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A Geometric Model for Differential K-Homology

Adnane Elmrabty[1] and Mohamed Maghfoul[2]

[1,2]Department of Mathematics, Faculty of Sciences
Ibn Tofail University, Kenitra, Morocco
[1]Email: adnane_elmrabty@yahoo.com
[2]Email: mmaghfoul@lycos.com

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Abstract

In this paper, we construct a differential refinement of K-homology, using the (M, E^{∇^E}, f)-picture of Baum-Douglas for K-homology and continuous currents. This leads to a geometric realization of K-homology with coefficients in $\mathbb{R}/\mathbb{Z}$ and a description of Freed-Lott differential K-theory through the relative eta invariant.

Keywords: *Atiyah-Patodi-Singer index theorem, differential K-characters, differential K-theory, geometric K-homology.*

1 Introduction

A classical theorem in algebraic topology asserts that for every generalized cohomology theory there exists a dual homology theory, which correspond to each other by a spectrum. An important example of such a duality is K-theory and K-homology. K-theory was introduced by Grothendieck in the sixties and it is a generalized cohomology theory defined in terms of vector bundles. For the dual homology theory, K-homology, there are two different popular models. The so-called analytic K-homology was proposed by Atiyah [2] in the framework of index theory and worked out by Kasparov [14] in the seventies. An alternative model, geometric K-homology, was introduced by Baum and Douglas [7] in 1982. One of the main advantages of this geometric formulation is that K-homology cycles encode the most primitive requisite objects that must

be carried by any D-brane, such as a $Spin^c$-structure and a Hermitian vector bundle. In 2007, Baum, Higson and Schick [8] proved that this geometric picture is indeed equivalent to the other definitions.
Besides the classical cohomology theories, there are also so-called differential cohomology theories, which combine cohomological information with differential form information. Motivated from physics, in the last decade, such differential extensions of K-theory have been studied extensively (see Bunke-Schick [9]). Consequently, as Bunke and Schick write in Section 4.10 of their survey [9], "it is very desirable to have differentiable extensions also of K-homology".

The present paper proposes a definition of differential K-homology by combining the geometric picture of Baum and Douglas for K-homology with continuous currents. More precisely, let X be a smooth compact manifold, and $K^{geo}(X)$ its geometric K-homology. If $Ch_* : K^{geo}(X) \to H_*^{dR}(X)$ denotes the homological Chern character, we define the differential refinement $\check{K}$ of K^{geo} as a *homotopy pullback*

$$\begin{array}{ccc} \check{K}(X) & \xrightarrow{i} & K^{geo}(X) \\ \Big\downarrow{\scriptstyle \mathcal{R}} & \circlearrowleft & \Big\downarrow{\scriptstyle Ch_*} \\ \Omega_*^{cl}(X) & \xrightarrow{Rham} & H_*^{dR}(X) \end{array}$$

with a commutative diagram

$$\begin{array}{ccc} \Omega_{*+1}(X) & \xrightarrow{a} & \check{K}(X) \\ \Big\downarrow{\scriptstyle \partial} & \swarrow{\scriptstyle \mathcal{R}} & \\ \Omega_*(X) & & \end{array}$$

The natural transformations i (the underlying homology class), a (the action of continuous currents) and $\mathcal{R}$ (the characteristic closed continuous current) are essential parts of the picture. We define the flat K-homology $\check{K}^f(X)$ as the kernel of the curvature $\mathcal{R} : \check{K}(X) \to \Omega_*(X)$ and a group $\check{K}^0(X)$ out of $\mathbb{R}/\mathbb{Z}$-valued homomorphisms on the odd part of $\check{K}(X)$ (see Subsection 4.2). We obtain two short exact sequences

$$0 \longrightarrow \check{K}^f(X) \hookrightarrow \check{K}(X) \xrightarrow{\mathcal{R}} \Omega_*^0(X) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow Hom(K_{odd}^{geo}(X), \mathbb{R}/\mathbb{Z}) \xrightarrow{i^*} \check{K}^0(X) \xrightarrow{a^*} \Omega_0^{even}(X) \longrightarrow 0 \; ,$$

where $\Omega_*^0(X)$ denotes the group of closed continuous currents on X whose de Rham homology class lies in the image of Ch_*, and $\Omega_0^*(X)$ denotes the group of closed real-valued differential forms on X with integer K-periods (Definition 4.1).

The main result of this paper (Subsection 4.2) is the construction of an explicit isomorphism between $\check{K}^0(X)$ and the Freed-Lott differential K-group $\hat{K}_{FL}(X)$.

The format of this paper will be as follows: In Section 2, we define the differential K-homology group $\check{K}(X)$ and its flat part, and we point out some of their properties. Section 3 is concerned with the given of a pairing between differential K-homology and the Freed-Lott differential K-theory, which agrees with the K-theoretical and the K-homological curvatures. Finally, in Section 4, we explicit the construction of an isomorphism between $\check{K}^0(X)$ and $\hat{K}_{FL}(X)$.

2 Differential K-Homology Groups

In this section, we define differential K-homology groups, taking inspiration from the (M, E^{∇^E}, f)-picture of Baum-Douglas for K-homology [7] and the work of Freed-Lott [12].

Definition 2.1. *Let X be a smooth compact manifold. A K-chain over X is a triple, $(W, \varepsilon^{\nabla^\varepsilon}, g)$, where*

- *W is a smooth compact $Spin^c$-manifold;*
- *ε is a Hermitian vector bundle over M carrying with a Hermitian connection ∇^ε; and*
- *$g : W \to X$ is a smooth map.*

Here, the $Spin^c$-condition on W means that the orthonormal frame bundle of W has a topological reduction to a principal $Spin^c$-bundle.
There are no connectedness requirements made upon W, and hence the bundle ε can have different fibre dimensions on the different connected components of W. It follows that the disjoint union,

$$(W, \varepsilon^{\nabla^\varepsilon}, g) \sqcup (W', \varepsilon'^{\nabla^{\varepsilon'}}, g') := (W \sqcup W', \varepsilon \sqcup \varepsilon'^{\nabla^\varepsilon \sqcup \nabla^{\varepsilon'}}, g \sqcup g'),$$

is a well-defined operation on the set of K-chains over X.
The boundary $\partial(W, \varepsilon^{\nabla^\varepsilon}, g)$ of a K-chain $(W, \varepsilon^{\nabla^\varepsilon}, g)$ is the K-cycle $(\partial W, \varepsilon|_{\partial W}{}^{\nabla^\varepsilon|_{\partial W}}, g|_{\partial W})$. A K-cycle is a K-chain with empty boundary.
A K-cycle (M, E^{∇^E}, f) is called even (resp. odd), if all connected components of M are of even (resp. odd) dimension.
Two K-cycles (M, E^{∇^E}, f) and $(M', E'^{\nabla^{E'}}, f')$ over X are isomorphic, if there exists a diffeomorphism $h : M \to M'$ such that

- h preserves the $Spin^c$-structures;

- $h^*E' \cong E$; and
- the diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ X & & \end{array}$$

commutes.

Recall that on any smooth n-manifold X one has the de Rham chain complex

$$\Omega_n(X) \xrightarrow{\partial} \Omega_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_0(X)$$

where $\Omega_p(X)$ is the space of continuous real-valued p-currents on X and the current $\partial\phi$ is given on differential forms by $\partial\phi(w) = \phi(dw)$ where d is the exterior derivative. Let $\Omega_*(X)$ denote $\Omega_*(X) := \oplus_{p\geq 0}\Omega_p(X)$. If $\Omega_{even}(X)$ and $\Omega_{odd}(X)$ denote, respectively, $\oplus_{k\geq 0}\Omega_{2k}(X)$ and $\oplus_{k\geq 0}\Omega_{2k+1}(X)$, then the group $\Omega_*(X) = \Omega_{even}(X) \oplus \Omega_{odd}(X)$ has a natural $\mathbb{Z}_2$-grading.

Definition 2.2. *Let X be a smooth compact manifold. A differential K-cycle over X is a pair, (ϑ, ϕ), where*

- *ϑ is a K-cycle over X; and*
- *$\phi \in \frac{\Omega_*(X)}{img(\partial)}$.*

A differential K-cycle (ϑ, ϕ) is called even (resp. odd), if ϑ is even (resp. odd) and $\phi \in \frac{\Omega_{odd}(X)}{img(\partial)}$ (resp. $\phi \in \frac{\Omega_{even}(X)}{img(\partial)}$).

Let E be a smooth Hermitian vector bundle over a smooth compact manifold M. The geometric Chern form of a Hermitian connection ∇ on E is the closed real-valued even-degree differential form on M

$$ch(\nabla) := tr(e^{\frac{-\nabla^2}{2i\pi}}) = \sum_{j=1} ch_j(\nabla),$$

where $ch_j(\nabla) = \frac{1}{j!}tr\left(\frac{-\nabla^2}{2i\pi}\right)^j$.
If ∇_1 and ∇_2 are two Hermitian connections on E, there is a canonically-defined Chern-Simons class $CS(\nabla_1, \nabla_2) \in \frac{\Omega^{odd}(M)}{img(d)}$ [15] such that

$$dCS(\nabla_1, \nabla_2) = ch(\nabla_1) - ch(\nabla_2).$$

This implies that the de Rham cohomology class of $ch(\nabla)$ does not depend on the choice of ∇. This class will be denoted by $Ch(E)$ and called the Chern character of E.

