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Variational Discretization and Mixed Methods for Semilinear Parabolic Optimal Control Problem

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Abstract

In this paper we study the variational discretization and mixed finite element methods for optimal control problem governed by semilinear parabolic equations. The space discretization of the state variable is done using usual mixed finite elements. The state and the co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is not discreted. Then we derive a priori error estimates both for the coupled state and the control approximation.

Keywords: *A priori error estimates, semilinear parabolic optimal control problem, variational discretization, mixed finite element methods*

1 Introduction

Optimal control problems governed by semilinear parabolic equations is an important problem in engineering applications. The finite element method was undoubtedly the most widely used numerical method in computing optimal control problems. There have been extensive studies in convergence of the finite element approximation of optimal control problems. For optimal control problems governed by linear elliptic equations, a priori error estimates of the standard finite element discretization were established long ago, see, for example, Falk [10]. The authors presented error estimates of finite element approximations of state constrained convex parabolic boundary control problems in [1]. Then, Malanowski in [21] established a priori error estimates for the finite element approximations to convex constrained optimal control systems. In [2] the authors considered the finite element approximation of a distributed optimal control problem governed by a semilinear elliptic partial differential equation, where pointwise constraints on the control were given. Casas studied the numerical approximation of distributed semilinear optimal control problems and proved that the L^2-error estimates were of order $o(h)$, which was optimal according to the $C^{0,1}(\bar{\Omega})$-regularity of the optimal control in [3]. While the a priori error analysis for finite element discretization of optimal control problems governed by elliptic equations was discussed in many publications, see, e.g., [13, 26], there were only few published results on this topic for parabolic problems. Meidner and Vexler proposed a priori error estimates for space-time finite element discretization of parabolic optimal control problems without control constraints in [22]. The space discretization of the state variable was done using usual conforming finite elements, whereas the time discretization was based on discontinuous Galerkin methods. Some recent progress in a priori error estimates can be found in [15, 24], but there were only few published results on this topic for nonlinear optimal control problems.

In many control problems, the objective functional contains gradient of the state variables. Thus the accuracy of gradient is important in numerical approximation of the state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods, see, for example, [16]. However, there was only very limited research work on analyzing such elements for optimal control problems. Recently, we have derived a priori error estimates, a posteriori error estimates and superconvergence for quadratic optimal control problems using mixed finite element methods in [4, 5, 6, 7, 8, 18, 19, 20, 27].

In [14], the author first presents the variational discretization concept for optimal control problems with control constraints, with implicitly utilizes the first order optimality conditions and the discretization of the state and adjoint equations for the discretization of the control instead of discretizing the space of admissible controls.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\| v \|_{m,p}^p = \sum_{|\alpha|\leq m} \| D^\alpha v \|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $| v |_{m,p}^p = \sum_{|\alpha|=m} \| D^\alpha v \|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\| v \|_{L^s(J;W^{m,p}(\Omega))} = \left(\int_0^T ||v||_{W^{m,p}(\Omega)}^s dt\right)^{\frac{1}{s}}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. The details can be found in [17].

In this paper we study a priori error estimates of the variational discretization and mixed finite element methods for optimal control problem governed by semilinear parabolic equations. We focus our attention on the following semilinear parabolic optimal control problem:

$$\min_{u(t)\in K\subset U}\left\{\frac{1}{2}\int_0^T\left(\|\ \vec{p}-\vec{p}_d\ \|^2+\|\ y-y_d\ \|^2+\|\ u\ \|^2\right)dt\right\} \tag{1}$$

subject to the state equation

$$y_t(x,t)+\mathrm{div}\vec{p}(x,t)+\phi(y(x,t))=f(x,t)+Bu(x,t),\ x\in\Omega,\ t\in J, \tag{2}$$
$$\vec{p}(x,t)=-A(x)\nabla y(x,t),\ \ x\in\Omega, \tag{3}$$
$$y(x,t)=0,\ x\in\partial\Omega,\ t\in J,\ y(x,0)=y_0(x),\ x\in\Omega, \tag{4}$$

where the bounded open set $\Omega\subset\mathbf{R}^2$, is 2 regular convex polygon with boundary $\partial\Omega$, $J=(0,T]$, $f\in L^2(J;L^2(\Omega))$, and $U=L^2(J;L^2(\Omega))$. For any $R>0$ the function $\phi(\cdot)\in W^{2,\infty}(-R,R)$, $\phi'(y)\in L^2(\Omega)$ for any $y\in L^2(J;H^1(\Omega))$, and $\phi'(y)\geq 0$. Here, $A(x)\in H^1(\Omega)$ and K denotes the admissible set of the control variable, defined by

$$K=\left\{u(x,t)\in L^2(J;L^2(\Omega)):\ u(x,t)\geq 0\ a.e.\ x\in\Omega,\ t\in J\right\}. \tag{5}$$

The outline of this paper is as follows. In Section 2, we construct the variational discretization and mixed finite element discretization for the optimal control problem (1)-(4). In Section 3, we derive a priori error estimates for the variational discretization and fully discrete mixed finite element approximation of the semilinear parabolic optimal control problem. Finally, we analyze the conclusion in section 4.

