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# Some Partial Metric Spaces of Sequences and Functions

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**Abstract**

*In this paper, we investigate some notions of the classical sets of sequences and functions by using the partial metrics with respect to the partial ordering. Also, we examine the completeness of these spaces and obtain the alpha-, beta- and gamma-duals of some of these. We investigate the relationships between these sets and their classical forms and give some properties including definitions, propositions and various kinds of partial metric spaces. Finally, we show that each of the sets forms a vector space on the real field and present some results on the completeness of these partial metric spaces.*

**Keywords:** *Sequence and function spaces, metric space, partial metric space, complete partial metric space.*

# 1 Introduction

A partially ordered set (or poset) is a pair $(X, \sqsubseteq)$ such that $\sqsubseteq$ is a partial ordering on $X$. For each partial metric space $(X, p)$ let $\sqsubseteq_p$ be the binary relation over $X$ such that $x \sqsubseteq_p y$ (to be read, $x$ is part of $y$) if and only if $p(x, x) = p(x, y)$. For the partial metric $\max(\min)\{a, b\}$ over the nonnegative reals, $\sqsubseteq_{\max}$ ($\sqsubseteq_{\min}$) is the usual ordering $\geq$ ($\leq$). For intervals, $[a, b] \sqsubseteq_p [c, d]$ if and only if $[c, d]$ is a subset of $[a, b]$.

By $\omega$, we denote the space of all real valued sequences and any subspace of $w$ is called a *sequence space*. Firstly, we define the classical sets $\ell_\infty(P)$, $c(P)$, $c_0(P)$ and $\ell_q(P)$ consisting of the bounded, convergent, null and $q$-absolutely summable sequences by using the partial metric $p$ with respect to the partial ordering $\sqsubseteq_p$, as follows:

$$\begin{aligned}
\ell_\infty(P) &:= \left\{x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} \left\{p^s(x_k, 0)\right\} < \infty\right\}, \\
c(P) &:= \left\{x = (x_k) \in \omega : \exists l \in \mathbb{R} \ni \lim_{k \to \infty} p^s(x_k, l) = 0\right\}, \\
c_0(P) &:= \left\{x = (x_k) \in \omega : \lim_{k \to \infty} p^s(x_k, 0) = 0\right\}, \\
\ell_q(P) &:= \left\{x = (x_k) \in \omega : \sum_{k=0}^{\infty} p^s(x_k, 0)^q < \infty\right\}, \quad (1 \leq q < \infty),
\end{aligned}$$

where the distance function $p^s$ denotes the usual metric with $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ induced by the partial metric $p$. One can show that $c(P)$, $c_0(P)$ and $\ell_\infty(P)$ are complete metric spaces with the partial metric $p_\infty$ with respect to the partial ordering $\sqsubseteq_p$ defined by

$$p_\infty(x, y) := \sup_{k \in \mathbb{N}} \left\{p^s(x_k, y_k)\right\},$$

where $x = (x_k)$ and $y = (y_k)$ are the elements of the sets $c_0(P)$, $c(P)$ or $\ell_\infty(P)$. Also, the space $\ell_q(P)$ is complete metric space with the partial metric $p_q$ defined by

$$p_q(x, y) := \left[\sum_{k=0}^{\infty} p^s(x_k, y_k)^q\right]^{1/q}, \quad (1 \leq q < \infty),$$

where $x = (x_k)$ and $y = (y_k)$ are the points of $\ell_q(P)$.

Secondly, we construct the sets $bs(P)$, $cs(P)$ and $cs_0(P)$ consisting of the sets of all bounded, convergent, null series by using the partial metric $p$, as follows:

$$\begin{aligned}
bs(P) &:= \left\{x=(x_k)\in w : \sup_{n\in\mathbb{N}} p^s\left(\sum_{k=0}^{n} x_k, 0\right) < \infty\right\},\\
cs(P) &:= \left\{x=(x_k)\in w : \exists l\in\mathbb{R} \ni \lim_{n\to\infty} p^s\left(\sum_{k=0}^{n} x_k, l\right) = 0\right\},\\
cs_0(P) &:= \left\{x=(x_k)\in w : \lim_{n\to\infty} p^s\left(\sum_{k=0}^{n} x_k, 0\right) = 0\right\}.
\end{aligned}$$

One can conclude that the spaces $bs(P)$, $cs(P)$ and $cs_0(P)$ are complete metric spaces with the partial metric $P_\infty$ with respect to the partial ordering $\sqsubseteq_p$ defined by

$$P_\infty(x,y) := \sup_{n\in\mathbb{N}}\left\{p^s\left(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k\right)\right\},$$

where $x=(x_k)$ and $y=(y_k)$ are the elements of the sets $bs(P)$, $cs(P)$ or $cs_0(P)$.

Thirdly, we introduce the space $bv(P)$, $bv_q(P)$ and $bv_\infty(P)$ consisting of sequences of q-bounded variation by using the partial metric $p$ with respect to the partial ordering $\sqsubseteq_p$, as follows:

$$\begin{aligned}
bv(P) &:= \left\{x=(x_k)\in w : \sum_{k=0}^{\infty} p^s\left[(\Delta x)_k, 0\right] < \infty\right\},\\
bv_q(P) &:= \left\{x=(x_k)\in w : \sum_{k=0}^{\infty} p^s\left[(\Delta x)_k, 0\right]^q < \infty\right\},\\
bv_\infty(P) &:= \left\{x=(x_k)\in w : \sup_{k\in\mathbb{N}}\{p^s\left[(\Delta x)_k, 0\right]\} < \infty\right\}.
\end{aligned}$$

One can easily see that the sets $bv(P)$, $bv_q(P)$ and $bv_\infty(P)$ are complete metric spaces with the following partial metrics,

$$\begin{aligned}
P_\Delta(x,y) &:= \sum_{k=0}^{\infty}\left\{p^s\big[(\Delta x)_k, (\Delta y)_k\big]\right\},\\
P_q^\Delta(x,y) &:= \left\{\sum_{k=0}^{\infty} p^s\big[(\Delta x)_k, (\Delta y)_k\big]^q\right\}^{1/q},\\
P_\infty^\Delta(x,y) &:= \sup_{k\in\mathbb{N}}\left\{p^s\big[(\Delta x)_k, (\Delta y)_k\big]\right\},
\end{aligned}$$

respectively, where $x = (x_k)$ and $y = (y_k)$ are the elements of the sets $bv(P)$, $bv_q(P)$ or $bv_\infty(P)$, and $(\Delta x)_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Finally, we give the classical sets $B[a,b]$ and $C[a,b]$ consisting of the bounded and continuous functions defined on $[a,b]$, by using the partial metric $p$ with respect to the partial ordering $\sqsubseteq_p$, as follows:

$$\begin{aligned} B[a,b] &:= \left\{ f | f : [a,b] \overset{\text{bounded}}{\longrightarrow} \mathbb{R}^+ \right\} \\ C[a,b] &:= \left\{ f | f : [a,b] \overset{\text{continuous}}{\longrightarrow} \mathbb{R}^+ \right\}. \end{aligned}$$

It can be shown by a routine verification that $B[a,b]$ and $C[a,b]$ are complete partial metric spaces with the partial metric $p_1$ and $p_2$ defined by

$$\begin{aligned} p_1(f,g) &:= \sup_{t\in[a,b]} p^s[f(t), g(t)], \\ p_2(f,g) &:= \max_{t\in[a,b]} p^s[f(t), g(t)], \end{aligned}$$

respectively, where $f, g$ are bounded and continuous functions on $[a,b]$.

The main purpose of the present paper is to study the corresponding sets $\ell_\infty(P)$, $c(P)$, $c_0(P)$, $\ell_q(P)$, $bs(P)$, $cs(P)$, $cs_0(P)$, $bv(P)$, $bv_q(P)$, $bv_\infty(P)$ of sequences and the sets $C[a,b]$ and $B[a,b]$ of functions to the classical spaces. The rest of this paper is organized, as follows:

In section 2, some required definitions and consequences related with non-zero self distance, partial order sets, weighted space, quasi-metric space, partial Hausdorff metric and some topological properties are given. Section 3 is devoted to the completeness of the sets $\ell_\infty(P)$, $c(P)$, $c_0(P)$, $\ell_q(P)$, $bs(P)$, $cs(P)$, $cs_0(P)$, $bv(P)$, $bv_q(P)$, $bv_\infty(P)$ of sequences and the sets $C[a,b]$, $B[a,b]$ of functions with the partial metrics by taking into account the partially ordering together some related examples. Additionally, in this section we define the norm function with respect to the partial metric $p^s$ induced by the partial metric $p$, and we show that the sets $\ell_\infty(P)$, $c(P)$, $c_0(P)$ and $\ell_q(P)$ of sequence are Banach spaces with the related norms. In the final section of the paper, we also define the alpha-, beta- and gamma-duals of the sets $\ell_\infty(P)$, $c(P)$, $c_0(P)$, $\ell_1(P)$, $bs(P)$, $cs(P)$ and $cs_0(P)$ of sequences.