Let M be a smooth $Spin^c$-manifold. Let S be the spinor bundle associated with the $Spin^c$-structure of M. We denote by $L := S \times_\gamma \mathbb{C}$ the Hermitian line bundle over M associated with S by the homomorphism $\gamma : Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \mapsto U(1)$ which is trivial on the $Spin(n)$-factor and is the square on the $U(1)$-factor. If ∇^L is a Hermitian connection on L, then the Todd form of the Levi-Civita connection ∇^M on M is defined by

$$Td(\nabla^M) := e^{\frac{ch_1(\nabla^L)}{2}} \wedge \hat{A}(\nabla^M),$$

where $\hat{A}(\nabla^M)$ is the $\hat{A}$-polynomial in the Pontryagin forms of ∇^M, defined by using the multiplicative sequence associated with the series [16]

$$\frac{x/2}{sinh(x/2)}.$$

Definition 2.3. *(Isomorphism). Let X be a smooth compact manifold. Two differential K-cycles $(M, E^{\nabla^E}, f, \phi)$ and $(M', E'^{\nabla^{E'}}, f', \phi')$ over X are isomorphic, if there exists an isomorphism $h : M \to M'$ between the two K-cycles (M, E^{∇^E}, f) and $(M', E'^{\nabla^{E'}}, f')$ such that*

$$\phi - \phi' = \int_{M\times[0,1]} Td(\nabla^{M\times[0,1]})ch(B)(f \circ p)^*,$$

where B is the connection on the pullback of E by the projection $p : M\times[0, 1] \to M$ given by $B = (1 - t)\nabla^E + th^*\nabla^{E'} + dt\frac{d}{dt}$.

The set of isomorphism classes of differential K-cycles over X is denoted $\check{C}(X)$. It is an abelian semigroup under the operation of addition,

$$(\vartheta, \phi) + (\vartheta', \phi') := (\vartheta \sqcup \vartheta', \phi + \phi').$$

Definition 2.4. *(Bordism). Two differential K-cycles $(M, E^{\nabla^E}, f, \phi)$ and $(M', E'^{\nabla^{E'}}, f', \phi')$ over X are bordant, if there exists a K-chain $(W, \varepsilon^{\nabla^\varepsilon}, g)$ over X such that the two K-cycles $(M\sqcup M'^-, E \sqcup E'^{\nabla^E\sqcup\nabla^{E'}}, f\sqcup f')$ and $\partial(W, \varepsilon^{\nabla^\varepsilon}, g)$ are isomorphic and $\phi - \phi' = [\int_W Td(\nabla^W)ch(\nabla^\varepsilon)g^*]$, where M'^- denotes M' with its $Spin^c$-structure reversed [7]. A differential K-cycle z over X is called a boundary in X if there exists a K-chain $(W, \varepsilon^{\nabla^\varepsilon}, g)$ over X such that $z = (\partial W, \varepsilon|_{\partial W}^{\nabla^\varepsilon|_{\partial W}}, g|_{\partial W}, [\int_W Td(\nabla^W)ch(\nabla^\varepsilon)g^*])$.*

We have one more operation on differential K-cycles to introduce.
Let $(M, E^{\nabla^E}, f, \phi)$ be a differential K-cycle over X, and let H be a $Spin^c$-Euclidean vector bundle over M with even-dimensional fibers and ∇^H an Euclidean connection on H. Let 1 denote the trivial rank-one real vector bundle.

The direct sum $H \oplus 1$ is a $Spin^c$-vector bundle, and moreover the total space of this bundle may be equipped with a $Spin^c$-structure in a canonical way. This is because its tangent bundle fits into an exact sequence

$$0 \to \pi^*[H \oplus 1] \to T(H \oplus 1) \to \pi^*[TM] \to 0$$

where π is the projection from $H \oplus 1$ onto M.
Let us now denote by $\hat{M}$ the unit sphere bundle of the bundle $H \oplus 1$. Since $\hat{M}$ is the boundary of the disk bundle, we may equip it with a natural $Spin^c$-structure by first restricting the given $Spin^c$-structure on the total space of $H \oplus 1$ to the disk bundle, and then taking the boundary of this $Spin^c$-structure to obtain a $Spin^c$-structure on the sphere bundle.
Let $S = S_- \oplus S_+$ be the $\mathbb{Z}_2$-graded spinor bundle associated with the $Spin^c$-structure of H with a fixed Hermitian connection $\nabla^S = \nabla^{S_-} \oplus \nabla^{S_+}$. Let $\mathcal{S}_-$ and $\mathcal{S}_+$ denote, respectively, the pullbacks of S_- and S_+ to the total space of H. Since $\hat{M}$ consists of two copies of the ball bundle of H glued together by the identity map of the sphere bundle $\mathbb{S}(H)$ of H, the clutching of $\mathcal{S}_+$ and $\mathcal{S}_-$ using Clifford multiplication over $\mathbb{S}(H)$ yields a new vector bundle $\hat{H}$ over $\hat{M}$. Let $\nabla^{\hat{H}}$ be the Hermitian connection on $\hat{H}$ induced by ∇^H and ∇^S.

Definition 2.5. *(Vector bundle modification). The process of obtaining the differential K-cycle* $(\hat{M}, \hat{H} \otimes \pi^* E^{\nabla^{\hat{H}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi)$ *from* $(M, E^{\nabla^E}, f, \phi)$ *is called vector bundle modification.*

We are now ready to define the differential K-homology $\check{K}(X)$ of X.

Definition 2.6. *The differential K-homology* $\check{K}(X)$ *of* X *is the group obtained from quotienting* $\check{C}(X)$ *by the equivalence relation* $\sim$ *generated by the relations of*

(i) direct sum:

$$(M, E^{\nabla^E}, f, \phi) + (M, E'^{\nabla^{E'}}, f, \phi') \sim (M, E \oplus E'^{\nabla^E \oplus \nabla^{E'}}, f, \phi + \phi');$$

(ii) bordism*; and*

(iii) vector bundle modification*.*

The group operation is induced by addition of differential K-cycles. We denote the differential homology class of a differential K-cycle $(M, E^{\nabla^E}, f, \phi)$ by $[M, E^{\nabla^E}, f, \phi]$. The inverse of a class $[M, E^{\nabla^E}, f, \phi] \in \check{K}(X)$ is equal to $[M^-, E^{\nabla^E}, f, -\phi]$, and the neutral element of $\check{K}(X)$ is represented by any boundary in X.

Since the equivalence relation $\sim$ preserves the parity of the dimension of M in differential K-cycles $(M, E^{\nabla^E}, f, \phi)$, one can define the subgroup $\check{K}_{even}(X)$ (resp. $\check{K}_{odd}(X)$) consisting of classes of even (resp. odd) differential K-cycles. Then $\check{K}(X) = \check{K}_{even}(X) \oplus \check{K}_{odd}(X)$ has a natural $\mathbb{Z}_2$-grading.

The construction of differential K-homology is functorial. If $\rho : X \to Y$ is a smooth map between two smooth compact manifolds, then the induced homomorphism

$$\check{\rho} : \check{K}(X) \to \check{K}(Y)$$

of $\mathbb{Z}_2$-graded abelian groups is given on classes of differential K-cycles $[M, E^{\nabla^E}, f, \phi] \in \check{K}(X)$ by

$$\check{\rho}[M, E^{\nabla^E}, f, \phi] := [M, E^{\nabla^E}, \rho \circ f, \phi \circ \rho^*],$$

where $\rho^* : \Omega^*(Y) \to \Omega^*(X)$ is the pullback map.

We can measure the size of $\check{K}$ by inserting it in a certain exact sequence.