2 Variational Discretization and Mixed Methods

First, we introduce the co-state parabolic equation

$$-z_t-\mathrm{div}(A(\nabla z+\vec{p}-\vec{p}_d))+\phi'(y)z=y-y_d,\quad x\in\Omega, \tag{6}$$

with the conditions

$$z(x,t)=0,\ x\in\partial\Omega,\ t\in J;\quad z(x,T)=0,\ x\in\Omega.$$

Next, we assume that the two given functions $\vec{p}_d$, y_d are continuously differentiable with respect to t, moreover, $y_d\in L^2(J;H^2(\Omega))$, $\vec{p}_d\in(L^2(J;H^2(\Omega)))^2$.

We now describe the variational discretization and mixed finite element approximation of semilinear parabolic optimal control problem (1)-(4). Let $\vec{V}=H(\mathrm{div})=\{\vec{v}\in(L^2(\Omega))^2,\mathrm{div}\vec{v}\in L^2(\Omega)\}$ endowed with the norm given by $\|\ \vec{v}\|_{H(\mathrm{div})}=(\|\vec{v}\|^2_{0,\Omega}+\|\mathrm{div}\vec{v}\|^2_{0,\Omega})^{1/2}$. We denote $W=L^2(\Omega)$.

We recast (1)-(4) as the following weak form: find $(\vec{p}, y, u) \in \vec{V} \times W \times K$ such that

$$\min_{u \in K \subset U} \left\{ \frac{1}{2} \int_0^T \left(\|\vec{p} - \vec{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right) dt \right\} \tag{7}$$

$$(A^{-1}\vec{p}, \vec{v}) - (y, \mathrm{div}\vec{v}) = 0, \ \forall \vec{v} \in \vec{V}, \tag{8}$$

$$(y_t, w) + (\mathrm{div}\vec{p}, w) + (\phi(y), w) = (f + Bu, w), \ \forall w \in W, \tag{9}$$

$$y(x, 0) = y_0(x), \ \forall x \in \Omega. \tag{10}$$

It is well known (see, e.g., [17]) that the optimal control problem (7)-(10) has a solution $(\vec{p}, y, u)$, and that a triplet $(\vec{p}, y, u)$ is the solution of (7)-(10) if and only if there is a co-state $(\vec{q}, z) \in \vec{V} \times W$ such that $(\vec{p}, y, \vec{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\vec{p}, \vec{v}) - (y, \mathrm{div}\vec{v}) = 0, \ \forall \vec{v} \in \vec{V}, \tag{11}$$

$$(y_t, w) + (\mathrm{div}\vec{p}, w) + (\phi(y), w) = (f + Bu, w), \ \forall w \in W, \tag{12}$$

$$y(x, 0) = y_0(x), \ \forall x \in \Omega, \tag{13}$$

$$(A^{-1}\vec{q}, \vec{v}) - (z, \mathrm{div}\vec{v}) = -(\vec{p} - \vec{p}_d, \vec{v}), \ \forall \vec{v} \in \vec{V}, \tag{14}$$

$$-(z_t, w) + (\mathrm{div}\vec{q}, w) + (\phi'(y)z, w) = (y - y_d, w), \ \forall w \in W, \tag{15}$$

$$z(x, T) = 0, \ \forall x \in \Omega, \tag{16}$$

$$\int_0^T (B^* z + u, \tilde{u} - u)_U dt \geq 0, \ \forall \tilde{u} \in K, \tag{17}$$

where B^* is the adjoint operator of B and $(\cdot, \cdot)_U$ is the inner product of U. In the rest of the paper, we shall simply write the product as $(\cdot, \cdot)$ whenever no confusion should be caused.

We also assume that both parabolic equations (2) and (6) have sufficiently regularity and $u \in L^2(J; W^{1,\infty}(\Omega))$, $y, \ z \in L^2(J; H^2(\Omega))$, $\vec{p}, \ \vec{q} \in (L^2(J; H^2(\Omega)))^2$.

Let T_h be regular triangulation of Ω. They are assumed to satisfy the angle condition which means that there is a positive constant C such that

$$C^{-1} h_\tau^2 \leq |\tau| \leq C h_\tau^2, \quad \forall \tau \in \mathrm{T}_h,$$

where $|\tau|$ is the area of τ and h_τ is the diameter of τ. Let $h = \max h_\tau$. In addition C or c denotes a general positive constant independent of h.

