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Unique Common Fixed Point Theorems For Compatible Mappings In Complete Metric Space

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Abstract

In this paper, we have studied unique common fixed point theorems for two pairs of compatible mappings and compatible of type (A) in complete metric space.

Keywords: *Complete metric space, Continuous map, Compatible mapping, Compatible of type(A)*

1 Introduction

The concept of common fixed point theorem for commuting mappings have been investigated by Jungck[3, 4, 5], who generalized the Banach's fixed point theorem [9]. The generalization of commutativity, given by Jungck[3], is called compatible mapping. Sharma and Patidar [8], also generalized the notion of commutativity and resulting mappings were called as compatible of type(A). The object of this paper is to generalize some unique common fixed point theorems given by Fisher[1], Pant[7], Cho & Murthy[11], Shukla & Tiwari[2], Singh & Singh[10] and Lohani & Badshah[6] using compatible mapping and compatible of type (A) in complete metric space.

Definition 1.1. *Two mappings A and B from a metric space (X, d) into itself are called commuting on X if*

$$d(ABx, BAx) = 0 \ for \ all \ x \in X.$$

Definition 1.2. Two mappings A and B from a metric space (X, d) into itself are called weakly commuting on X if

$$d(ABx, BAx) \leq d(Ax, Bx) \ for \ all \ x \in X.$$

Commuting mappings are weakly commuting but the converse is not necessarily true. The following example illustrate this fact.

Example 1.1. Consider two mappings A and $B : X \to X$, where $X = [0, 1]$ with Euclidean metric d, such that

$$Ax = \frac{2x}{5 - 3x}, \ \ Bx = \frac{5x}{4}$$

for all $x \in X$. Then, for any $x \in X$, we have

$$d(ABx, BAx) = \left|\frac{10x}{20 - 15x} - \frac{40x}{5 - 3x}\right| = \left|\frac{50 - 770x + 600x^2}{5(4 - 3x)(5 - 3x)}\right|$$

$$\leq \left|\frac{2x}{5 - 3x} - \frac{5x}{4}\right| = \left|\frac{-25 + 23x}{4(5 - 3x)}\right| = d(Ax, Bx).$$

Clearly, A and B are weakly commuting mappings on x whereas they are not commuting mappings on X. Since, we have

$$ABx = \frac{2x}{4 - 3x} < \frac{40x}{5 - 3x} = BAx$$

for any non-zero $x \in X$.
This shows that $d(ABx, BAx) \neq 0$ i.e., A and B are not commuting.

Definition 1.3. *Two mappings A and B from a metric space (X, d) into itself are called compatible on X if*

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \ for \ some \ points \ x \in X.$$

Clearly, if A and B are compatible mappings on X with $d(Ax, Bx) = 0$ for some $x \in X$, then we have

$$d(ABx, BAx) = 0.$$

Note that weakly commuting mappings are compatible but the converse is not necessarily true.

Definition 1.4. *Two mappings A and B from a metric space (X, d) into itself are called compatible of type (A) if*

$$\lim_{n\to\infty} d(BAx_n, AAx_n) = 0$$

and

$$\lim_{n\to\infty} d(ABx_n, BBx_n) = 0,$$

whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \ for \ some \ z \in X.$$

Lemma 1.1. [4] *Let A and B be compatible mappings from a metric space (X, d) into itself. Suppose that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \ for \ some \ x \in X.$$

If A continuous, then

$$\lim_{n\to\infty} BAx_n = Ax.$$

Now, Let A, B, C and D be mappings from a complete metric space (X, d) into itself satisfying the following conditions

$$A(X) \subset D(X),\ \ B(X) \subset C(X) \tag{1}$$

and

$$d(Ax, By) \leq \alpha\Big[\frac{\{d(Cx, Ay)\}^{m+1} + \{d(Dy, By)\}^{m+1}}{\{d(Cx, Ax)\}^{m} + \{d(Dy, By)\}^{m}}\Big] + \beta d(Cx, Dy) \tag{2}$$

for all $x, y \in X$, where α, $\beta \geq 0$, $\alpha + \beta < 1$ and $m \geq 1$.
Then, for an arbitrary point $x_0 \in X$ by equation (1), we choose a point $x_1 \in X$ such that $Dx_1 = Ax_0$ and for this point x_1, there exist a point $x_2 \in X$ such that $Cx_2 = Bx_1$ and so on. Proceeding in the similar fashion, we can define a sequence $\{y_n\}$ in X, such that

$$y_{2n+1} = Dx_{2n+1} = Ax_{2n} \ and \ y_{2n} = Cx_{2n} = Bx_{2n-1}. \tag{3}$$

Lemma 1.2. [5] *Let A, B, C and D be mappings from a complete metric space (X, d) into itself satisfying the equations (1) and (2). Then the sequence $\{y_n\}$ defined by equation (3) is a cauchy sequence in X.*