Let $\Omega^0_*(X)$ denote the group of closed real-valued continuous currents on X whose de Rham homology class lies in the image of $Ch_* : K^{geo}_*(X) \to \frac{\Omega^{cl}_*(X)}{img(\partial)}$ with $Ch_*[M, E^{\nabla^E}, f] = [\int_M Td(\nabla^M) ch(\nabla^E) f^*]$.
Let $a : \Omega_*(X) \to \check{K}_{*+1}(X)$ be the additive map that associates with each $\phi \in \Omega_*(X)$ the class $[\emptyset, \emptyset, \emptyset, -[\phi]] \in \check{K}_{*+1}(X)$. If $\phi \in \Omega^0_*(X)$, then there exists a K-cycle (M, E^{∇^E}, f) over X such that $[\phi] = [\int_M Td(\nabla^M) ch(\nabla^E) f^*]$. It follows that $(\emptyset, \emptyset, \emptyset, -[\phi]) = (\partial M^-, E|_{\partial M^-}^{\nabla^E|_{\partial M^-}}, f|_{\partial M^-}, [\int_{M^-} Td(\nabla^M) ch(\nabla^E) f^*])$, and then ϕ represents the zero element of $\check{K}_{*+1}(X)$. Hence, a induces a well-defined homomorphism from $\frac{\Omega_*(X)}{\Omega^0_*(X)}$ into $\check{K}_{*+1}(X)$, still denoted by a. Moreover, we have a short exact sequence

$$0 \to \frac{\Omega_{*-1}(X)}{\Omega^0_{*-1}(X)} \xrightarrow{a} \check{K}_*(X) \xrightarrow{i} K^{geo}_*(X) \to 0,$$

where i is the forgetful homomorphism.
This, together with the fact that the only K-cycles on pt are $(pt, \mathbb{C}^k, id_{pt})$, implies that

$$\check{K}_{even}(pt) = K^{geo}_{even}(pt) \cong \mathbb{Z} \quad \text{and} \quad \check{K}_{odd}(pt) \cong \mathbb{R}/\mathbb{Z}.$$

We have two short exact sequences

$$0 \to \mathbb{R}/\mathbb{Z} \to \check{K}_{even}(S^1) \to \mathbb{Z} \to 0 \;\; \textit{and}$$

$$0 \to \frac{Hom(C^\infty(S^1), \mathbb{R})}{Hom_0(C^\infty(S^1), \mathbb{R})} \to \check{K}_{odd}(S^1) \to \mathbb{Z} \to 0,$$

where $Hom_0(C^\infty(S^1), \mathbb{R})$ denotes the group of homomorphisms $\phi : C^\infty(S^1) \to \mathbb{R}$ where $\phi(1) \in \mathbb{Z}$. The second exact sequence above implies that a lift to $\mathbb{R}$ of the homomorphism that associates with each closed curve $\gamma \in C^\infty(S^1)$ the holonomy around γ, $H(\gamma) \in SO(2) \cong \mathbb{R}/\mathbb{Z}$, induces a class in $\check{K}_{odd}(S^1)$ which depend only on H.

Let us now construct an index map $\widetilde{\eta} : \check{K}_{odd}(X) \to \mathbb{R}/\mathbb{Z}$.
We first recall the construction of the eta invariant.

Let M be an $2p-1$-dimensional smooth closed $Spin^c$-manifold. Let E be a Hermitian vector bundle over M with a fixed Hermitian connection. Denote by D_E the closure of the Dirac operator acting on the spinor bundle S on M with coefficients in E. The operator D_E is odd for the $\mathbb{Z}_2$-grading

$$S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E),$$

and we shall denote by D_E^+ the operator D_E acting from $S^+ \otimes E$ to $S^- \otimes E$. The spectrum $(\lambda_i)_{i\in I}$ of D_E is a discrete subset of $\mathbb{R}$. The eta function of D_E is then defined by

$$\eta(s, D_E) := \sum_{\substack{\lambda_i \neq 0 \\ i \in I}} \lambda_i |\lambda_i|^{-(s+1)}, \quad Re(s) \gg 0.$$

From the classical spectral estimates, the above series is known to be absolutely convergent in the half-plane $Re(s) > 2p-1$. Furthermore, it can be extended to a meromorphic function on the complex plane with simple poles [3, 4, 5]. We denote also by $s \mapsto \eta(s, D_E)$ this extension. An important result due to Atiyah, Patodi and Singer [3] states that the residue of the function $s \mapsto \eta(s, D_E)$ at zero is trivial. The number $\eta(0, D_E)$ is thus well defined. The eta (spectral) invariant of the operator D_E is then by definition

$$\eta_E := \eta(0, D_E).$$

The eta invariant is a measure of the spectral asymmetry of D_E.

Now let W be an even-dimensional smooth compact $Spin^c$-manifold with boundary. Let ε be a Hermitian vector bundle over W with a Hermitian connection ∇^ε. Suppose that the metric and connection are constant in the normal direction near the boundary and denote by D_ε the closure of the Dirac operator acting on the spinor bundle on W with coefficients in ε with respect to the global Szegö boundary condition considered in [3]. Near the boundary, we have

$$D_\varepsilon \simeq \sigma(\frac{\partial}{\partial t} + D_{\varepsilon|\partial W}),$$

where σ is a bundle isomorphism (Clifford multiplication by the inward unit vector).
The Atiyah-Patodi-Singer index theorem relates the Fredholm index of D^+_ε with topological and spectral invariants. More precisely, we have

$$Ind(D^+_\varepsilon) = \int_W Td(\nabla^W)ch(\nabla^\varepsilon) - \bar{\eta}_{\varepsilon|_{\partial W}},$$

where $\bar{\eta}_{\varepsilon|_{\partial W}} := \frac{\eta_{\varepsilon|_{\partial W}} + dim\ Ker(D_{\varepsilon|_{\partial W}})}{2}$.

Proposition 2.7. *There is an index map*

$$\widetilde{\eta} : \check{K}_{odd}(X) \to \mathbb{R}/\mathbb{Z}$$

given through the eta invariant.

Proof. Let $(M, E^{\nabla^E}, f, \phi)$ be an odd differential K-cycle over X. Set

$$\widetilde{\eta}(M, E^{\nabla^E}, f, \phi) := \bar{\eta}_E - \phi(1) \mod \mathbb{Z}.$$

The map $\widetilde{\eta}$ is obviously additive. We show that $\widetilde{\eta}$ is compatible with the equivalence relation on differential K-cycles. Compatibility with relation (i) from Definition 2.6 is straightforward.
Let $(W, \varepsilon^{\nabla^\varepsilon}, g)$ be an even K-chain over X. The Atiyah-Patodi-Singer index theorem [3, 4, 5] implies that

$$\bar{\eta}_{\varepsilon|_{\partial W}} - \int_W Td(\nabla^W)ch(\nabla^\varepsilon) = -Ind(D^+_\varepsilon) \in \mathbb{Z}.$$

Then $\widetilde{\eta}$ is compatible with the relation (ii) of bordism. So the proof reduces to showing that $\widetilde{\eta}$ is compatible with the relation (iii) of vector bundle modification.
Let (M, E^{∇^E}, f) be an odd K-cycle over X, and let $H \to M$ be an even $Spin^c$-vector bundle of dimension $2p$. We consider the smooth closed manifold $\hat{M}$ which has been defined above (Definition 2.5) and which is an $\mathbb{S}^{2p}$-fibration over M,

$$\pi : \hat{M} \to M.$$

If $S_{\mathbb{S}^{2p}} = S^+_{\mathbb{S}^{2p}} \oplus S^-_{\mathbb{S}^{2p}}$ and $S_M = S^+_M \oplus S^-_M$ are the spinor bundles associated with the $Spin^c$-structures on the tangent vector bundles $T\mathbb{S}^{2p}$ and TM respectively, then the spinor bundle $S_{\hat{M}}$ associated with the tangent vector bundle $T\hat{M}$ is isomorphic to the graded tensor product vector bundle $\tilde{S}_{\mathbb{S}^{2p}} \hat{\otimes} \tilde{S}_M$, where $\tilde{S}_{\mathbb{S}^{2p}}$ and $\tilde{S}_M$ are corresponding lifts to $\hat{M}$. Let B be the Bott bundle over $\mathbb{S}^{2p}$ (see [1] for the construction of this element). We denote by D_B the self-adjoint Dirac operator on $\mathbb{S}^{2p}$ with coefficients in B. The index of D^+_B is equal to 1.

According to [6], we get out of D_B a differential operator $\widetilde{D}_B$ on $\hat{M}$ acting on smooth sections of the vector bundle $S_{\hat{M}} \otimes \hat{H} \otimes \pi^* E$. In the same way and following the same reference [6], we get out of the Dirac operator on M twisted by E, D_E, a differential operator $\widetilde{D}_E$ over $\hat{M}$ acting on smooth sections of $S_{\hat{M}} \otimes \hat{H} \otimes \pi^* E$.
The sharp product of $\widetilde{D}_B$ and $\widetilde{D}_E$ yields an elliptic differential operator $\widetilde{D}_B \sharp \widetilde{D}_E$ acting on sections of $S_{\hat{M}} \otimes \hat{H} \otimes \pi^* E$. This operator can be identified with the Dirac operator on $\hat{M}$ twisted by the vector bundle $\hat{H} \otimes \pi^* E$:

$$D_{\hat{H} \otimes \pi^* E} = \widetilde{D}_B \sharp \widetilde{D}_E.$$

We can work locally and assume that the fibration $\pi : \hat{M} \to M$ is trivial: π is the projection $\mathbb{S}^{2p} \times M \to M$. The Hilbert space on which $D_{\hat{H} \otimes \pi^* E}$ acts is the graded tensor product

$$L^2(\mathbb{S}^{2p} \times M, S_{\hat{M}} \otimes \hat{H} \otimes \pi^* E) = L^2(\mathbb{S}^{2p}, S_{\mathbb{S}^{2p}} \otimes B) \hat{\otimes} L^2(M, S_M \otimes E).$$