## 2 Preliminaries, Background and Notation

In 1992, a partial metric space is introduced as a generalisation of the notion of metric space defined in 1906 by Maurice Frechet such that the distance of a point from itself is not necessarily zero. This notion has a wide array of applications not only in many branches of mathematics, but also in the field of computer domain and semantics. Motivated by the needs of computer science for non Hausdorff Scott topology, one show that much of the essential structure of metric spaces, such as Banach's contraction mapping theorem, can be generalised to allow for the possibility of non zero self-distances $d(x,x)$.

Nonzero self-distance is thus motivated by experience from computer science, and seen to be plausible for the example of finite and infinite sequences. The question we now ask is whether nonzero self-distance can be introduced to any metric space. That is, is there a generalization of the metric space axioms to introduce nonzero self-distance such that familiar metric and topological properties are retained? The following is suggested.

**Proposition 2.1** [11] (Nonzero self-distance) *Let $S^\omega$ be the set of all infinite sequences $x = (x_0, x_1, x_2, \dots)$ over a set $S$. For all such sequences $x$ and $y$, let $d_s(x,y) = 2^{-k}$, where $k$ is the largest number (possibly $\infty$) such that $x_i = y_i$ for each $i < k$. Thus $d_s(x,y)$ is defined to be $1$ over $2$ to the power of the length of the longest initial sequence common to both $x$ and $y$. It can be shown that $(S^\omega, d_s)$ is a metric space.*

*To be interested in an infinite sequence $x$ they would want to know how to compute it, that is, how to write a computer program to print out the values $x_0$, then $x_1$, then $x_2$, and so on. As $x$ is an infinite sequence, its values cannot be printed out in any finite amount of time, and so computer scientists are interested in how the sequence $x$ is formed from its parts, the finite sequences $(x_0), (x_0, x_1), (x_0, x_1, x_2)$ and so on. After each value $x_k$ is printed, the finite sequence $x = (x_0, x_1, x_2, \dots, x_k)$ represents that part of the infinite sequence produced so far. Each finite sequence is thus thought of in computer science as being a partially computed version of the infinite sequence $x$, which is totally computed. Suppose now that the above definition of $d_s$ is extended to $S^*$, the set of all finite sequences over $S$. If $x$ is a finite sequence, then $d_s(x,x) = 2^{-k}$ for some number $k < \infty$ which is not zero, since $x_j = x_j$ can only hold if $x_j$ is defined. Thus, axiom P1 does not hold for finite sequences. This raises an intriguing contrast between 20th century mathematics of which the theory of metric spaces is our working example and the contemporary experience of computer science. The truth of the statement $x = x$ is surely unchallenged in mathematics, while in computer scienceits truth can only be asserted to the extent to which $x$ is computed.*

**Definition 2.2** [7] *Let $X$ be a non-empty set and $p$ be a function from $X \times X$ to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair $(X,p)$ is called a partial metric space and $p$ is a partial metric for $X$, if the following partial metric axioms are satisfied for all $x, y, z \in X$:*

*(P1)* $x = y$ *if and only if* $p(x,x) = p(x,y) = p(y,y)$.

*(P2)* $0 \le p(x,x) \le p(x,y)$.

*(P3)* $p(x, y) = p(y, x)$.

*(P4)* $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Each partial metric space thus gives rise to a metric space with the additional notion of nonzero self-distance introduced. Also, a partial metric space is a generalization of a metric space; indeed, if an axiom $p(x, x) = 0$ is imposed, then the above axioms reduce to their metric counterparts. Thus, a metric space can be defined to be a partial metric space in which each self-distance is zero.

It is clear that $p(x, y) = 0$ implies $x = y$ from (P1) and (P2). But, $x = y$ does not imply $p(x, y) = 0$, in general. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family open $p$-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

**Remark 2.3** [5] *Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(.,.)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, e.g., $x_n \to 2$ and $x_n \to 3$ as $n \to \infty$.*

**Proposition 2.4** *[12] If $p$ is a partial metric on $X$, then the function $p^s$ defined by*

$$\begin{array}{rcrcl} p^s & : & X \times X & \longrightarrow & \mathbb{R}^+ \\ & & (x, y) & \longmapsto & p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \end{array}$$

*is a usual metric on $X$. For example, in $(\mathbb{R}^-, p)$ where $p$ is the usual partial metric on $\mathbb{R}^-$, we obtain the usual distance in $\mathbb{R}^-$ since for any $x, y \in \mathbb{R}^-$, $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) = x + y - 2\min\{x, y\} = |x - y|$.*

**Definition 2.5** [11] *A partial ordering on $X$ is a binary relation $\sqsubseteq$ on $X$ such that*

*(i)* $x \sqsubseteq x$ *(reflexivity).*

*(ii) If* $x \sqsubseteq y$ *and* $y \sqsubseteq x$ *then* $x = y$ *(antisymmetry).*

*(iii) If* $x \sqsubseteq y$ *and* $y \sqsubseteq z$ *then* $x \sqsubseteq z$ *(transitivity).*

*A partially ordered set (or poset) is a pair $(X, \sqsubseteq)$ such that $\sqsubseteq$ is a partial ordering on $X$. For each partial metric space $(X, p)$; let $\sqsubseteq_p$ be the binary relation over $X$ such that $x \sqsubseteq_p y$ (to be read, $x$ is part of $y$) if and only if $p(x, x) = p(x, y)$. Then, it can be shown that $(X, \sqsubseteq_p)$ is a poset.*

**Definition 2.6** [5] *Let $X$ be a nonempty set. Then, $(X, p, \preceq)$ is called an ordered (partial) metric space if*

*(i)* $(X, p)$ *is a (partial) metric space,*

*(ii)* $(X, \preceq)$ *is a partially ordered set.*

**Definition 2.7** [5] *Let $(X, \preceq)$ be a partially ordered set. Then, the following statements hold:*

*(a) The elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.*

*(b) A subset $\mathcal{K}$ of $X$ is said to be well ordered if every two elements of $\mathcal{K}$ are comparable.*

*(c) A mapping $f : X \to X$ is called nondecreasing with respect to $\preceq$ if $x \preceq y$ implies $f(x) \preceq f(y)$.*

For the partial metric $\max\{a, b\}$ over the nonnegative reals, $\sqsubseteq_{\max}$ is reduced to the usual ordering $\geq$. For intervals, $[a, b] \sqsubseteq_p [c, d]$ if and only if $[c, d]$ is a subset of $[a, b]$. Thus the notion of a partial metric extends that of a metric by introducing nonzero self-distance which can be used to define the relation is part of which, for example, can be applied to model the output from a computer program.

**Definition 2.8** (cf. [8, 12, 13, 1]) *Let $(x_n)$ be a sequence in a partial metric space $(X, p)$. Then, we say that*

*(a) A sequence $(x_n)$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.*

*(b) A sequence $(x_n)$ is a Cauchy sequence if there exists (and is finite) $\lim_{m,n\to\infty} p(x_n, x_m)$.*

*(c) A partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $(x_n)$ in $X$ converges, with respect to the topology $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$. It is easy to see that every closed subset of a complete partial metric space is complete.*

*(d) A mapping $f : X \to X$ is called to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subset B_p(f(x_0), \varepsilon)$.*

*(e) A sequence $(x_n)$ in a partial metric space $(X, p)$ converges to a point $x \in X$, for any $\epsilon > 0$ such that $x \in B_p(x, \epsilon)$, there exists $n_0 \geq 1$ so that, $x_n \in B_p(x, \epsilon)$ for any $n \geq n_0$.*

A sequence $(x_n)$ in a partial metric space $(X, p)$, is called 0-Cauchy, if $\lim_{m,n\to\infty} p(x_m, x_n) = 0$. We say that $(X, p)$ is 0-complete if every 0-Cauchy sequence in $X$ converges, with respect to $p$, to a point $x \in X$ such that $p(x, x) = 0$. Note that every 0-Cauchy sequence in $(X, p)$ is Cauchy in $(X, p^s)$, and that every complete partial metric space is 0-complete. A paradigm for partial metric spaces is the pair $(X, p)$, where $X = \mathbb{Q} \cap [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \geq 0$ which provides an example of an incomplete 0-complete partial metric space.

**Lemma 2.9** [12] *Let $(X, p)$ be a partial metric space. Then,*

*(i) $(x_n)$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, p^s)$.*

*(ii) A partial metric space $(X, p)$ is complete if and only if the metric space $(X, p^s)$ is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{m,n\to\infty} p(x_n, x_m)$.*

In the partial metric space $(\mathbb{R}^-, p)$, the limit of the sequence $(-1/n)$ is 0 since one has $\lim_{n\to\infty} p^s(-1/n, 0)$, where $p^s$ is the usual metric induced by $p$ on $\mathbb{R}^-$.