2 Main Results

Theorem 2.1. Let A, B, C and D be mappings from a complete metric space (X, d) into itself satisfying the equations (1) and (2). If any one of the A, B, C and D is continuous and pairs A, C and B, D are compatible on X. Then A, B, C and D have a unique common fixed point in X.

Proof: Let $\{y_n\}$ be a sequence in X defined by the equation (3), then by Lemma(1.2), sequence $\{y_n\}$ is cauchy sequence. Since (X, d) is complete metric space so sequence $\{y_n\}$ is converges to some point $u \in X$. Consequently, the subsequence $\{Ax_{2n}\}$, $\{Cx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Dx_{2n+1}\}$ of the sequence $\{y_n\}$ also converges to u.
We assume that C is continuous. Since A and C are compatible mappings on X, then Lemma(1.1) gives that

$$C^2x_{2n} \ and \ ACx_{2n} \to Cu \ as \ n \to \infty. \tag{4}$$

Consider,

$$d(ACx_{2n}, Bx_{2n-1})$$
$$\leq \alpha\Big[\frac{\{d(C^2x_{2n}, ACx_{2n})\}^{m+1} + \{d(Dx_{2n-1}, Bx_{2n-1})\}^{m+1}}{\{d(C^2x_{2n}, ACx_{2n})\}^{m} + \{d(Dx_{2n-1}, Bx_{2n-1})\}^{m}}\Big] + \beta d(C^2x_{2n}, Dx_{2n-1})$$
$$\leq \alpha\big[d(C^2x_{2n}, ACx_{2n}) + d(Dx_{2n-1}, Bx_{2n-1})\big] + \beta d(C^2x_{2n}, Dx_{2n-1}).$$

Using equation (4) and subsequences of sequence $\{y_n\}$ converging to u, in above equation, we have

$$d(Cu, u) \leq \alpha\big[d(Cu, Cu) + d(u, u)\big] + \beta d(Cu, u)$$
$$\Rightarrow \quad (1-\beta)d(Cu, u) \leq 0$$
$$\Rightarrow \quad d(Cu, u) = 0 \ as \ \beta \neq 1.$$

Therefore

$$Cu = u. \tag{5}$$

Again, consider

$$d(Au, Bx_{2n-1})$$
$$\leq \alpha\Big[\frac{\{d(Cu, Au)\}^{m+1} + \{d(Dx_{2n-1}, Bx_{2n-1})\}^{m+1}}{\{d(Cu, Au)\}^{m} + \{d(Dx_{2n-1}, Bx_{2n-1})\}^{m}}\Big] + \beta d(Cu, Dx_{2n-1})$$
$$\leq \alpha\big[d(Cu, Au) + d(Dx_{2n-1}, Bx_{2n-1})\big] + \beta d(Cu, Dx_{2n-1}).$$

Using equations (4) & (5) and subsequences of sequence $\{y_n\}$ converging to u, in above equation, we have

$$d(Au, u) \leq \alpha\big[d(u, Au) + d(u, u)\big] + \beta d(u, u)$$

$$\Rightarrow \quad (1-\alpha)d(Au,u) \leq 0$$
$$\Rightarrow \quad d(Au,u) = 0 \ as \ \alpha \neq 1.$$

We have

$$Au = u. \tag{6}$$

Since $A(X) \subset D(X)$, therefore there exist a point v in X, such that

$$u = Au = Dv. \tag{7}$$

Now, consider

$$d(u,Bv) = d(Au,Bv)$$
$$\leq \alpha\Big[\frac{\{d(Cu,Au)\}^{m+1} + \{d(Dv,Bv)\}^{m+1}}{\{d(Cu,Av)\}^{m} + \{d(Dv,Bv)\}^{m}}\Big] + \beta d(Cu,Dv)$$
$$\leq \alpha\Big[d(Cu,Au) + d(Dv,Bv)\Big] + \beta d(Cu,Dv),$$

using equations (5), (6), & (7), we get

$$d(u,Bv) \leq \alpha\Big[d(u,u) + d(u,Bv)\Big] + \beta d(u,u)$$

$$\Rightarrow \quad (1-\alpha)d(u,Bv) \leq 0$$
$$\Rightarrow \quad d(u,Bv) = 0 \ as \ \alpha \neq 1.$$