If we split the first factor, $L^2(\mathbb{S}^{2p}, S_{\mathbb{S}^{2p}} \otimes B)$, as $\ker(D_B^+)$ plus its orthogonal complement, then we obtain a corresponding direct sum decomposition of $L^2(\mathbb{S}^{2p} \times M, S_{\hat{M}} \otimes \hat{H} \otimes \pi^* E)$. We therefore obtain a decomposition of $D_{\hat{H} \otimes \pi^* E}$ as a direct sum of two operators. Since the kernel of D_B^+ is one-dimensional, the first operator acts on $\ker(D_B^+) \hat{\otimes} L^2(M, S_M \otimes E) \cong L^2(M, S_M \otimes E)$ and is equal to D_E. The second operator has a antisymmetric spectrum. To see this, if T is the partial isometry part of D_B^+ in the polar decomposition, and if γ is the grading operator on $L^2(M, S_M \otimes E)$, then the odd-graded involution $iT \hat{\otimes} \gamma$ on the Hilbert space $\ker(D_B^+)^\perp \hat{\otimes} L^2(M, S_M \otimes E)$ anticommutes with the restriction of $D_{\hat{H} \otimes \pi^* E}$ to $\ker(D_B^+)^\perp \hat{\otimes} L^2(M, S_M \otimes E)$. Furthermore, the kernel of $D^+_{\hat{H} \otimes \pi^* E}$ coincides with the kernel of D_E^+. Since the same relation holds for the adjoint, we deduce that

$$\widetilde{\eta}(M, E^{\nabla^E}, f, \phi) = \widetilde{\eta}(\hat{M}, \hat{H} \otimes \pi^* E^{\nabla^{\hat{H}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi).$$

□

Remark 2.8. *Let us consider the collapse map $\epsilon : X \to pt$. We show that the index map $\epsilon_* : \check{K}_{odd}(X) \to \check{K}_{odd}(pt) \cong \mathbb{R}/\mathbb{Z}$ is realized analytically by $\widetilde{\eta}$. Let $(M, E^{\nabla^E}, f, \phi)$ be an odd differential K-cycle over X. Let $H \cong \mathbb{S}^3 \to \mathbb{C}P^1 \cong \mathbb{S}^2$ be the Hopf hyperplane bundle with the natural connection form $\omega = \bar{z}_1 dz_1 + \bar{z}_2 dz_2$ where z_1,z_2 are standard complex coordinates on $\mathbb{C}^2$. Following a theorem due to Michael Hopkins, there is a positive integer k such that the K-cycle $(M \times S^{2k}, E \otimes H^{k^{\nabla^E \otimes \omega^k}}, M \times S^{2k} \to pt)$ is the boundary of a*

K-chain $(W, \varepsilon^{\nabla^\varepsilon}, W \to pt)$*. It follows that*

$$\begin{aligned}
\epsilon_*[M, E^{\nabla^E}, f, \phi] &= [M, E^{\nabla^E}, M \to pt, \phi(1)] \\
&= [M \times S^{2k}, E \otimes H^{k^{\nabla^E \otimes \omega^k}}, M \times S^{2k} \to pt, \phi(1)] \\
&= [\partial W, \varepsilon|_{\partial W}^{\nabla^\varepsilon|_{\partial W}}, \partial W \to pt, \phi(1)] \\
&= [\emptyset, \emptyset, \emptyset, -\int_W Td(\nabla^W) ch(\nabla^\varepsilon) + \phi(1)] \\
&= [\emptyset, \emptyset, \emptyset, -\bar{\eta}_{E \otimes H^k} + \phi(1)] \\
&= [\emptyset, \emptyset, \emptyset, -Ind(D_H^+)^k \times \bar{\eta}_E + \phi(1)] \\
&= [\emptyset, \emptyset, \emptyset, -\bar{\eta}_E + \phi(1)] \\
&= a(\widetilde{\eta}[M, E^{\nabla^E}, f, \phi]).
\end{aligned}$$

Definition 2.9. *Let* $(M, E^{\nabla^E}, f, \phi)$ *be a differential K-cycle over* X*. The curvature* $\mathcal{R}(M, E^{\nabla^E}, f, \phi)$ *of* $(M, E^{\nabla^E}, f, \phi)$ *is the real-valued current on* X *given by*

$$\mathcal{R}(M, E^{\nabla^E}, f, \phi) := \int_M Td(\nabla^M) ch(\nabla^E) f^* - \partial \phi.$$

Proposition 2.10. *The curvature defined above induces a group homomorphism*

$$\mathcal{R} : \check{K}(X) \to \Omega_*(X).$$

Proof. It is obvious that $\mathcal{R}$ is compatible with the relation (i) from Definition 2.6. Let $(W, \varepsilon^{\nabla^\varepsilon}, g)$ be a K-chain over X. Stokes' theorem implies that

$$\begin{aligned}
\mathcal{R}(\partial W, \varepsilon|_{\partial W}{}^{\nabla^\varepsilon|_{\partial W}}, g|_{\partial W}, \int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*) &= \int_{\partial W} \big(Td(\nabla^W) ch(\nabla^\varepsilon) g^*(\cdot)\big)\,|_{\partial W} \\
&\quad - \int_W d\left(Td(\nabla^W) ch(\nabla^\varepsilon) g^*(\cdot)\right) = 0.
\end{aligned}$$

On the other hand, let $\pi : \hat{M} \to M$ be the even unit sphere bundle constructed out of an even-dimensional $Spin^c$-vector bundle H over M. Let us compute the curvature $\mathcal{R}(\hat{M}, \hat{H} \otimes \pi^* E^{\nabla^{\hat{H}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi)$ of the modification $(\hat{M}, \hat{H} \otimes \pi^* E^{\nabla^{\hat{H}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi)$ of $(M, E^{\nabla^E}, f, \phi)$. Denote by $\pi_!$ integration of differential forms along the fibers of π

$$\pi_! : \Omega^*(\hat{M}) \to \Omega^{*-2p}(M),$$

where $2p = dim(\hat{M}) - dim(M)$ is the dimension of the fibers of π. We first observe that

$$\int_{\hat{M}} Td(\nabla^{\hat{M}}) ch(\nabla^{\hat{H}} \otimes \pi^* \nabla^E)(f \circ \pi)^* = \int_M \Big(\pi_!(Td(\nabla^{\hat{M}}) ch(\nabla^{\hat{H}}))\Big)\, ch(\nabla^E) f^*.$$

But $\pi_!(Td(\nabla^{\hat{M}})ch(\nabla^{\hat{H}}))$ coincides with the Todd form of the Levi-Civita connection on M. More precisely, we can work locally and assume that the fibration $\pi : \hat{M} \to M$ is trivial. So $Td(\nabla^{\hat{M}}) = \pi^* Td(\nabla^M) \wedge p^* Td(\nabla^{\mathbb{S}^{2p}})$. Here, p is the projection $\mathbb{S}^{2p} \times M \to \mathbb{S}^{2p}$. Thus

$$\mathcal{R}(\hat{M}, \hat{H} \otimes \pi^* E^{\nabla^{\hat{H}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi) = \int_{\mathbb{S}^{2p}} Td(\nabla^{\mathbb{S}^{2p}}) ch(\nabla^{\hat{H}}|_{\mathbb{S}^{2p}}) \times \int_M Td(\nabla^M) \wedge ch(\nabla^E) f^* - \partial\phi.$$

However, the Atiyah-Singer index theorem in $\mathbb{S}^{2p}$ shows that $\int_{\mathbb{S}^{2p}} Td(\nabla^{\mathbb{S}^{2p}}) \wedge ch(\nabla^{\hat{H}}|_{\mathbb{S}^{2p}})$ is equal to 1. □

Note that $\mathcal{R} \circ a = \partial$. Moreover, we have a short exact sequence

$$0 \longrightarrow \frac{\Omega^{cl}_{*-1}(X)}{\Omega^0_{*-1}(X)} \xrightarrow{a} \check{K}_*(X) \xrightarrow{(\mathcal{R},i)} R_*(X) \longrightarrow 0 ,$$

where $R_*(X) = \{(\phi, \vartheta) \in \Omega^0_*(X) \times K^{geo}_*(X) \mid [\phi] = Ch_*(\vartheta)\}$.
In particular, if X is a smooth compact oriented manifold which has trivial de Rham cohomology, then $x \in \check{K}_*(X)$ is determined uniquely by $(\mathcal{R}(x), i(x))$.

Definition 2.11. *Denote by $\check{K}^f(X)$ the kernel of $\mathcal{R}$.*

The group $\check{K}^f(X)$ has a natural $\mathbb{Z}_2$-grading, and we have

$$\check{K}^f_{even}(pt) = 0 \quad \text{and} \quad \check{K}^f_{odd}(pt) = \check{K}_{odd}(pt) \cong \mathbb{R}/\mathbb{Z}.$$

Let $\rho : X \to Y$ be a smooth map between two smooth compact manifolds. Since $\check{\rho} : \check{K}(X) \to \check{K}(Y)$ satisfies $\mathcal{R} \circ \check{\rho} = \rho_* \circ \mathcal{R}$, it induces a well-defined homomorphism $\check{K}^f(X) \to \check{K}^f(Y)$ of $\mathbb{Z}_2$-graded abelian groups, also denoted by $\check{\rho}$. It follows that the groups $\check{K}^f_{even}(X)$ and $\check{K}^f_{odd}(X)$ are functorial in X.