**Lemma 2.10** [5] *Let $(X, p)$ be a partial metric space, $f : X \to X$ be a given mapping. Suppose that $f$ is continuous at $x_0 \in X$ and for each sequence $(x_n)$, if $x_n \to x_0$ in $(X, \tau_p)$ then $f(x_n) \to f(x_0)$ holds in $(X, \tau_p)$.*

**Definition 2.11** [1] *Suppose that $(X_1, p_1)$ and $(X_2, p_2)$ are partial metric spaces with induced metrics $p_1^s$ and $p_2^s$ respectively. Then the function $f : (X_1, p_1) \to (X_2, p_2)$ is said to be continuous if both $f : (X_1, \tau_{p_1}) \to (X_2, \tau_{p_2})$ and $f : (X_1, p_1^s) \to (X_2, p_2^s)$ are respectively continuous in the sense of topological and metric spaces.*

**Definition 2.12** [11] *A sequence $x = (x_n)$ of points in a partial metric space $(X, p)$ is Cauchy if there exists $a \geq 0$ such that for each $\epsilon > 0$ there exists $k$ such that $|p(x_n, x_m) - a| < \epsilon$ for all $n, m > k$. In other words, a sequence $x = (x_n)$ in a partial metric space $(X, p)$ is Cauchy if $\lim_{n,m\to\infty} p(x_n, x_n) = a$ implies $a = 0$ whenever $(X, p)$ is a metric space.*

**Definition 2.13** [11] *A sequence $x = (x_n)$ in a partial metric space $(X, p)$ converges to $y$ in $X$ if*

$$p(y, y) = \lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, y).$$

**Lemma 2.14** [10] *Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space $(X, p)$ such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

**Definition 2.15** *A sequence $(x_n)$ in a partial metric space $(X,p)$ is bounded if and only if there exists $M>0$ such that $p^s(x_n,0)\leq M$.*

Now, we give some definitions about the sets of bounded or continuous functions by taking into account the partial order $\sqsubseteq_p$ on $[a,b]$.

**Definition 2.16** *The sequence $\{f_n(t)\}$ of functions is said to be uniformly convergent to $f(t)$ on $[a,b]$, if for every $\epsilon>0$ and there exists $n_0=n_0(\epsilon)\in\mathbb{N}$, depending only on $\epsilon$, such that $p^s(f_n(t),0)<\epsilon$ for every $n>n_0$.*

**Lemma 2.17** *Let $(X,p)$ be a partial metric space and $f$ be a function from $X$ to $Y$. The function $f$ is said to be bounded if and only if there exists $M>0$ such that $p^s(f(t),0)\leq M$.*

**Lemma 2.18** *Let $(X,p)$ be partial metric space and $\{f_n(t)\}$ be a sequence of continuous functions on $I$. If $\{f_n(t)\}$ uniformly converges to $f(t)$ on $I$, then the function $f(t)$ is continuous on $I$.*

Partial metric spaces arose from the need to develop a version of the contraction fixed point theorem which would work for partially computed sequences as well as totally computed ones. Since then much research has been aimed at extrapolating away from computer science in order to develop a mathematics of posets for metric spaces. To discover more about the properties of partial metric spaces we now look at equivalent formulations.

**Definition 2.19** [11] (Equivalent partial metric spaces) *A weighted metric space is a triple $(X,d,|\cdot|)$ such that $(X,d)$ is a metric space. Then,*

*(i)* $0\leq |x|$,

*(ii)* $|x|-|y|\leq d(x,y)$

*for all $x,y\in X$. Thus, a weighted metric space is a metric space with a non-negative real number assigned to each point as a weight. Let $(X,d,|\cdot|)$ be a weighted metric space and let*

$$p(x,y)=\frac{|x|+|y|+d(x,y)}{2}.$$

*Then $(X,p)$ is a partial metric space and $p(x,x)=|x|$. Conversely, if $(X,p)$ is a partial metric space, then $(X,d,|\cdot|)$, where (as before) $p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$ and $|x|=p(x,x)$, is a weighted metric space. It can be seen that from either space we can move to the other and back again. In a weighted metric space the ordering can be defined by $x\sqsubseteq_p y$ if $|x|=d(x,y)+|y|$. Note that any metric space can be trivially weighted by defining $|x|=0$ for each $x$. Thus a partial metric space combines the metric notion of distance, weight, and poset in a single formalism.*

**Definition 2.20** [11] *A quasi-metric $q$ on $X$, defined by $q : X \times X \to \mathbb{R}$ which has the following properties for $x, y, z \in X$,*

*(Q1)* $0 \leq q(x, y)$.

*(Q2) If* $x = y$ *then* $q(x, y) = 0$.

*(Q3) If* $q(x, y) = q(y, x) = 0$ *then* $x = y$.

*(Q4)* $q(x, z) \leq q(x, y) + q(y, z)$.

*Since quasi-metrics are not in general symmetric, we revise our definition of indistancy to be $q(x, y) = q(y, x) = 0$. Thus, in quasi-metric spaces equality is identified with indistancy. A metric space (X, d) can be formed by defining $d(x, y) = q(x, y) + q(y, x)$. For any quasi-metric $q$, a partial order $\sqsubseteq_q$ is described by $x \sqsubseteq_p y$ if and only if $q(x, y) = 0$.*

Each partial metric induces a quasi-metric in a natural way. In fact, partial metrics are equivalent to weighted quasi-metrics *[12]*. Their topology is the topology of the associated quasi-metric. It is well known that each second-countable $T_0$ space is quasi-metrizable. This does not hold for partial metrics. Kunzi and Vajner [4] provide a subtle discussion of which spaces are partial metrizable. Every quasi-metric generates a quasi-uniformity in the usual way. Conversely, every countably based quasi-uniformity with associated $T_0$ topology can be generated in such a way. It is not known whether this is also true for partial metrics. This connection between posets and quasi-metric spaces can be related to partial metric spaces as follows.

**Definition 2.21** [11] *A weighted quasi-metric space is a triple $(X, q, |\cdot| : X \to \mathbb{R})$ such that $(X, q)$ is a quasi-metric space and $0 \leq |x|$ for each $x$ in X, and $|x| + q(x, y) = |y| + q(y, x)$ for all $x$ and $y$ in X. If we define $p(x, y) = |x| + q(x, y)$ then $(X, p)$ is a partial metric space. Conversely, if $(X, p)$ is a partial metric space then $(X, q_s, |\cdot|_p)$ where $q_s(x, y) = p(x, y) - p(x, x)$ and $|x|_p = p(x, x)$, is a weighted quasi-metric space. A weightless point of a weighted quasi-metric space is a point of zero weight. With these definitions, for any partial metric, $\sqsubseteq_p = \sqsubseteq_{q_s}$. Every quasi-metric space has not a weight function $|\cdot|$.*

**Definition 2.22** [14] *Let $(X, p)$ and $(Y, p')$ be two partial metric spaces. A mapping $f : X \to Y$ is said to be an isometry if $p'[f(x), f(y)] = p(x, y)$ for all $x, y \in X$.*

**Definition 2.23** [14] *Two partial metric spaces $(X, p)$ and $(Y, p')$ are called isometric if there is an isometry from $X$ onto $Y$.*

**Definition 2.24** [6] (Partial Hausdorff metric) *Let $(X, p)$ be a partial metric space. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space $(X, p)$, induced by the partial metric $p$. Note that closedness is take from $(X, \tau_p)$ ($\tau_p$ is the topology induced by $p$) and boundedness is given as follows: $A$ is a bounded subset in $(X, p)$ if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(a, a) + M$.*

*For $A, B \in CB^p(X)$ and $x \in X$, define $p(x, A) = \inf\{p(x, a), a \in A\}$,$\delta_p(A, B) = \sup\{p(a, B), a \in A\}$ and $\delta_p(B, A) = \sup\{p(b, A), b \in B\}$. Finally, we say that*

$$H_p(A, B) := \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

*It is immediate to check that $p(x, A) = 0 \Rightarrow p^s(x, A) = 0$ where $p^s(x, A) = \inf\{p^s(x, a), a \in A\}$.*

**Remark 2.25** [6] *Let $(X, p)$ be a partial metric space and $A$ any nonempty set in $(X, p)$, then $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of $A$ with respect to the partial metric $p$. Note that $A$ is closed in $(X, p)$ if and only if $A = \overline{A}$.*

**Proposition 2.26** [6] *Let $(X, p)$ be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following:*

*(i)* $\delta_p(A, A) = \sup\{p(a, a), a \in A\}$.

*(ii)* $\delta_p(A, A) \leq \delta_p(A, B)$.

*(iii)* $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$.

*(iv)* $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

**Proposition 2.27** [6] *Let $(X, p)$ be a partial metric space. For any $A, B, C \in CB^p(X)$, we have*

*(i)* $H_p(A, A) \leq H_p(A, B)$.

*(ii)* $H_p(A, B) \leq H_p(B, A)$.

*(iii)* $H_p(A, B) = 0$ *implies that* $A = B$.

*(vi)* $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$.