Thus, we have

$$Bv = u. \tag{8}$$

From equations (5), (6), (7) & (8), we have

$$Dv = Bv = u = Au = Cu. \tag{9}$$

Since B and D are compatible on X, then

$$d(BDv, DBv) = 0$$

$$\Rightarrow \quad DBv = BDv. \tag{10}$$

From equations (9) & (10), we get

$$Du = DBv = BDv = Bu. \tag{11}$$

Moreover, by the equation (2), we have

$$d(u,Du) = d(Au,Bu)$$

$$\leq \alpha\Big[\frac{\{d(Cu,Au)\}^{m+1} + \{d(Du,Bu)\}^{m+1}}{\{d(Cu,Au)\}^{m} + \{d(Du,Bu)\}^{m}}\Big] + \beta d(Cu,Du)$$

$$\leq \alpha\big[d(Cu, Au) + d(Du, Bu)\big] + \beta d(Cu, Du)$$
$$= \beta d(Cu, Du)$$
$$= \beta d(u, Du)$$
$$\Rightarrow \quad (1-\beta)d(u, Du) \leq 0$$
$$\Rightarrow \quad d(u, Du) = 0 \ as \ \beta \neq 1.$$

We have

$$Du = u. \tag{12}$$

Since $Bu = Du$, so $Bu = u$. Thus u is a common fixed point of A, B, C and D.
Similarly, we can prove the result, when any one of the A, B and D is continuous.This prove the result.

We shall prove the uniqueness of the common fixed point for this.
Suppose u and z be two common fixed points of A, B, C and D. Then from equation (2), we have

$$d(u, z) = d(Au, Bz)$$
$$\leq \alpha\Big[\frac{\{d(Cu, Au)\}^{m+1} + \{d(Dz, Bz)\}^{m+1}}{\{d(Cu, Au)\}^{m} + \{d(Dz, Bz)\}^{m}}\Big] + \beta d(Cu, Dz)$$
$$\leq \alpha\big[d(Cu, Au) + d(Dz, Bz)\big] + \beta d(Cu, Dz)$$
$$= \alpha\big[d(u, u) + d(z, z)\big] + \beta d(u, z)$$
$$\Rightarrow \quad (1-\beta)d(u, z) \leq 0$$
$$\Rightarrow \quad d(u, z) = 0 \ as \ \beta \neq 1.$$

Finally, we get

$$u = z.$$

Thus, u is unique common fixed point of A, B, C and D.

Theorem 2.2. Let A, B, C and D be mappings from a complete metric space (X, d) into itself. Suppose that any one of A, B, C and D is continuous and for some positive integer p, q, r and t, which satisfy the following conditions

$$A^p(X) \subset D^t(X) \ and \ B^q(X) \subset C^r(X) \tag{13}$$

and

$$d(A^p x, B^q y) \leq \alpha\Big[\frac{\{d(C^r x, A^p x)\}^{m+1} + \{d(D^t y, B^q y)\}^{m+1}}{\{d(C^r x, A^p x)\}^{m} + \{d(D^t y, B^q y)\}^{m}}\Big] + \beta d(C^r x, D^t y) \tag{14}$$

for all $x, y \in X$*, where* α, $\beta \geq 0$, $\alpha + \beta < 1$ *and* $m \geq 1$.
Suppose that A *&* C *and* B *&* D *are compatible on* X*. Then* A, B, C *and* D *have a unique common fixed point in* X*.*

Proof: Proof of this theorem is similar to the proof of theorem (2.1).