Proposition 2.12. *The functor $\check{K}^f$ is homotopy invariant.*

Proof. Let $\rho_k : X \to Y$, $k = 0, 1$, be two smooth homotopic maps. Let $(M, E^{\nabla^E}, f, [\phi])$ be a differential K-cycle over X with zero curvature. We check that $\check{\rho}_0[M, E^{\nabla^E}, f, [\phi]] = \check{\rho}_1[M, E^{\nabla^E}, f, [\phi]]$ in $\check{K}^f(Y)$. Let $\rho : [0,1] \times X \to Y$ be a smooth homotopy between ρ_0 and ρ_1. Let $i_k : X \to \{k\} \times X \subset [0,1] \times X$,

$k = 0, 1$, be the inclusions. For every $w \in \Omega^*(Y)$,

$$\begin{aligned}
\phi \circ \rho_1^*(w) - \phi \circ \rho_0^*(w) &= \phi(i_1^*(\rho^* w) - i_0^*(\rho^* w)) \\
&= \phi(d \int_{[0,1]} \rho^* w + \int_{[0,1]} d\rho^* w) \\
&= (\partial\phi)(\int_{[0,1]} \rho^* w) + \partial(\phi \circ (p_X : [0,1] \times X \to X)_! \circ \rho^*)(w) \\
&= \int_M Td(\nabla^M) ch(\nabla^E) f^*(\int_{[0,1]} \rho^* w) \\
&\quad + \partial(\phi \circ (p_X : [0,1] \times X \to X)_! \circ \rho^*)(w) \\
&= \int_{[0,1]\times M} Td(\nabla^{[0,1]\times M}) ch({p_M}^* \nabla^E)(\rho \circ (id_{[0,1]} \times f))^*(w) \\
&\quad + \partial(\phi \circ (p_X : [0,1] \times X \to X)_! \circ \rho^*)(w).
\end{aligned}$$

Here, $p_M : [0,1] \times M \to M$ and $p_X : [0,1] \times X \to X$ are projections. Then $([0,1] \times M, {p_M}^* E^{{p_M}^* \nabla^E}, \rho \circ (id_{[0,1]} \times f))$ is a bordism between $(M, E^{\nabla^E}, \rho_0 \circ f, [\phi] \circ \rho_0^*)$ and $(M, E^{\nabla^E}, \rho_1 \circ f, [\phi] \circ \rho_1^*)$. □

Remark 2.13. *Note that we have a short exact sequence*

$$0 \longrightarrow \check{K}_*^f(X) \hookrightarrow \check{K}_*(X) \xrightarrow{\mathcal{R}} \Omega_*^0(X) \longrightarrow 0 .$$

So, when X is contractible, the surjective homomorphism $\mathcal{R} : \check{K}_{even}(X) \to \Omega^0_{even}(X)$ turn out to be an isomorphism.

Now, we will define a homomorphism $Ch_*^{\mathbb{R}/\mathbb{Q}} : \check{K}_*^f(X) \to \frac{\Omega^{cl}_{*+1}(X, \mathbb{R}/\mathbb{Q})}{img(\partial)}$ which fits into a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \frac{\Omega^{cl}_{*+1}(X,\mathbb{R})}{img(\partial)} & \xrightarrow{a} & \check{K}_*^f(X) & \xrightarrow{i} & K_*^{geo}(X) & \longrightarrow \cdots \\
& \Big\downarrow {\scriptstyle -Id} & \circlearrowleft & \Big\downarrow {\scriptstyle Ch_*^{\mathbb{R}/\mathbb{Q}}} & \circlearrowleft & \Big\downarrow {\scriptstyle Ch_*} & \\
\cdots \longrightarrow & \frac{\Omega^{cl}_{*+1}(X,\mathbb{R})}{img(\partial)} & \longrightarrow & \frac{\Omega^{cl}_{*+1}(X,\mathbb{R}/\mathbb{Q})}{img(\partial)} & \longrightarrow & \frac{\Omega^{cl}_{*}(X,\mathbb{Q})}{img(\partial)} & \longrightarrow \cdots
\end{array}$$

where the bottom row is a Bockstein sequence. Upon tensoring everything with $\mathbb{Q}$, it follows from the five-lemma that $Ch_*^{\mathbb{R}/\mathbb{Q}}$ is a rational isomorphism.

We define $Ch_*^{\mathbb{R}/\mathbb{Q}}$ on $\check{K}_*^f(X)$. Let $(M, E^{\nabla^E}, f, \phi)$ be a differential K-cycle over X with zero curvature. Then the class of (M, E^{∇^E}, f) in $K_*^{geo}(X)$ has vanishing Chern character. Thus there is a positive integer k such that $k(M, E^{\nabla^E}, f)$ is the boundary of a K-chain $(W, \varepsilon^{\nabla^\varepsilon}, g)$. It follows from the definitions that $[\frac{1}{k} \int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*] - \phi$ is an element of $\frac{\Omega^{cl}_{*+1}(X,\mathbb{R})}{img(\partial)}$. Let $Ch_*^{\mathbb{R}/\mathbb{Q}}(M, E^{\nabla^E}, f, \phi)$

be the image of $[\frac{1}{k}\int_W Td(\nabla^W)ch(\nabla^{\varepsilon})g^*]-\phi$ under the natural homomorphism $\frac{\Omega^{cl}_{*+1}(X,\mathbb{R})}{img(\partial)} \to \frac{\Omega^{cl}_{*+1}(X,\mathbb{R}/\mathbb{Q})}{img(\partial)}$. We show that $Ch^{\mathbb{R}/\mathbb{Q}}_*(M,E^{\nabla^E},f,\phi)$ is independent of the choices of k and $(W,\varepsilon^{\nabla^{\varepsilon}},g)$. Suppose that k' is another positive integer such that $k'(M,E^{\nabla^E},f)$ is the boundary of a K-chain $(W',{\varepsilon'}^{\nabla^{\varepsilon'}},g')$. Then

$$\begin{aligned}(kk')\left([\frac{1}{k}\int_W Td(\nabla^W)ch(\nabla^{\varepsilon})g^*]-[\frac{1}{k'}\int_{W'} Td(\nabla^{W'})ch(\nabla^{\varepsilon'})g'^*]\right) &= [\int_{k'W} Td(\nabla^{k'W}) \\ &\quad \wedge ch(\nabla^{k'\varepsilon})(k'g)^*] \\ &\quad -[\int_{kW'} Td(\nabla^{kW'}) \\ &\quad \wedge ch(\nabla^{k\varepsilon'})(kg')^*] \\ &= Ch_*[P,V^{\nabla^V},j]\end{aligned}$$

where (P,V^{∇^V},j) is the K-cycle obtained by gluing the two K-chains $k'(W,\varepsilon^{\nabla^{\varepsilon}},g)$ and $k(W',{\varepsilon'}^{\nabla^{\varepsilon'}},g')$ along their boundary via the isomorphism $k'\partial(W,\varepsilon^{\nabla^{\varepsilon}},g) \overset{\cong}{\to} kk'(M,E^{\nabla^E},f) \overset{\cong}{\to} k\partial(W',{\varepsilon'}^{\nabla^{\varepsilon'}},g')$. Then $[\frac{1}{k}\int_W Td(\nabla^W)ch(\nabla^{\varepsilon})g^*]$ $-[\frac{1}{k'}\int_{W'} Td(\nabla^{W'})ch(\nabla^{\varepsilon'})g'^*]$ is the same, up to multiplication by rational numbers, as the image of $Ch_*[P,V^{\nabla^V},j] \in \frac{\Omega^{cl}_{*+1}(X,\mathbb{Q})}{img(\partial)}$, and so vanishes when mapped into $\frac{\Omega^{cl}_{*+1}(X,\mathbb{R}/\mathbb{Q})}{img(\partial)}$. Thus $Ch^{\mathbb{R}/\mathbb{Q}}_*(M,E^{\nabla^E},f,\phi)$ does not depend on choices of k and $(W,\varepsilon^{\nabla^{\varepsilon}},g)$. It is obvious that $Ch^{\mathbb{R}/\mathbb{Q}}_*$ extends to a linear map from $\check{K}^f_*(X)$ to $\frac{\Omega^{cl}_{*+1}(X,\mathbb{R}/\mathbb{Q})}{img(\partial)}$.

3 $\hat{K}_{FL}$-Module Structure

The purpose of this section is to construct an explicit pairing between the differential K-homology and the Freed-Lott differential K-theory $\hat{K}_{FL}$.
We first recall briefly the definition of $\hat{K}_{FL}$. For more details, see Freed-Lott [12].