**Remark 2.28** [6] *It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true, in general.*

# 3 Completeness of Some Spaces of Sequences and Functions with Respect to the Partial Metric

**Proposition 3.1** [1] *Let $x, y \in X$ and define the partial distance functions $p$ by*

$$\begin{array}{rlcl} p: & X \times X & \longrightarrow & \mathbb{R}^+ \ (\mathbb{R}^-) \\ & (x,y) & \longmapsto & p(x,y) = \max\{x,y\} \ (-\min\{x,y\}) \end{array}$$

*for $X = \mathbb{R}^+$ and $X = \mathbb{R}^-$, respectively. Then, $(\mathbb{R}^+, p)$ is complete partial metric space; where the self-distance for any point $x \in \mathbb{R}^+$ is its value itself. The pair $(\mathbb{R}^-, p)$ is complete partial metric space for which $p$ is called the usual partial metric on $\mathbb{R}^-$; where the self-distance for any point $x \in \mathbb{R}^-$ is its absolute value.*

The open balls are of the form $B_p(x, \epsilon) = \{y \in \mathbb{R}^+ : \max\{x,y\} < \epsilon\} = (0, \epsilon)$ for all $x \in \mathbb{R}^+$ and $\epsilon > 0$ with $x \leq -\epsilon$ otherwise, if $x > \epsilon$ then $B_p(x, \epsilon) = \emptyset$. Suppose that $y \in B_p(x, \epsilon)$, then $\max\{x,y\} < \epsilon$ which implies that $y < \epsilon$. Similarly, the open balls are of the form $B_p(x, \epsilon) = \{y \in \mathbb{R}^- : -\min\{x,y\} < \epsilon\} = (-\epsilon, 0)$ for all $x \in \mathbb{R}^-$ and $\epsilon > 0$ with $x \geq -\epsilon$ otherwise, if $x < -\epsilon$ then $B_p(x, \epsilon) = \emptyset$. Suppose that $y \in B_p(x, \epsilon)$, then $-\min\{x,y\} < \epsilon$ which implies that $\min\{x,y\} > -\epsilon$, hence $y > -\epsilon$.

**Example 3.2** [10] *Let $X = [0,1] \cup [2,3]$ and define the distance function $p$ by*

$$p(x,y) = \left\{ \begin{array}{ccl} |x-y| & , & \{x,y\} \subset [0,1], \\ \max\{x,y\} & , & \{x,y\} \cap [2,3] \neq \emptyset, \end{array} \right.$$

*It is easy to check that $(X, p)$ is a complete partial metric space.*

**Example 3.3** [1] *Let $P_w$ denote the power set of the positive integers $w = \mathbb{N}_1$ with the subset ordering. The function $p : P_w \times P_w \to [0,1]$ such that*

$$p(x,y) = 1 - \sum_{n \in x \cap y} \frac{1}{2^n} \quad \textit{for any} \quad x, y \in P_w$$

*is a partial metric on $P_w$ and the space $P_w$ is complete with respect to its usual partial metric.*

**Example 3.4** [1] *Let $X^\infty$ be the set of finite and infinite sequences over a non-empty set $X$, with the prefix ordering $(a_0, a_1, ..., a_n) \leq (b_0, b_1, ..., b_m)$ if $n \leq m$ and $a_i = b_i$ for $i = 0, ..., n$. Denote the length of a sequence $x \in X^\infty$*

*by $l(x)$ with $l(\emptyset)=0$, which is the index of the last term of $x$ whose value is defined. Then the function $p: X^\infty \times X^\infty \to \mathbb{R}^+$ defined for any $x, y \in X^\infty$ by*

$$p(x,y) = 2^{-\sup\{i\in\mathbb{N}: i\leq \min\{l(x),l(y)\}\},\forall \quad 0\leq j<i, \quad x_j=y_j\}}$$

*is a partial metric on $X^\infty$, called the Baire partial metric. The value of the supremum is the first instance where the sequences differ (taking care if one sequence is shorter than the other).*

**Proposition 3.5** *Define $p_\infty$ on the space $\gamma(P)$ by*

$$\begin{array}{rcccl} p_\infty & : & \gamma(P)\times\gamma(P) & \longrightarrow & \mathbb{R}^+ \\ & & (x,y) & \longmapsto & p_\infty(x,y) = \sup\limits_{k\in\mathbb{N}}\{p^s(x_k,y_k)\}, \end{array}$$

*where $\gamma(P)$ denotes any of the spaces $\ell_\infty(P)$, $c(P)$ and $c_0(P)$, and $x=(x_k), y=(y_k)\in\gamma(P)$. Then, $(\gamma(P), p_\infty)$ is complete partial metric space with respect to the usual partial ordering in Definition 2.5.*

*Proof.* Since the proof is similar for the spaces $c(P)$ and $c_0(P)$, we prove the theorem only for the space $\ell_\infty(P)$. Let $x=(x_k)$, $y=(y_k)$ and $z=(z_k)\in \ell_\infty(P)$. Then,

(i) By using the axiom (P1) in Definition 2.2, it is trivial that

$$\begin{aligned} x=y &\Leftrightarrow p^s(x_k,y_k) = 2p(x_k,y_k) - p(x_k,x_k) - p(y_k,y_k) = 0 \\ &\Leftrightarrow p^s(x_k,y_k) = p^s(x_k,x_k) = p^s(y_k,y_k) \\ &\Leftrightarrow \sup_{k\in\mathbb{N}}\{p^s(x_k,y_k): k\in\mathbb{N}\} = \sup_{k\in\mathbb{N}}\{p^s(x_k,x_k): k\in\mathbb{N}\} \\ &= \sup_{k\in\mathbb{N}}\{p^s(y_k,y_k): k\in\mathbb{N}\} \Leftrightarrow p_\infty(x,y) = p_\infty(x,x) = p_\infty(y,y). \end{aligned}$$

(ii) By using the axiom (P2) in Definition 2.2, it folllows that

$$\begin{aligned} p_\infty(x,x) &= \sup_{k\in\mathbb{N}}\{2p(x_k,x_k) - p(x_k,x_k) - p(x_k,x_k)\} \geq 0, \\ p_\infty(x,y) &= \sup_{k\in\mathbb{N}}\{p^s(x_k,y_k)\} = \sup_{k\in\mathbb{N}}\{2p(x_k,y_k) - p(x_k,x_k) - p(y_k,y_k)\} \\ &= \sup_{k\in\mathbb{N}}\{[p(x_k,y_k) - p(x_k,x_k)] + [p(x_k,y_k) - p(y_k,y_k)]\} \\ &\geq \sup_{k\in\mathbb{N}}\{p^s(x_k,x_k)\} \geq 0 \Rightarrow 0 \leq p_\infty(x,x) \leq p_\infty(x,y). \end{aligned}$$

(iii) By using the axiom (P3) in Definition 2.2, it is clear that

$$\begin{aligned} p_\infty(x,y) = \sup_{k\in\mathbb{N}}\{p^s(x_k,y_k): k\in\mathbb{N}\} &= \sup_{k\in\mathbb{N}}\{p^s(y_k,x_k): k\in\mathbb{N}\} \\ &= p_\infty(y,x). \end{aligned}$$

(iv) By using the axiom (P4) in Definition 2.2 with the inequality

$$\begin{aligned} p^s(x_k, z_k) &= 2p(x_k, z_k) - p(x_k, x_k) - p(z_k, z_k) \\ &\leq 2[p(x_k, y_k) + p(y_k, z_k) - p(y_k, y_k)] - p(x_k, x_k) - p(z_k, z_k) \\ &= p^s(x_k, y_k) + p^s(y_k, z_k) - p^s(y_k, y_k), \end{aligned}$$

we have

$$\begin{aligned} p_\infty(x, z) &= \sup_{k\in\mathbb{N}} \{p^s(x_k, z_k)\} \leq \sup_{k\in\mathbb{N}}\{p^s(x_k, y_k) + p^s(y_k, z_k) - p^s(y_k, y_k)\} \\ &\leq \sup_{k\in\mathbb{N}} \{p^s(x_k, y_k) : k \in \mathbb{N}\} + \sup_{k\in\mathbb{N}} \{p^s(y_k, z_k) : k \in \mathbb{N}\} \\ &- \sup_{k\in\mathbb{N}} \{p^s(y_k, y_k) : k \in \mathbb{N}\} \leq p_\infty(x, y) + p_\infty(y, z) - p_\infty(y, y). \end{aligned}$$

Therefore, one can conclude that $(\ell_\infty(P), p_\infty)$ is a partial metric space on $\ell_\infty(P)$. It remains to prove the completeness of the space $\ell_\infty(P)$. Let $x_m = \left\{x_1^{(m)}, x_2^{(m)}, \ldots\right\}$ be any Cauchy sequence on $\ell_\infty(P)$. Then, for any $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ for all $m, r > m_0$ such that

$$p_\infty(x_m, x_r) = \sup_{k\in\mathbb{N}} p^s\left(x_k^{(m)}, x_k^{(r)}\right) < \epsilon.$$

A fortiori, for every fixed $k \in \mathbb{N}$ and for $m, r > m_0$

$$\left\{p^s\left(x_k^{(m)}, x_k^{(r)}\right) : k \in \mathbb{N}\right\} < \epsilon. \tag{1}$$