Theorem 2.3. Let A, B, C and D be four mappings of a complete metric space X into itself satisfying

$$\begin{aligned} d(Ax, By) \leq\ & \alpha\Big[\frac{d(Dy, By)d(Cx, Dy)}{d(Dx, Ax) + d(By, Dx)}\Big] \\ & + \beta\Big[\frac{d(Ax, Dx)d(Ay, Cy)}{d(Dx, Ax) + d(By, Dx)}\Big] \\ & + \gamma\Big[\frac{d(Dx, Bx)d(By, Dy)}{d(Dx, Ax) + d(By, Dx)}\Big] \\ & + \delta\Big[\frac{d(Cx, Dy)d(Ax, By)}{d(Dx, Ax) + d(By, Dx)}\Big] \end{aligned} \tag{15}$$

and

$$A(X) \subset D(X) \text{ and } B(X) \subset C(X) \tag{16}$$

for all $x, y \in X$ and $\alpha,\ \beta,\ \gamma,\ \delta \geq 0$ such that $\alpha + \beta + \gamma + \delta < 1$. Suppose that the pairs A, C and B, D are compatible of type (A) and any one of the A, B, C and D is continuous. Then A, B, C and D have a unique coomon fixed point in X.

Proof: We are given that (X, d) is a complete metric space, so every cauchy sequence in X is converges in X. We define a sequence $\{y_n\}$ in X, such that

$$Ax_{2n+1} = y_{2n+2}, Dx_{2n} = y_{2n} \text{ and } Bx_{2n+1} = y_{2n+2}, Cx_{2n} = y_{2n} \tag{17}$$

for $n = 1, 2, 3, \cdots\cdots$.
By putting $x = x_{2n}$ and $y = x_{2n+1}$ in (15), we have

$$\begin{aligned} d(Ax_{2n}, Bx_{2n+1}) \leq\ & \alpha\Big[\frac{d(Dx_{2n+1}, Bx_{2n+1})d(Cx_{2n}, Dx_{2n+1})}{d(Dx_{2n}, Ax_{2n}) + d(Bx_{2n+1}, Dx_{2n})}\Big] \\ & + \beta\Big[\frac{d(Ax_{2n}, Dx_{2n})d(Ax_{2n+1}, Cx_{2n+1})}{d(Dx_{2n}, Ax_{2n}) + d(Bx_{2n+1}, Dx_{2n})}\Big] \\ & + \gamma\Big[\frac{d(Dx_{2n}, Bx_{2n})d(Bx_{2n+1}, Dx_{2n+1})}{d(Dx_{2n}, Ax_{2n}) + d(Bx_{2n+1}, Dx_{2n})}\Big] \\ & + \delta\Big[\frac{d(Cx_{2n}, Dx_{2n+1})d(Ax_{2n}, Bx_{2n+1})}{d(Dx_{2n}, Ax_{2n}) + d(Bx_{2n+1}, Dx_{2n})}\Big]. \end{aligned} \tag{18}$$

Using equation (17) in equation (18), we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) \leq\ & \alpha\Big[\frac{d(y_{2n+1}, y_{2n+2})d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}\Big] \\ & + \beta\Big[\frac{d(y_{2n+1}, y_{2n})d(y_{2n+2}, y_{2n+1})}{d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}\Big] \\ & + \gamma\Big[\frac{d(y_{2n}, y_{2n+1})d(y_{2n+2}, y_{2n+1})}{d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}\Big] \\ & + \delta\Big[\frac{d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n+2})}{d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}\Big]. \end{aligned} \tag{19}$$

Using triangle inequality in (19), we have

$$d(y_{2n+1}, y_{2n+2}) \leq (\alpha + \beta + \gamma + \delta)d(y_{2n}, y_{2n+1}). \tag{20}$$

Taking $h = \alpha + \beta + \gamma + \delta$. Then we have

$$d(y_{2n+1}, y_{2n+2}) \leq h\, d(y_{2n}, y_{2n+1}). \tag{21}$$

Similarly, by putting $x = x_{2n-1}$ and $y = x_{2n}$ in (15), we have

$$d(y_{2n}, y_{2n+1}) \leq h\, d(y_{2n-1}, y_{2n}). \tag{22}$$

Similarly, continue this process, we have

$$d(y_{2n}, y_{2n+1}) \leq h^{2n}\, d(y_0, y_1). \tag{23}$$

For $k > n$ and using triangle inequality, we have

$$\begin{aligned} d(y_n, y_{n+k}) &\leq \sum_{i=1}^{k} d(y_{n+i-1}, y_{n+i}) \\ &\leq \sum_{i=1}^{k} h^{n+i-1}\, d(y_{n+i-1}, y_{n+i}) \\ &= \frac{h^n(1-h^k)}{1-h}\, d(y_0, y_1) \\ &\to 0 \ \ as\, n \to \infty. \end{aligned}$$