Let X be a smooth compact manifold. Let

$$0 \to F_1 \overset{i}{\to} F_2 \to F_3 \to 0$$

be a short exact sequence of Hermitian vector bundles over X, and let $s: F_3 \to F_2$ be a splitting map. Then $i \oplus s: F_1 \oplus F_3 \to F_2$ is an isomorphism. For all Hermitian connections ∇^{F_1},∇^{F_2},∇^{F_3} on F_1,F_2,F_3, respectively, we set

$$CS(\nabla^{F_1},\nabla^{F_2},\nabla^{F_3}) := CS((i\oplus s)^*\nabla^{F_2},\nabla^{F_1}\oplus\nabla^{F_3}).$$

The class $CS(\nabla^{F_1}, \nabla^{F_2}, \nabla^{F_3})$ does not depend on the choice of s, and

$$dCS(\nabla^{F_1}, \nabla^{F_2}, \nabla^{F_3}) = ch(\nabla^{F_2}) - ch(\nabla^{F_1}) - ch(\nabla^{F_3}).$$

A K-cocycle of Freed and Lott over X is a triple, (F, ∇^F, w), where F is a Hermitian vector bundle over X, ∇^F is a Hermitian connection on F, and $w \in \frac{\Omega^{odd}(X)}{img(d)}$ is a class of differential forms. The Freed-Lott differential K-theory group of X, $\hat{K}_{FL}(X)$, is the abelian group coming from the following generators and relations. The generators are K-cocycles of Freed-Lott over X, and the relations are $(F_2, \nabla^{F_2}, w_2) = (F_1 \oplus F_3, \nabla^{F_1} \oplus \nabla^{F_3}, w_1 + w_3)$ whenever there is a short exact sequence of Hermitian vector bundles over X,

$$0 \to F_1 \to F_2 \to F_3 \to 0,$$

and $w_2 = w_1 + w_3 - CS(\nabla^{F_1}, \nabla^{F_2}, \nabla^{F_3})$.

The group $\hat{K}_{FL}(X)$ carries a ring structure given by

$$m([F_1, \nabla^{F_1}, w_1], [F_2, \nabla^{F_2}, w_2]) := [F_1 \otimes F_2, \nabla^{F_1} \otimes \nabla^{F_2}, \\ ch(\nabla^{F_1}) \wedge w_2 + ch(\nabla^{F_2}) \wedge w_1 - w_1 \wedge dw_2].$$

We have a well-defined group homomorphism

$$R : \hat{K}_{FL}(X) \to \Omega^{even}(X)$$

with $R[F, \nabla^F, w] = ch(\nabla^F) - dw$. Let $\hat{K}^f_{FL}(X)$ denote the kernel of R. Note that we have a short exact sequence

$$0 \to \hat{K}^f_{FL}(X) \hookrightarrow \hat{K}_{FL}(X) \xrightarrow{R} \Omega^{even}_K(X) \to 0,$$

where $\Omega^*_K(X)$ denotes the group of closed differential forms whose de Rham cohomology class lies in the image of the Chern character.

Proposition 3.1. *There is a natural pairing*

$$\mu : \hat{K}_{FL}(X) \otimes \check{K}_*(X) \to \check{K}_*(X).$$

Proof. Let (F, ∇^F, w) be a K-cocycle of Freed-Lott over X, and let $(M, E^{\nabla^E}, f, \phi)$ be a differential K-cycle over X. Set

$$\mu((F, \nabla^F, w), (M, E^{\nabla^E}, f, \phi)) := [M, E \otimes f^*F^{\nabla^E \otimes f^*\nabla^F}, f, [\int_M Td(\nabla^M) ch(\nabla^E) \\ \wedge f^*(w \wedge \cdot)] + \phi(R[F, \nabla^F, w] \wedge \cdot)].$$

It is apparent that the map μ is biadditive. We show that μ is compatible with the equivalence relation $\sim$ from Definition 2.6 and the equivalence relation used to define the Freed-Lott differential K-theory. We check that μ is compatible with $\sim$. Compatibility with relations (i) and (iii) from Definition 2.6 is straightforward. Let (F, ∇^F, w) be a differential K-cocycle over X, and let $(W, \varepsilon^{\nabla^\varepsilon}, g)$ be a K-chain over X. We have

$$\mu((F,\nabla^F,w),(\partial W,\varepsilon|_{\partial W}^{\nabla^\varepsilon|_{\partial W}},g|_{\partial W},[\int_W Td(\nabla^W)ch(\nabla^\varepsilon)g^*])) = [\partial W, \varepsilon|_{\partial W}\otimes g|_{\partial W}^* F^{\nabla^\varepsilon|_{\partial W}\otimes g|_{\partial W}^*\nabla^F}, g|_{\partial W}, \phi],$$

where

$$\phi = [\int_{\partial W} Td(\nabla^{\partial W})ch(\nabla^{\varepsilon|_{\partial W}})g|_{\partial W}^*(w\wedge\cdot)]+[\int_W Td(\nabla^W)ch(\nabla^\varepsilon)g^*(R[F,\nabla^F,w]\wedge\cdot)].$$

It follows that

$$\begin{aligned}\phi &= [\int_W (Td(\nabla^W)ch(\nabla^\varepsilon)g^*(dw\wedge\cdot))] + [\int_W Td(\nabla^W)ch(\nabla^\varepsilon)g^*(R[F,\nabla^F,w]\wedge\cdot)]\\ &= [\int_W Td(\nabla^W)ch(\nabla^\varepsilon\otimes g^*\nabla^F)g^*(\cdot)].\end{aligned}$$

Hence, μ is compatible with the relation (ii) of bordism. So the proof reduces to showing that μ is compatible with the equivalence relation on K-cocycles of Freed-Lott. Let $(M, E^{\nabla^E}, f, \phi)$ be a differential K-cycle over X, and let (F, ∇^F, w) and $(F', \nabla^{F'}, w')$ be two K-cocycles of Freed-Lott over X, which define the same class in $\hat{K}_{FL}(X)$. Since the map $\mu(\cdot)(M, E^{\nabla^E}, f, \phi)$ is additive, we can assume that there exists an isomorphism of Hermitian vector bundles $h : F \to F'$ such that $CS(\nabla^F, h^*\nabla^{F'}) = w - w'$. We set

$$\Phi = [\int_M Td(\nabla^M)ch(\nabla^E)f^*(w\wedge\cdot)] + \phi(R[F,\nabla^F,w]\wedge\cdot),$$

and

$$\Psi = [\int_M Td(\nabla^M)ch(\nabla^E)f^*(w'\wedge\cdot)] + \phi(R[F',\nabla^{F'},w']\wedge\cdot).$$

The two K-cycles $(M, E\otimes f^*F^{\nabla^E\otimes f^*\nabla^F}, f)$ and $(M, E\otimes f^*F'^{\nabla^E\otimes f^*\nabla^{F'}}, f)$ are isomorphic and

$$\Phi - \Psi = [\int_M Td(\nabla^M)ch(\nabla^E)f^*((w-w')\wedge\cdot)].$$

Since $CS(\nabla^E \otimes f^*\nabla^F, \nabla^E \otimes f^*(h^*\nabla^{F'})) = ch(\nabla^E) \wedge f^*CS(\nabla^F, h^*\nabla^{F'})$ (see [16]), we have

$$\Phi - \Psi = [\int_M Td(\nabla^M)\left(\int_{[0,1]\times M/M} ch(B)\right) f^*],$$

where B is the connection as in Definition 2.3. It follows that

$$\begin{aligned}\Phi - \Psi &= [\int_M \int_{[0,1]\times M/M} Td(\nabla^{[0,1]\times M})ch(B)(f\circ p)^*]\\ &= [\int_{[0,1]\times M} Td(\nabla^{[0,1]\times M})ch(B)(f\circ p)^*].\end{aligned}$$

Then $\mu((F,\nabla^F,w),(M,E^{\nabla^E},f,\phi)) = \mu((F',\nabla^{F'},w'),(M,E^{\nabla^E},f,\phi))$. □

Let us consider the collapse map $\epsilon : X \to pt$. Note that we can define an index pairing

$$\begin{cases} \hat{K}_{FL}(X)\otimes \check{K}_{even}(X) \to \mathbb{Z} \\ \hat{K}_{FL}(X)\otimes \check{K}_{odd}(X) \to \mathbb{R}/\mathbb{Z}\end{cases}$$

as $\epsilon_* : \check{K}_*(X) \to \check{K}_*(pt)$ composed with $\mu : \hat{K}_{FL}(X)\otimes \check{K}_*(X) \to \check{K}_*(X)$.