Hence for every fixed $k \in \mathbb{N}$, by using completeness of $\mathbb{R}$, we say that $x_k^{(m)} = \left\{x_k^{(1)}, x_k^{(2)}, \ldots\right\}$ is a Cauchy sequence and is convergent. Now, we suppose that $\lim_{m\to\infty} x_k^{(m)} = x_k$ and $x = (x_1, x_2, \ldots)$. We must show that

$$\lim_{m\to\infty} p_\infty(x_m, x) = 0 \quad \text{and} \quad x \in \ell_\infty(P).$$

The constant $m_0 \in \mathbb{N}$ for all $m > m_0$, taking the limit as $r \to \infty$ in (1), we obtain

$$p^s\left[x_k^{(m)}, x_k\right] < \epsilon \tag{2}$$

for all $k \in \mathbb{N}$. Since $x_m = \left(x_k^{(m)}\right) \in \ell_\infty(P)$, there exists $M > 0$ such that $p^s\left[x_k^{(m)}, 0\right] \leq M$ for all $k \in \mathbb{N}$. Thus, (2) gives together with the triangle inequality for $m > m_0$ that

$$p^s(x_k, 0) \leq p^s\left[x_k, x_k^{(m)}\right] + p^s\left[x_k^{(m)}, 0\right] \leq \epsilon + M. \tag{3}$$

It is clear that (3) holds for every $k \in \mathbb{N}$ whose right-hand side does not involve $k$. This leads us to the consequence that $x = (x_k) \in \ell_\infty(P)$. Also, from (2) we obtain for $m > m_0$ that $p_\infty(x_m, x) = \sup_{k\in\mathbb{N}} p^s(x_k^{(m)}, x_k) \leq \epsilon$. This shows that $p_\infty(x_m, x) \to 0$ as $m \to \infty$. Since $(x_m)$ is an arbitrary Cauchy sequence, $\ell_\infty(P)$ is complete.

**Proposition 3.6** *Define the distance function $p_q$ by*

$$\begin{array}{rcl} p_q \ : \ \ell_q(P) \times \ell_q(P) & \longrightarrow & \mathbb{R}^+ \\ (x, y) & \longmapsto & p_q(x, y) = \left[\sum_{k=0}^{\infty} p^s(x_k, y_k)^q\right]^{1/q}, \quad (1 \leq q < \infty) \end{array}$$

*where $x = (x_k)$, $y = (y_k) \in \ell_q(P)$. Then, $(\ell_q(P), p_q)$ is complete partial metric space with respect to the usual partial ordering in Definition 2.5.*

*Proof.* It is obvious that $p_q$ satisfies the axioms (P1), (P2) and (P3). Let $x = (x_k)$, $y = (y_k)$ and $z = (z_k) \in \ell_q(P)$. Then, we derive by applying the Minkowski's inequality that

$$\begin{aligned} p_q(x, z) &= \left\{\sum_{k=0}^{\infty} [p(x_k, z_k) - p(x_k, x_k) + p(x_k, z_k) - p(z_k, z_k)]^q\right\}^{1/q} \\ &\leq \left\{\sum_{k=0}^{\infty} [p(x_k, z_k) - p(x_k, x_k)]^q\right\}^{1/q} + \left\{\sum_{k=0}^{\infty} [p(x_k, z_k) - p(z_k, z_k)]^q\right\}^{1/q} \\ &\leq \left\{\sum_{k=0}^{\infty} [p(x_k, y_k) + p(y_k, z_k) - p(y_k, y_k) - p(x_k, x_k)]^q\right\}^{1/q} + \\ &+ \left\{\sum_{k=0}^{\infty} [p(x_k, y_k) + p(y_k, z_k) - p(y_k, y_k) - p(z_k, z_k)]^q\right\}^{1/q} \\ &\leq \left\{\sum_{k=0}^{\infty} [p(x_k, y_k) - p(x_k, x_k)]^q\right\}^{1/q} + \left\{\sum_{k=0}^{\infty} [p(y_k, z_k) - p(y_k, y_k)]^q\right\}^{1/q} \\ &+ \left\{\sum_{k=0}^{\infty} [p(x_k, y_k) - p(y_k, y_k)]^q\right\}^{1/q} + \left\{\sum_{k=0}^{\infty} [p(y_k, z_k) - p(z_k, z_k)]^q\right\}^{1/q} \\ &\leq p_q(x, y) + p_q(y, z) - p_q(y, y). \end{aligned}$$

This shows that the axiom (P4) also holds. Therefore, one can conclude that $(\ell_q(P), p_q)$ is a partial metric space.

Since the proof is analogous for the cases $q = 1$ and $q = \infty$ we omit their detailed proof and we consider only case $1 < q < \infty$. It remains to prove the completeness of the space $\ell_q(P)$. Let $x_m = \left\{x_1^{(m)}, x_2^{(m)}, \ldots\right\}$ be any Cauchy sequence on $\ell_q(P)$. Then for every $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$p_q(x_m, x_r) = \left\{ \sum_{k=0}^{\infty} p^s \left[ x_k^{(m)}, x_k^{(r)} \right]^q \right\}^{1/q} < \epsilon \tag{4}$$

for all $m, r > m_0$. We obtain for each fixed $k \in \mathbb{N}$ from (4) that

$$p^s \left[ x_k^{(m)}, x_k^{(r)} \right] < \epsilon \tag{5}$$

for all $m, r > m_0$. By using the completeness of $\mathbb{R}$, we say that the sequence $x_k^{(m)} = \left\{ x_k^{(1)}, x_k^{(2)}, \ldots \right\}$ is a Cauchy sequence and is convergent for each fixed $k \in \mathbb{N}$, say to $x_k \in \mathbb{R}$.

Now, we suppose that $x_k^{(m)} \to x_k$ as $m \to \infty$ and $x = (x_k)$. We must show that

$$\lim_{m \to \infty} p_q(x_m, x) = 0 \quad \text{and} \quad x \in \ell_q(P).$$

We have from (5) for each $j \in \mathbb{N}$ and $m, r > m_0$ that

$$\sum_{k=0}^{j} p^s \left[ x_k^{(m)}, x_k^{(r)} \right]^q < \epsilon^q. \tag{6}$$

Take any $m > m_0$. Let us pass to limit firstly $r \to \infty$ and next $j \to \infty$ in (6) to obtain $p_q(x_m, x) < \epsilon$. By using the inclusion (3) and Minkowski's inequality for each $j \in \mathbb{N}$ that

$$\left[ \sum_{k=0}^{j} p^s(x_k, 0)^q \right]^{1/q} \leq \left\{ \sum_{k=0}^{j} p^s \left[ x_k^{(m)}, x_k \right]^q \right\}^{1/q} + \left\{ \sum_{k=0}^{j} p^s \left[ x_k^{(m)}, 0 \right]^q \right\}^{1/q} < \infty$$

which implies that $x \in \ell_q(P)$. Since $p_q(x_m, x) \leq \epsilon$ for all $m > m_0$ it follows that $\lim_{m\to\infty} p_q(x_m, x) = 0$. Since $(x_m)$ is an arbitrary Cauchy sequence, the space $(\ell_q(P), p_q)$ is complete. This step concludes the proof.

**Theorem 3.7** *$n$-dimensional Euclidian space $\mathbb{R}^n$ consisting of all ordered $n$-tuples of real numbers, is a partial metric space with the metric $p$ with respect to the usual partial ordering in Definition 2.5, defined by*

$$p(x, y) = \sqrt{\sum_{k=1}^{n} p^s(x_k, y_k)^2}; \quad x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n. \tag{7}$$

*Proof.* Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$. Then, (P1), (P2) and (P3) are obvious. To prove (P4), we use Minkowski's inequality with $q = 2$ in Proposition 3.6. This step concludes the proof.

**Definition 3.8** *Let $X$ be a vector space over the field $\mathbb{R}$ and $\|\cdot\|$ be a function from $X$ to $\mathbb{R}^+$ satisfying the following norm axioms: For $x$, $y \in X$ and $\alpha \in \mathbb{R}$,*

*(N1)* $\|x\| = 0 \Leftrightarrow x = 0$,

*(N2)* $\|\alpha x\| = |\alpha|\|x\|$,

*(N3)* $\|x + y\| \leq \|x\| + \|y\|$.

*Then, $(X, \|\cdot\|)$ is said a normed space. It is trivial that a norm $\|\cdot\|$ on $X$ defines a metric $p^s$, induced by the partial metric $p$ with respect to the usual partial ordering in Definition 2.5, on $X$ which is given by*

$$p^s(x, y) = \|x - y\|; \quad (x, y \in X).$$

Now, we can give the theorem on the completeness of the metric space $(\mathbb{R}^n, p)$.