Hence $\{y_n\}$ is a cauchy sequence in X, so by completeness of X, sequence $\{y_n\}$ is converges to a point z in X. Also, every subsequences of sequence $\{y_n\}$ are also converges to z in X. Then we have

$$Ax_{2n} = Dx_{2n+1} \to z \ and\ Bx_{2n} = Cx_{2n+1} \to z \ as\ n \to \infty. \tag{24}$$

Since A and C are compatible of type (A) and suppose A is continuous map on X. Then, we have

$$AAx_{2n} \to Az \text{ and } CAx_{2n} \to Az \text{ as } n \to \infty. \tag{25}$$

Now, putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (15). We have

$$\begin{aligned} d(AAx_{2n}, Bx_{2n+1}) \leq\ & \alpha\Big[\frac{d(Dx_{2n+1}, Bx_{2n+1})d(CAx_{2n}, Dx_{2n+1})}{d(DAx_{2n}, AAx_{2n}) + d(Bx_{2n+1}, DAx_{2n})}\Big] \\ & + \beta\Big[\frac{d(AAx_{2n}, DAx_{2n})d(Ax_{2n+1}, Cx_{2n+1})}{d(DAx_{2n}, AAx_{2n}) + d(Bx_{2n+1}, DAx_{2n})}\Big] \\ & + \gamma\Big[\frac{d(DAx_{2n}, BAx_{2n})d(Bx_{2n+1}, Dx_{2n+1})}{d(DAx_{2n}, AAx_{2n}) + d(Bx_{2n+1}, DAx_{2n})}\Big] \\ & + \delta\Big[\frac{d(CAx_{2n}, Dx_{2n+1})d(AAx_{2n}, Bx_{2n+1})}{d(DAx_{2n}, AAx_{2n}) + d(Bx_{2n+1}, DAx_{2n})}\Big]. \end{aligned} \tag{26}$$

Using equation (24) and (25) in equation (26), we have

$$d(Az, z) \leq \delta d(Az, z)$$

$$\Rightarrow \quad (1 - \delta)d(Az, z) \leq 0$$

$$\Rightarrow \quad d(Az, z) = 0 \text{ as } \delta \neq 1.$$

Which gives

$$Az = z. \tag{27}$$

Similarly, by putting $x = Cx_{2n}$ and $y = x_{2n+1}$ in (15). Suppose A and C are compatible of type(A) and C is continuous on X. Then, we have

$$Cz = z. \tag{28}$$

Similarly, we can show that, if B, D are compatible of type (A) and either B or D are continuous. Then

$$Bz = Dz = z. \tag{29}$$

Therefore, from equation (27), (28) & (29), we have

$$Az = Bz = Cz = Dz = z. \tag{30}$$

Thus z is a common fixed point of A, B, C and D.

We shall prove the uniqueness of the common fixed point for this. Suppose z and w be two common fixed points of A, B, C and D.

$$i.e. \quad Az = Bz = Cz = Dz = z \text{ and } Aw = Bw = Cw = Dw = w. \tag{31}$$

Then from (15), we have

$$\begin{aligned} d(z,w) = d(Az,Bw) \leq \alpha\Big[\frac{d(Dw,Bw)d(Cz,Dw)}{d(Dz,Az)+d(Bw,Dz)}\Big] \\ + \beta\Big[\frac{d(Az,Dz)d(Aw,Cw)}{d(Dz,Az)+d(Bw,Dz)}\Big] \\ + \gamma\Big[\frac{d(Dz,Bz)d(Dw,Bw)}{d(Dz,Az)+d(Bw,Dz)}\Big] \\ + \delta\Big[\frac{d(Cz,Dw)d(Az,Bw)}{d(Dz,Az)+d(Bw,Dz)}\Big]. \end{aligned} \tag{32}$$

Using equation (30), we have

$$d(z,w) \leq \ \delta \ d(z,w)$$

$$\Rightarrow \quad (1-\delta)d(z,w) \leq 0$$

$$\Rightarrow \quad d(z,w) = 0 \ as \ n \to \infty.$$

Thus, we have

$$z \ = \ w. \tag{33}$$

Thus A, B, C and D have the unique common fixed point in X.

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