If X is a smooth closed $Spin^c$-manifold, then we can define a homomorphism $\jmath : \hat{K}_{FL}(X) \to \check{K}(X)$ by setting

$$\jmath([F,\nabla^F,w]) := [X, F^{\nabla^F}, id_X, [\int_X Td(\nabla^X) w \wedge \cdot]].$$

Remark 3.2. *Let $(F,\nabla^F,0)$ be a K-cocycle of Freed-Lott over S^1. Since $\partial D^2 = S^1$, the underlying $Spin^c$-structure of S^1 is given by the boundary $Spin^c$-structure and the vector bundle F is topologically trivial. Therefore we can find a Hermitian vector bundle on D^2 carrying with a Hermitian connection $(F',\nabla^{F'})$ which restricts to (F,∇^F) on the boundary. Since S^1 has the bounding $Spin^c$-structure, the Dirac operator is invertible and has a symmetric spectrum. Then $\bar{\eta}_F = 0$, and we get*

$$\begin{aligned}\epsilon_*\circ\jmath([F,\nabla^F,0]) &= [\partial D^2, F'|_{\partial D^2}^{\nabla^{F'}|_{\partial D^2}}, \partial D_2 \to pt, 0]\\ &= [\emptyset,\emptyset,\emptyset, -\int_{D^2} ch(\nabla^{F'})] = a(\bar{\eta}_F) = 0.\end{aligned}$$

The triviality of $[S^1, F^{\nabla^F}, S^1 \to pt, 0]$ is analog to the relation in [10, Corollary 4.6, p. 51] involving the suspension functor.

The pairing μ and the homomorphism $\jmath$ are related by the following commutative diagram

$$\begin{array}{ccc} \hat{K}_{FL}(X)\otimes\hat{K}_{FL}(X) & \xrightarrow{m} & \hat{K}_{FL}(X) \\ \downarrow{\scriptstyle id\otimes\jmath} & \circlearrowleft & \downarrow{\scriptstyle\jmath} \\ \hat{K}_{FL}(X)\otimes\check{K}(X) & \xrightarrow{\mu} & \check{K}(X). \end{array}$$

A relation between the K-theoretical curvature R and the K-homological curvature $\mathcal{R}$ is illustrated by the following commutative square

$$\begin{array}{ccc} \hat{K}_{FL}(X) & \xrightarrow{\jmath} & \check{K}(X) \\ \uparrow & \circlearrowleft & \uparrow \\ \hat{K}^f_{FL}(X) & \xrightarrow{\jmath} & \check{K}^f(X) \end{array}$$

Let us now define a relation between $\mu:\hat{K}_{FL}(X)\otimes\check{K}_*(X)\to\check{K}_*(X)$ and the cap product in de Rham (co)homology, in commutative diagram terms. If $\Omega^p(X)\otimes\Omega_q(X)\to\Omega_{q-p}(X)$ denotes the pairing $(w,\phi)\mapsto\phi(w\wedge\cdot)$, then the following diagram

$$\begin{array}{ccc} \hat{K}_{FL}(X)\otimes\check{K}_*(X) & \xrightarrow{\mu} & \check{K}_*(X) \\ \downarrow{\scriptstyle R\otimes\mathcal{R}} & & \downarrow{\scriptstyle\mathcal{R}} \\ \Omega^{even}(X)\otimes\Omega_*(X) & \longrightarrow & \Omega_*(X) \end{array}$$

commutes. To see this, let $x:=[F,\nabla^F,w]\in\hat{K}_{FL}(X)$ and $\xi:=[M,E^{\nabla^E},f,\phi]\in\check{K}_*(X)$. For every $v\in\Omega^*(X)$,

$$\begin{aligned} \mathcal{R}(\mu(x,\xi))(v) &= \int_M Td(\nabla^M)ch(\nabla^E)ch(f^*\nabla^F)\wedge f^*(v) \\ &\quad-\int_M Td(\nabla^M)ch(\nabla^E)f^*(w\wedge dv)-\phi(R(x)\wedge dv) \\ &= \int_M Td(\nabla^M)ch(\nabla^E)(f^*(ch(\nabla^F))-f^*(dw))\wedge f^*(v) \\ &\quad-\phi(R(x)\wedge dv) \\ &= \mathcal{R}(\xi)(R(x)\wedge v). \end{aligned}$$

We can define an index pairing $\widetilde{\alpha}:\hat{K}_{FL}(X)\to Hom(\check{K}_{odd}(X),\mathbb{R}/\mathbb{Z})$ as $\widetilde{\eta}$ composed with $\mu:\hat{K}_{FL}(X)\otimes\check{K}_{odd}(X)\to\check{K}_{odd}(X)$: for every $[F,\nabla^F,w]\in\hat{K}_{FL}(X)$ and $[M,E^{\nabla^E},f,\phi]\in\check{K}_{odd}(X)$,

$$\begin{aligned} \widetilde{\alpha}([F,\nabla^F,w])([M,E^{\nabla^E},f,\phi]) &= \bar{\eta}_{E\otimes f^*F}-\int_M Td(\nabla^M)ch(\nabla^E)f^*(w) \\ &\quad-\phi(R[F,\nabla^F,w]) \mod \mathbb{Z}. \end{aligned}$$

4 The Isomorphism $\hat{K}_{FL}(X) \cong \check{K}^0(X)$

This section is concerned with the construction of a model of differential K-theory through differential K-homology.

4.1 The Index Pairing $\beta : \widetilde{K}^*(X) \to Hom(\check{K}_*(X), \mathbb{R}/\mathbb{Z})$

In this subsection, we construct an index pairing between the differential K-homology $\check{K}$ and the differential K-characters $\widetilde{K}$.
We first recall the definition of $\widetilde{K}$ (see [11]).

Let X be a smooth compact manifold. Denote by $C_*(X)$ the set of equivalence classes of K-cycles over X, for the equivalence relation generated by direct sum and vector bundle modification. It is obvious that $C_*(X)$ is a semigroup under the addition operation given by disjoint union,

$$(M, E^{\nabla^E}, f) + (M', E'^{\nabla^{E'}}, f') := (M \sqcup M', E \sqcup E'^{\nabla^E \sqcup \nabla^{E'}}, f \sqcup f').$$

We define a homomorphism $j : \Omega^*(X) \to Hom(C_*(X), \mathbb{R})$ by setting

$$j(w)(M, E^{\nabla^E}, f) := \int_M Td(\nabla^M) ch(\nabla^E) f^*(w).$$

Definition 4.1. *Let $w \in \Omega^*(X)$ be a real differential form.*

- *The set of K-periods of w is the subset $j(w)(C_*(X))$ of $\mathbb{R}$.*
- *The set of closed differential forms on X with integer K-periods will be denoted by $\Omega_0^*(X)$.*

The set $\Omega_0^*(X)$ is an abelian group for the sum of differential forms. Stokes' theorem assures that $img[d : \Omega^{*-1}(X) \to \Omega^*(X)] \subset \Omega_0^*(X)$. Moreover, the Atiyah-Singer index theorem and the surjectivity of the usual Atiyah-Singer homomorphism $K(X) \to Hom(K_{even}^{geo}(X), \mathbb{Z})$ implie that $\Omega_0^{even}(X) = \Omega_K^{even}(X)$.

Definition 4.2. *A differential K-character on X is a semigroup homomorphism $h : C_*(X) \to \mathbb{R}/\mathbb{Z}$ such that*

$$h(\partial(W, \varepsilon^{\nabla^\varepsilon}, g)) = \overline{\int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*(\delta(h))} \in \mathbb{R}/\mathbb{Z}$$

for some $\delta(h) \in \Omega_0^*(X)$ and for all K-chain $(W, \varepsilon^{\nabla^\varepsilon}, g)$ over X, where $\overline{\lambda} := \lambda$ mod $\mathbb{Z}$.

The set of differential K-characters on X is denoted by $\widetilde{K}(X)$. It is an abelian group, which has a natural $\mathbb{Z}_2$-grading:

$$\widetilde{K}(X) = \widetilde{K}^{even}(X) \oplus \widetilde{K}^{odd}(X).$$

Let h be a differential K-character on X. The differential form $\delta(h)$ in the above definition does only depend on h. Thus we have a group homomorphism of degree 1

$$\delta : \widetilde{K}(X) \to \Omega_0^*(X).$$

Note that a differential form $w \in \Omega^*(X)$ determines a differential K-character on X by setting

$$h_w(M, E^{\nabla^E}, f) := \overline{j(w)(M, E^{\nabla^E}, f)}.$$

It is easy to check that $\delta(h_w) = dw$.

Now we construct an index pairing

$$\widetilde{K}^*(X) \to Hom(\check{K}_*(X), \mathbb{R}/\mathbb{Z}).$$

Proposition 4.3. *There is an index pairing*

$$\beta : \widetilde{K}^*(X) \to Hom(\check{K}_*(X), \mathbb{R}/\mathbb{Z}).$$

Proof. Let β be the map that associates with each differential K-character h on X and differential K-cycle $(M, E^{\nabla^E}, f, \phi)$ over X of the same parity the class $h(M, E^{\nabla^E}, f) - \overline{\phi(\delta(h))} \in \mathbb{R}/\mathbb{Z}$.

The map β is obviously biadditive. The only thing to check is that β is compatible with the relations from Definition 2.6. Compatibility with relations (i) and (iii) from Definition 2.6 is straightforward. So the proof reduces to showing that β is compatible with the relation (ii) of bordism. Let h be a differential K-character on X and $(W, \varepsilon^{\nabla^\varepsilon}, g)$ a K-chain over X of opposite parity. Then

$$\begin{aligned}
\beta(h)(\partial W, \varepsilon|_{\partial W}{}^{\nabla^\varepsilon|_{\partial W}}, g|_{\partial W}, [\int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*]) &= h(\partial W, \varepsilon|_{\partial W}{}^{\nabla^\varepsilon|_{\partial W}}, g|_{\partial W}) \\
&\quad - \overline{\int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*(\delta(h))} \\
&= \overline{\int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*(\delta(h))} \\
&\quad - \overline{\int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*(\delta(h))} \\
&= 0.
\end{aligned}$$

□

4.2 The Main Result

Let X be a smooth compact manifold. We construct a subgroup of $Hom(\check{K}_{odd}(X), \mathbb{R}/\mathbb{Z})$ and an isomorphism between it and $\hat{K}_{FL}(X)$.