**Theorem 3.9** $(\mathbb{R}^n, p)$ *is complete.*

*Proof.* It is known by Theorem 3.7 that $p$ defined by (7) is a partial metric on $\mathbb{R}^n$. Suppose that $(x_m) = \left\{x_k^{(m)}\right\}$ is a Cauchy in $\mathbb{R}^n$, where $x_m = \left\{x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, \ldots, x_n^{(m)}\right\}$ for each fixed $m \in \mathbb{N}$. Since $(x_m)$ is Cauchy, for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$p(x_m, x_r) = \sqrt{\sum_{k=1}^{n} p^s(x_k^{(m)}, x_k^{(r)})^2} < \varepsilon \tag{8}$$

with the partial ordering in Definition 2.5, for all $m, r > n_0$. We have $p^s(x_k^{(m)}, x_k^{(r)}) < \varepsilon$ for all $m, r > n_0$. This shows for each fixed $k \in \{1, 2, \ldots, n\}$ that $\left\{x_k^{(1)}, x_k^{(2)}, \ldots\right\}$ is a convergent sequence with $x_k^{(m)} \to x_k$, as $m \to \infty$. Using these $n$ limits, we define $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$. From (8), letting $r \to \infty$ it is obtained that $p(x_m, x) \leq \varepsilon$ for all $m > n_0$ which shows that $(x_m)$ converges in $\mathbb{R}^n$. Consequently $(\mathbb{R}^n, p)$ is a complete metric space.

It is trivial that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with respect to the algebraic operations $(+)$ addition and scalar multiplication $(\cdot)$ defined on $\mathbb{R}^n$, as follows:

$$\begin{array}{rrcl} + : & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (x, y) & \longmapsto & x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ & & & \\ \cdot : & \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (\alpha, x) & \longmapsto & \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n), \end{array}$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Since $\mathbb{R}^n$ is a complete metric space with the metric $p$ defined by (7) induced by the norm $\|\cdot\|$, as a direct consequence of Theorem 3.9, we have:

**Corollary 3.10** *$\mathbb{R}^n$ is a Banach space with the norm $\|\cdot\|_2$ defined by*

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} p^s(x_k,0)^2}; \quad x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n.$$

Since it is known by Proposition 3.5 that the spaces $\ell_\infty(P)$, $c(P)$ and $c_0(P)$ are complete metric spaces with the partial metric $p_\infty$ induced by the norm $\|\cdot\|_\infty$, defined by

$$\|x\|_\infty = \sup_{k\in\mathbb{N}} p^s(x_k,0); \quad x=(x_k)\in\gamma, \quad \gamma\in\{\ell_\infty(P),c(P),c_0(P)\} \tag{9}$$

we have:

**Corollary 3.11** *The spaces $\ell_\infty(P)$, $c(P)$ and $c_0(P)$ are Banach spaces with the norm $\|\cdot\|_\infty$ defined by (9).*

Since it is known by Proposition 3.6 that the space $\ell_q(P)$ is complete metric spaces with the metric $p_q$ induced by the norm $\|\cdot\|_q$, defined by

$$\|x\|_q = \left[\sum_{k=0}^{\infty} p^s(x_k,0)^q\right]^{1/q}; \quad (x=(x_k)\in\ell_q(P), \quad q\geq 1), \tag{10}$$

we have:

**Corollary 3.12** *The space $\ell_q(P)$ is a Banach space with the norm $\|\cdot\|_q$ defined by (10).*

**Proposition 3.13** *Define $P_\infty$ on the space $\mu(P)$ by*

$$\begin{aligned} P_\infty \ : \ \mu(P)\times\mu(P) &\longrightarrow \mathbb{R}^+ \\ (x,y) &\longmapsto P_\infty(x,y)=\sup_{n\in\mathbb{N}} p^s\left(\sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k\right), \end{aligned}$$

*where $\mu(P)$ denotes any of the spaces $bs(P)$, $cs(P)$ and $cs_0(P)$, and $x=(x_k), y=(y_k)\in\mu(P)$. Then, $(\mu(P),P_\infty)$ is a complete partial metric space with respect to the usual partial ordering in Definition 2.5.*

*Proof.* Since the proof is similar to Proposition 3.5, one can easily establish that $(\mu(P),P_\infty)$ is a complete partial metric space. So, we leave it to the reader.

**Proposition 3.14** *Define the distance functions $P_\Delta(x,y)$, $P_q^\Delta(x,y)$ and $P_\infty^\Delta(x,y)$ by*

$$\begin{aligned}
P_\Delta(x,y) &:= \sum_{k=0}^{\infty} p^s\left[(\Delta' x)_k, (\Delta' y)_k\right], \quad (\Delta' x)_k = x_k - x_{k+1} \\
P_q^\Delta(x,y) &:= \sum_{k=0}^{\infty} \{p^s\left[(\Delta x)_k, (\Delta y)_k\right]^q\}^{1/q}, \quad (\Delta x)_k = x_k - x_{k-1} \quad \textit{and } x_{-1} = 0 \\
P_\infty^\Delta(x,y) &:= \sup_{k\in\mathbb{N}} \; p^s\left[(\Delta x)_k, (\Delta y)_k\right],
\end{aligned}$$

*where $x = (x_k)$, $y = (y_k)$ are the element of the spaces $bv(P)$, $bv_q(P)$ or $bv_\infty(P)$, respectively. Then, $(bv(P), P_\Delta)$, $(bv_q(P), P_q^\Delta)$ and $(bv_\infty(P), P_\infty^\Delta)$ are complete metric spaces with respect to the usual partial ordering in Definition 2.5.*

*Proof.* Since the proof is similar for the spaces $bv(P)$ and $bv_\infty(P)$, we prove the theorem only for the space $bv_q(P)$. One can easily establish that $P_q^\Delta$ defines a metric on $bv_q(P)$ which is a routine verification, so we leave it to the reader. Also the proof is analogous for the cases $q = 1$ and $q = \infty$ we omit their detailed proof and we consider only the case $1 < q < \infty$. Let $x^i = \left\{x_0^{(i)}, x_1^{(i)}, \dots\right\}$ be any Cauchy sequence on $bv_q(P)$. Then for every $\epsilon > 0$, there exists a positive integer $n_0(\epsilon) \in \mathbb{N}$ for all $i, j > n_0$ such that

$$P_q^\Delta(x^i, x^j) := \sum_{n=0}^{\infty} \left\{p^s\left[(\Delta x)_n^i, (\Delta x)_n^j\right]^q\right\}^{1/q} < \epsilon \tag{11}$$

where $(\Delta x)_n = x_n - x_{n-1}$ and $x_{-1} = 0$. We obtain for each fixed $n \in \mathbb{N}$ from (11) that

$$p^s\left[(\Delta x)_n^i, (\Delta x)_n^j\right] < \epsilon \tag{12}$$

for all $i, j > n_0(\epsilon)$ which leads us to the fact that the sequence $\{(\Delta x)_n^i\}$ is a Cauchy sequence and is convergent. Now, we suppose that $(\Delta x)_n^i \to (\Delta x)_n$ as $n \to \infty$. We have from (12) for each $m \in \mathbb{N}$ and $i, j > n_0(\epsilon)$ that

$$\sum_{k=0}^{m} p^s\left[(\Delta x)_k^i, (\Delta x)_k^j\right]^q \leq P_q^\Delta(x^i, x^j)^q < \epsilon^q. \tag{13}$$

Take any $i > n_0(\epsilon)$. Let us pass to limit firstly $j \to \infty$ and next $m \to \infty$ in (13) to obtain $P_q^\Delta(x^i, x) \leq \epsilon$. Finally, by using Minkowski's inequality for each $m \in \mathbb{N}$ that

$$\left\{\sum_{k=0}^{m} p^s(\Delta x)_k, 0)^q\right\}^{1/q} \leq P_q^\Delta(x^i, x) + P_q^\Delta(x^i, 0) \leq \epsilon + P_q^\Delta(x^i, 0) < \infty$$

which implies that $x \in bv_q(P)$. Since $P_q^{\Delta}(x^i, x) \leq \epsilon$ for all $i > n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$. Since $(x^i)$ is an arbitrary Cauchy sequence, the space $bv_q(P)$ is complete. This step concludes the proof.

