.

[17] T.M. Elzaki and S.M. Elzaki, On the Elzaki transform and higher order ordinary differential equations, *Advan. Theor. Appl. Math.*, 6(1) (2011), 107-113.

[18] T.M. Elzaki and S.M. Elzaki, Application of new transform “Elzaki transform” to partial differential equations, *Glob. J. Pure. Appl. Mat.*, 7(1) (2011), 65-70.

[19] T.M. Elzaki and S.M. Elzaki, On the connections between Laplace and Elzaki transforms, *Advan. Theor. Appl. Math.*, 6(1) (2011), 1-10.