Let

$$0 \to \frac{\Omega_{*-1}(X)}{\Omega^0_{*-1}(X)} \xrightarrow{a} \check{K}_*(X) \xrightarrow{i} K^{geo}_*(X) \to 0$$

be the short exact sequence relating differential K-homology and geometric K-homology. Since the contravariant functor $Hom_{\mathbb{Z}}(\cdot, \mathbb{R}/\mathbb{Z})$ is left-exact, we obtain an exact sequence

$$0 \to Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z}) \xrightarrow{i^*} Hom(\check{K}_{odd}(X), \mathbb{R}/\mathbb{Z}) \xrightarrow{a^*} Hom(\frac{\Omega_{even}(X)}{\Omega^0_{even}(X)}, \mathbb{R}/\mathbb{Z}).$$

We may regard $\Omega^{even}_0(X) \subseteq Hom(\frac{\Omega_{even}(X)}{\Omega^0_{even}(X)}, \mathbb{R}/\mathbb{Z})$ by evaluation homomorphism. Set

$$\check{K}^0(X) := \{\chi \in Hom(\check{K}_{odd}(X), \mathbb{R}/\mathbb{Z}) \mid a^*(\chi) \in \Omega^{even}_0(X)\}.$$

Theorem. *The group $\check{K}^0(X)$ is isomorphic to $\hat{K}_{FL}(X)$.*

Proof. Note that the group $\check{K}^0(X)$ fits into an exact sequence

$$0 \to Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z}) \xrightarrow{i^*} \check{K}^0(X) \xrightarrow{a^*} \Omega^{even}_0(X)$$

and the index pairings $\widetilde{\alpha} : \hat{K}_{FL}(X) \to Hom(\check{K}_{odd}(X), \mathbb{R}/\mathbb{Z})$ and $\beta : \widetilde{K}^{odd}(X) \to Hom(\check{K}_{odd}(X), \mathbb{R}/\mathbb{Z})$ from section 3 and subsection 4.1 take values in $\check{K}^0(X)$. Let us check that a^* is surjective. Let $v \in \Omega^{even}_0(X)$. We define a homomorphism from the semigroup of the K-boundaries over X to $\mathbb{R}/\mathbb{Z}$ by setting

$$h(\partial(W, \varepsilon^{\nabla^\varepsilon}, g)) := \int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^*(v) \mod \mathbb{Z}.$$

Since $\mathbb{R}/\mathbb{Z}$ is divisible, h can be extended to a differential K-character $\tilde{h} \in \widetilde{K}^{odd}(X)$ with $\delta(\tilde{h}) = v$. Then $\beta(\tilde{h}) \in \check{K}^0(X)$ with $a^*(\beta(\tilde{h})) = v$.
It follows that we have a short exact sequence

$$0 \to Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z}) \xrightarrow{i^*} \check{K}^0(X) \xrightarrow{a^*} \Omega^{even}_0(X) \to 0.$$

Following the universal coefficient theorem [17]

$$0 \to Ext(K^{geo}_{even}(X), \mathbb{R}/\mathbb{Z}) \to \hat{K}^f_{FL}(X) \xrightarrow{\alpha} Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z}) \to 0,$$

together with the fact that $\mathbb{R}/\mathbb{Z}$ is divisible, the groups $\hat{K}^f_{FL}(X)$ and $Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z})$ are isomorphic via α. The isomorphism α is given by

$$\alpha([F, \nabla^F, w])([M, E^{\nabla^E}, f]) = \bar{\eta}_{E \otimes f^*F} - \int_M Td(\nabla^M) ch(\nabla^E) f^*(w) \mod \mathbb{Z}$$

(see [15]). Thus, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{K}^f_{FL}(X) & \hookrightarrow & \hat{K}_{FL}(X) & \xrightarrow{R} & \Omega^{even}_K(X) & \longrightarrow & 0 \\
 & & \alpha \downarrow \cong & \circlearrowleft & \widetilde{\alpha} \downarrow & \circlearrowleft & \| & & \\
0 & \longrightarrow & Hom(K^{geo}_{odd}(X), \mathbb{R}/\mathbb{Z}) & \xrightarrow{i^*} & \check{K}^0(X) & \xrightarrow{a^*} & \Omega^{even}_0(X) & \longrightarrow & 0
\end{array}
$$

in which the rows are exact sequences. The five-lemma argument shows that $\widetilde{\alpha}$ is an isomorphism. □

The following example illustrates how differential K-theory classes arise in geometry.

Example. *Let $SO(2) \to E \to M$ be a circle bundle over M with connection ∇. Let $\omega \in \Omega^2(M)$ denote its curvature form. Since $\frac{1}{2\pi}\omega$ represents the real Euler class, $\frac{1}{2\pi}\omega \in \Omega^2_0(M)$.*
Now we define an $\mathbb{R}/\mathbb{Z}$-valued homomorphism $\widetilde{H}$ on the semigroup of 1-differential K-cycles over a smooth compact manifold X. Let x be a 1-differential K-cycle over X and choose a closed curve γ, a 2-K-chain ζ over X and $\phi \in \frac{\Omega_{even}(X)}{img(\partial)}$ so that $x = (\gamma + \partial\zeta, \phi)$. We set

$$
\widetilde{H}(x) := e^{-2\pi i} H(\gamma) + \overline{\frac{1}{2\pi}\int_\zeta \omega(\zeta) - \phi(1)},
$$

where $H(\gamma)$ is the holonomy around γ and $\overline{\lambda} := \lambda \mod \mathbb{Z}$. It is clear that $\widetilde{H}$ is a linear map which is trivial on the boundaries $(\partial\zeta, \frac{1}{2\pi}\int_\zeta \omega(\zeta))$ with $a^*(\widetilde{H}) = 1$. So $\widetilde{H}$ determines an element in $\check{K}^0(X)$, and then gives rise to a class in the Freed-Lott differential K-group $\hat{K}_{FL}(X)$.

References

[1] M.F. Atiyah, Bott periodicity and the index of elliptic operators, *Quart. J. Math. Oxford*, 19(1968), 113-140.

[2] M.F. Atiyah, Global theory of elliptic operators, *In Proc. of the International Symposium on Functional Analysis*, Tokyo, University of Tokyo Press, (1970).

[3] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry-I, *Math. Proc. Cambridge Philos. Soc.*, 77(1975), 43-69.

[4] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry-II, *Math. Proc. Cambridge Philos. Soc.*, 78(1975), 405-432.

[5] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry-III, *Math. Proc. Cambridge Philos. Soc.*, 79(1976), 71-99.

[6] M.F. Atiyah and I.M. Singer, The index of elliptic operators-I, *Ann. of Math.*, 87(1968), 484-530.

[7] P. Baum and R. Douglas, K-homology and index theory, In R. Kadison (editor), Operator algebras and applications, *Proceedings of Symposia in Pure Mathematics*, Providence, RI, *American Mathematical Society*, 38(1982), 117-173.

[8] P. Baum, N. Higson and T. Schick, On the equivalence of geometric and analytic K-homology, *Pure Appl. Math. Q.*, 3(1) (2007), 1-24.

[9] U. Bunke and T. Schick, *Differential K-Theory: A Survey, Global Differential Geometry*, Springer, Heidelberg, (2012), 303-358.

[10] U. Bunke and T. Schick, Smooth K-theory, *Astrisque*, 328(2009), 45-135.

[11] J. Cheeger and J. Simons, Differential characters and geometric invariants, in Geometry and Topology (1983/84, College Park, Md.), *Lecture Notes in Math.*, Springer, Berlin, 1167(1985), 50-80.

[12] D.S. Freed and J. Lott, An index theorem in differential K-theory, *Geometry and Topology*, 14(2010), 903-966.

[13] F. Hirzebruch, Topological methods in algebraic geometry (new ed.), *Classics in Mathematics*, Springer-Verlag, Berlin, (1995).

[14] G.G. Kasparov, Topological invariants of elliptic operators I: K-homology, *Math. USSR Izvestija*, 9(1975), 751-792.

[15] J. Lott, R/Z index theory, *Comm. Anal. Geom.*, 2(2) (1994), 279-311.

[16] J. Simons and D. Sullivan, Structured vector bundles define differential K-theory, *Quanta of Maths*, *Clay Math. Proc.*, Providence, RI, *Amer. Math. Soc.*, 11(2010), 579-599.

[17] Z. Yosimura, Universal coefficient sequences for cohomology theories of CW-spectra-II, *Osaka J. Math.*, 16(1979), 201-217.