**Proposition 3.15** *Let $f, g \in B[a,b]$. Define the distance function $p_1$ by*

$$\begin{array}{rcccl} p_1 & : & B[a,b] \times B[a,b] & \longrightarrow & \mathbb{R}^+ \\ & & (f,g) & \longmapsto & p_1(f,g) := \sup_{t\in[a,b]} \ p^s[f(t), g(t)]. \end{array}$$

*Then, $(B[a,b], p_1)$ is a partial metric space with the partial order $\sqsubseteq_p$ on $B[a,b]$.*

*Proof.* One can easily establish that $p_1$ defines a partial metric on $B[a,b]$ which is a routine verification, so we omit its details. Suppose that the sequence $\{f_n(t)\}$ of bounded functions be any Cauchy sequence in the space $B[a,b]$. Then, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ for all $m, n > n_0$ such that

$$p_1(f_n, f_m) = \sup_{t\in[a,b]} \ p^s\left[f_n(t), f_m(t)\right] < \epsilon.$$

A fortiori, for every fixed $n \in \mathbb{N}$ and for $m, n > n_0$

$$p^s\left[f_n(t), f_m(t)\right] < \epsilon. \tag{14}$$

By taking into account the completeness of $\mathbb{R}$, we conclude that $\{f_n(t)\}$ is a Cauchy sequence and is convergent. Now, we suppose that $f_n(t) \to f(t)$ as $n \to \infty$ for all $t \in [a,b]$. We must show that

$$\lim_{n\to\infty} p_1(f_n, f) = 0 \ \text{ and } \quad f \in B[a,b].$$

Let $n_0 \in \mathbb{N}$ be fixed such that $m > n_0$, taking the limit for $m \to \infty$ in (14), we obtain

$$p^s\left[f_n(t), f(t)\right] < \epsilon \tag{15}$$

for all $t \in [a,b]$. Since $\{f_n(t)\} \in B[a,b]$, there exists $M > 0$ such that $p^s\left[f_n(t), 0\right] \leq M$. Thus, (15) gives together with the triangle inequality of Definition 2.2 for $m > n_0$ that

$$p^s[f(t), 0] \leq p^s[f(t), f_n(t)] + p^s[f_n(t), 0] \leq \epsilon + M.$$

That is to say that the sequence $f \in B[a,b]$. Other hand, using the inequality (15) and Definition 2.16, the sequence $\{f_n(t)\}$ of functions converges to $f(t)$ uniformly on $[a,b]$. Hence, the partial metric space $(B[a,b], p_1)$ is complete.

We give the following proposition without proof. Since the proof can also be obtained in the similar way used in the proof of Proposition 3.15, we omit the detail.

**Proposition 3.16** *Let $f, g \in C[a,b]$. Define the distance functions $p_2$ and $p_3$ by*

$$\begin{array}{rcccl} p_2 & : & C[a,b] \times C[a,b] & \longrightarrow & \mathbb{R}^+ \\ & & (f,g) & \longrightarrow & p_2(f,g) = \max\limits_{t \in [a,b]} \{p^s(f(t), g(t))\}, \end{array}$$

$$\begin{array}{rcccl} p_3 & : & C[a,b] \times C[a,b] & \longrightarrow & \mathbb{R}^+ \\ & & (f,g) & \longmapsto & p_3(f,g) := \int_a^b p^s(f(t), g(t))dt. \end{array}$$

*Then, $p_2$ and $p_3$ are the partial metrics on $C[a,b]$ with the partial order $\sqsubseteq_p$ and the space $C[a,b]$ is complete with respect to the metric $p_2$ while it is incomplete with respect to the metric $p_3$.*

# 4 The Duals of the Sets of Sequence with the Partial Metric

Firstly, we define the alpha-, beta- and gamma-duals of a set $\mu(P) \subset \omega$ which are respectively denoted by $\{\mu(P)\}^\alpha$, $\{\mu(P)\}^\beta$ and $\{\mu(P)\}^\gamma$, as follows:

$$\begin{aligned} \{\mu(P)\}^\alpha &:= \Big\{x = (x_k) \in w : (x_k y_k) \in \ell_1(P) \text{ for all } (y_k) \in \mu(P)\Big\}, \\ \{\mu(P)\}^\beta &:= \Big\{x = (x_k) \in w : (x_k y_k) \in cs(P) \text{ for all } (y_k) \in \mu(P)\Big\}, \\ \{\mu(P)\}^\gamma &:= \Big\{x = (x_k) \in w : (x_k y_k) \in bs(P) \text{ for all } (y_k) \in \mu(P)\Big\}. \end{aligned}$$

**Theorem 4.1** *The following statements hold:*

*(i)* $\{\ell_\infty(P)\}^\alpha = \{c(P)\}^\alpha = \{c_0(P)\}^\alpha = \ell_1(P)$.

*(ii)* $\{\ell_1(P)\}^\alpha = \ell_\infty(P)$.

*Proof.* (i) Since one can prove in the similar way that $\{c(P)\}^\alpha = \{c_0(P)\}^\alpha = \ell_1(P)$, we only show that $\{\ell_\infty(P)\}^\alpha = \ell_1(P)$. Let $y = (y_k) \in \ell_1(P)$ and $x = (x_k) \in \ell_\infty(P)$. Then, $\sup_{k\in\mathbb{N}} p^s(x_k, 0) < \infty$. Therefore,

$$\sum_{k=0}^{\infty} p^s(y_k x_k, 0) \leq \left[\sup_{k\in\mathbb{N}} p^s(x_k, 0)\right] \sum_{k=0}^{\infty} p^s(y_k, 0) < \infty, \tag{16}$$

since $y \in \ell_1(P)$ and this implies that $y \in \{\ell_\infty(P)\}^\alpha$. Hence, the inclusion $\ell_1(P) \subseteq \{\ell_\infty(P)\}^\alpha$ holds.

Conversely, let $y = (y_k) \in \{\ell_\infty(P)\}^\alpha$ and $x = (1, 1, \ldots, 1, \ldots) \in \ell_\infty(P)$. Then, using the inequality (16)

$$\sum_{k=0}^{\infty} p^s(y_k, 0) = \sum_{k=0}^{\infty} p^s(y_k x_k, 0) < \infty,$$

since $y \in \{\ell_\infty(P)\}^\alpha$ and this implies that $y \in \ell_1(P)$. Hence, the inclusion $\{\ell_\infty(P)\}^\alpha \subseteq \ell_1(P)$ holds. Therefore, the inclusions $\ell_1(P) \subseteq \{\ell_\infty(P)\}^\alpha$ and $\{\ell_\infty(P)\}^\alpha \subseteq \ell_1(P)$ give that $\{\ell_\infty(P)\}^\alpha = \ell_1(P)$.

(ii) Let $y = (y_k) \in \ell_\infty(P)$ and $x = (x_k) \in \ell_1(P)$. Then, $\sum_k p^s(x_k, 0) < \infty$. Now,

$$\sum_{k=0}^{\infty} p^s(x_k y_k, 0) \leq \left[ \sup_{k \in \mathbb{N}} \ p^s(y_k, 0) \right] \sum_{k=0}^{\infty} p^s(x_k, 0) < \infty$$

and $y \in \{\ell_1(P)\}^\alpha$. Therefore, the inclusion $\ell_\infty(P) \subseteq \{\ell_1(P)\}^\alpha$ holds.

Conversely, suppose that $y = (y_k) \in \{\ell_1(P)\}^\alpha$ and $x = (1, 1, \ldots, 1, 0, 0, \ldots) \in \ell_1(P)$. Thus, it is immediate that

$$\sup_{k \in \mathbb{N}} \ p^s(y_k, 0) = \sup_{k \in \mathbb{N}} \ p^s(x_k y_k, 0) \leq \sum_{k=0}^{\infty} p^s(x_k y_k, 0) < \infty$$

which gives that $y \in \ell_\infty(P)$. This means that the inclusion $\{\ell_\infty(P)\}^\alpha \subseteq \ell_1(P)$ holds. Therefore, by combining the inclusions $\ell_\infty(P) \subseteq \{\ell_1(P)\}^\alpha$ and $\{\ell_1(P)\}^\alpha \subseteq \ell_\infty(P)$ we obtain the desired result $\{\ell_1(P)\}^\alpha = \ell_\infty(P)$.

**Theorem 4.2** *Let $1 < q < \infty$ with $q^{-1} + r^{-1} = 1$. Then, $\{\ell_q(P)\}^\alpha = \{\ell_q(P)\}^\beta = \ell_r(P)$.*

*Proof.* Suppose that $1 < q < \infty$ with $q^{-1} + r^{-1} = 1$ and $y = (y_k) \in \{\ell_q(P)\}^\alpha / \ell_r(P)$. Then, $\sum_k p^s(y_k x_k, 0) < \infty$ for all $x = (x_k) \in \ell_q(P)$. Then, we can find an increasing sequence $(n_k)$ of positive integers such that $\left[p^s(y_{n_k}, 0)\right]^q > k^2$ for all $k \in \mathbb{N}_1$. Define the sequence $x = (x_n)$ by

$$x_n = \begin{cases} 1/y_{n_k} & , \quad n = n_k, \\ 0 & , \quad n \neq n_k, \end{cases}$$

where $k \in \mathbb{N}_1$. Then, $x \in \ell_q(P)$ but

$$\sum_{k=0}^{\infty} p^s(y_k x_k, 0) = \sum_{k=0}^{\infty} p^s(y_{n_k}, 0) \ p^s(y_{n_k}, 0)^{-1} = 1 + 1 + \cdots = \infty$$

which is a contradiction. Therefore, $y \in \ell_r(P)$ and we have $\{\ell_q(P)\}^\alpha \subseteq \ell_r(P)$.

Conversely, suppose that $x = (x_k) \in \ell_q(P)$ and $y = (y_k) \in \ell_r(P)$. Then, one can see by Holder's inequality

$$\sum_{k=0}^{\infty} p^s(y_k x_k, 0) \leq \left\{ \sum_{k=0}^{\infty} [p^s(y_k, 0)]^r \right\}^{1/r} \left\{ \sum_{k=0}^{\infty} [p^s(x_k, 0)]^q \right\}^{1/q} < \infty$$

which means that $y \in \{\ell_q(P)\}^\alpha$. Hence, the inclusion $\ell_r(P) \subseteq \{\ell_q(P)\}^\alpha$ holds.

**Lemma 4.3** [3] *If $X$ is solid, then $X^\alpha = X^\beta = X^\gamma$.*

Although one can directly derive the beta- and gamma-duals of the spaces $\ell_\infty(P)$, $c_0(P)$ and $\ell_q(P)$ which are solid, by combining the results of Theorem 4.1, Theorem 4.2 and Lemma 4.3, we give their detailed proof.

**Theorem 4.4** *The $\alpha-$dual of the sets $cs(P)$, $bs(P)$ and $bv_1(P)$ of sequences is the set $\ell_1(P)$.*

*Proof.* We prove the case $\{cs(P)\}^\alpha = \ell_1(P)$ and the rest can be obtained similarly. Let $x = (x_k) \in cs(P)$ and $y = (y_k) \in \ell_1(P)$. Then, $\sum_k p^s(y_k, 0) < \infty$ and we have

$$\sum_{k=0}^{\infty} p^s(y_k x_k, 0) \leq \left[ \sup_{n \in \mathbb{N}} \ p^s \left( \sum_{k=1}^{n} x_k, 0 \right) \right] \sum_{k=0}^{\infty} p^s(y_k, 0) < \infty.$$

Therefore, $y \in \{cs(P)\}^\alpha$ which gives that $\ell_1(P) \subseteq \{cs(P)\}^\alpha$.

Conversely, let $y = (y_k) \in \{cs(P)\}^\alpha / \ell_1(P)$. Then to every natural number $i$, we can find an odd $n_i$ with $n_i < n_{i+1}$ and $\sum_{k=n_i+1}^{n_{i+1}} p^s(y_k, 0) > 2^i$ for all $i \in \mathbb{N}$. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} (-1)^k 2^{-i/2} & , \quad n_i < k \leq n_{i+1}, \\ 0 & , \quad otherwise, \end{cases}$$

where $i \in \mathbb{N}$. Then, $x = (x_k) \in \{cs(P)\}$ but $\sum_k \left\{p^s(y_k x_k, 0)\right\} = \infty$. This contradicts that $y \in \{cs(P)\}^\alpha$, and so $y = (y_k)$ must be in $\ell_1(P)$ which gives the inclusion $\{cs(P)\}^\alpha \subseteq \ell_1(P)$.

**Theorem 4.5** *The following statements hold:*

*(i)* $\{cs(P)\}^\beta = bv_1(P)$.

*(ii)* $\{bv(P)\}^\beta = cs(P)$.

*Proof.* Since Part (ii) can be similarly proved, we consider only Part (i). Let $x = (x_k) \in \{cs(P)\}^\beta$ and $w = (w_k) \in c_0(P)$. Define the sequence $y = (y_k) \in cs(P)$ by $y_k = w_k - w_{k+1}$ for all $k \in \mathbb{N}$. Therefore, $\sum_k x_k y_k$ converges, but

$$\sum_{k=1}^{n}(w_k - w_{k+1})x_k = \sum_{k=0}^{n-1} w_k(x_k - x_{k-1}) - w_{n+1}x_n, \quad (x_{-1} = 0)$$

and the inclusion $\ell_1(P) \subset cs(P)$ yields that $(x_k) \in \{cs(P)\}^\beta \subset \{\ell_1(P)\}^\beta = \ell_\infty(P)$ which implies that

$$\sum_k (w_k - w_{k+1})x_k = \sum_k w_k(x_k - x_{k-1}), \quad (x_{-1} = 0).$$

Hence, $(x_k - x_{k-1}) \in \{c_0(P)\}^\beta = \{c_0(P)\}^\alpha = \ell_1(P)$, i.e., $x \in bv_1(P)$. Therefore, $\{cs(P)\}^\beta \subseteq bv_1(P)$.

Conversely, suppose that $x = (x_k) \in bv_1(P)$. Then, $(x_k - x_{k-1}) \in \ell_1(P)$. Further, if $y = (y_k) \in cs(P)$, the sequence $(w_n)$ defined by $w_n = \sum_{k=1}^n y_k$ for all $n \in \mathbb{N}$, is an element of the space $c(P)$. Since $\{c(P)\}^\alpha = \ell_1(P)$, the series $\sum_k w_k(x_k - x_{k+1})$ is convergent. Also, we have

$$\sum_{k=m}^{n} p^s\left[(w_k - w_{k-1})x_k, 0\right] \leq \sum_{k=m}^{n-1} p^s\left[w_k(x_k - x_{k+1}), 0\right] + w_n x_n - w_{m-1}x_m. \quad (17)$$

Since $(w_n) \in c(P)$ and $(x_k) \in bv_1(P) \subset c(P)$, second and third terms in the right-hand side of the inequality (17) tend to zero, as $m, n \to \infty$. Hence, the series $\sum_k (w_k - w_{k-1})x_k$, that is, $\sum_k x_k y_k$ converges and so, $bv_1(P) \subseteq \{cs(P)\}^\beta$. Thus, $\{cs(P)\}^\beta = bv_1(P)$ .

**Theorem 4.6** *The following statements hold:*

*(i)* $\{bs(P)\}^\gamma = \{cs(P)\}^\gamma = bv_1(P)$.

*(ii)* $\{bv_1(P)\}^\gamma = bs(P)$.

*Proof.* We prove only Part (i) for $\{cs(P)\}^\gamma$, the rest can be proved along similar lines.

By Theorem 4.5, we have $bv_1(P) \subseteq \{cs(P)\}^\beta$ and since $\{cs(P)\}^\beta \subset \{cs(P)\}^\gamma$, so $bv_1(P) \subset \{cs(P)\}^\gamma$. We need to show that $\{cs(P)\}^\gamma \subset bv_1(P)$. Let $x = (x_n) \in \{cs(P)\}^\gamma$ and $y = (y_n) \in c_0(P)$. Then, for the sequence, $(w_n) \in cs(P)$ defined by $w_n = y_n - y_{n+1}$ for all $n \in \mathbb{N}$, we can find a constant $K > 0$ such that $\sum_{k=1}^n p^s(x_k w_k, 0) \leq K$ for all $n \in \mathbb{N}$. Since $(y_n) \in c_0(P)$

and $(x_n) \in \{cs(P)\}^\gamma \subset \ell_\infty(P)$, there exists a constant $M > 0$ such that $p^s(x_n y_n, 0) \leq M$ for all $n \in \mathbb{N}$. Now,

$$\sum_{k=1}^{n} p^s\left[y_k(x_k - x_{k-1}), 0\right] \leq \sum_{k=0}^{n+1} p^s\left[x_k(y_k - y_{k+1}), 0\right] + y_{n+2}x_{n+1} \leq K + M.$$

Hence $(x_k - x_{k-1}) \in \{c_0(P)\}^\gamma = \ell_1(P)$, i.e., $(x_k) \in bv_1(P)$. Therefore, since the inclusion $\{cs(P)\}^\gamma \subset bv_1(P)$ holds we conclude that $\{cs(P)\}^\gamma = bv_1(P)$.

# 5 Concluding Remarks

Partial metrics are more flexible than metrics, they generate partial orders and their topological properties are more general than the ones for metrics, argued by the fact that the self distance of each point need not be zero. They are enormously useful in partially defined information for the study of domains and semantics in computer science.

We succeeded in developing the mathematical concepts of partial metrics which are equivalent to weightable quasi-metrics in the sense that a partial metric can be interpreted as a weightable quasi-metric and conversely a weighted quasi-metric can be considered as a partial metric.

The partial metric spaces were assigned every property from their induced metric spaces, allowing us to construct the completion of an incomplete partial metric space by first completing their induced metric space, using the set of all Cauchy sequences in the space of interest. The demonstration was similar to the one for completing metric spaces, involving notions of isometry and topological properties.

In this research some new sequence and function spaces are introduced by using the notion of partial metric with respect to the partial order, and shown that the given spaces were partially complete. In addition, this work presents a new tool for the description and analysis of partial metric spaces. The potential applications of the obtained results include the establishment of new sequence and function spaces on partial metric which are interesting topics for our future works. Of course, it will be meaningful to determine the alpha-, beta- and gamma-duals of the partial metric sequence spaces $\ell_\infty(P)$, $c(P)$ and $\ell_q(P)$. We should record that one can study on the domain of some triangle matrices in the partial metric sequence spaces $\ell_\infty(P)$, $c(P)$ and $\ell_q(P)$ which is a new development on the theory of sequence spaces.

Furthermore our research has highlighted the merits of working with logic and mathematics in contemporary computer science. Now we have an analogous but even more ambitious task of finding ways to have humans and machines think together as one new intelligent form, rather than trying to be the ultimate largest automatic theorem prover.

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