Let $\vec{V}_h \times W_h \subset \vec{V} \times W$ denote the Raviart-Thomas space [25] of the lowest order associated with the triangulation T_h of Ω, namely, $\vec{V}(\tau) = \{\vec{v} \in P_0^2(\tau) + x \cdot P_0(\tau)\}$, $W(\tau) = P_0(\tau), \forall \tau \in \mathrm{T}_h$, where P_k denotes the space of polynomials of total degree at most k, $x = (x_1, x_2)$ which treated as a vector, and

$$\vec{V}_h := \{\vec{v}_h \in \vec{V} : \ \forall \tau \in \mathrm{T}_h, \quad \vec{v}_h|_\tau \in \vec{V}(\tau)\},$$
$$W_h := \{w_h \in W : \ \forall \tau \in \mathrm{T}_h, \quad w_h|_\tau \in W(\tau)\}.$$

The mixed finite element discretization of (7)-(10) is as follows: compute $(\vec{p}_h, y_h, u_h) \in \vec{V}_h \times W_h \times K$ such that

$$\min_{u_h \in K} \left\{ \frac{1}{2} \int_0^T \left(\|\vec{p}_h - \vec{p}_d\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2 \right) dt \right\} \tag{18}$$

$$(A^{-1}\vec{p}_h, \vec{v}_h) - (y_h, \mathrm{div}\vec{v}_h) = 0, \ \forall \vec{v}_h \in \vec{V}_h, \tag{19}$$

$$(y_{ht}, w_h) + (\mathrm{div}\vec{p}_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \ \forall w_h \in W_h, \tag{20}$$

$$y_h(x, 0) = Y(x, 0), \ \forall x \in \Omega, \tag{21}$$

where $Y(x, 0)$ is the elliptic mixed methods projection into the finite dimensional space W_h of the initial data function $y_0(x)$.

The optimal control problem (18)-(21) again has a solution $(\vec{p}_h, y_h, u_h)$, and that a triplet $(\vec{p}_h, y_h, u_h)$ is the solution of (18)-(21) if and only if there is a co-state $(\vec{q}_h, z_h) \in \vec{V}_h \times W_h$ such that $(\vec{p}_h, y_h, \vec{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\vec{p}_h, \vec{v}_h) - (y_h, \mathrm{div}\vec{v}_h) = 0, \tag{22}$$

$$(y_{ht}, w_h) + (\mathrm{div}\vec{p}_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \tag{23}$$

$$y_h(x, 0) = Y(x, 0), \ \forall x \in \Omega, \tag{24}$$

$$(A^{-1}\vec{q}_h, \vec{v}_h) - (z_h, \mathrm{div}\vec{v}_h) = -(\vec{p}_h - \vec{p}_d, \vec{v}_h), \tag{25}$$

$$-(z_{ht}, w_h) + (\mathrm{div}\vec{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \tag{26}$$

$$z_h(x, T) = 0, \ \forall x \in \Omega, \tag{27}$$

$$(B^* z_h + u_h, \tilde{u}_h - u_h) \geq 0, \tag{28}$$

where $\vec{v} \in \vec{V}_h$, $w \in W_h$, $\tilde{u} \in K$.

We now consider the time discretization of the difference methods. Let $\triangle t > 0$, $N = T/\triangle t \in$ Z, and $t^n = n\triangle t$, $n \in$ Z. Also, let

$$\psi^n = \psi^n(x) = \psi(x, t^n), \qquad d_t\psi^n = \frac{\psi^n - \psi^{n-1}}{\triangle t}.$$

We define for $1 \leq p < \infty$ the discrete time dependent norms

$$|||\psi|||_{L^p(J;H^s(\Omega))} := \left(\sum_{n=1}^{N} \triangle t \|\psi^n\|_s^p \right)^{\frac{1}{p}},$$

and the standard modification for $p = \infty$.

Then we define the fully discrete finite element solution $(\vec{p}_h^n, y_h^n, \vec{q}_h^{n-1}, z_h^{n-1}, u_h^n)$ satisfies

$$(A^{-1}\vec{p}_h^n, \vec{v}) - (y_h^n, \mathrm{div}\vec{v}) = 0, \tag{29}$$

$$(d_t y_h^n, w) + (\mathrm{div}\vec{p}_h^n, w) + (\phi(y_h^n), w) = (f + Bu_h^n, w), \tag{30}$$

$$y_h^0(x) = Y(x, 0),\ \forall x \in \Omega, \tag{31}$$

$$(A^{-1}\vec{q}_h^{n-1}, \vec{v}) - (z_h^{n-1}, \mathrm{div}\vec{v}) = -(\vec{p}_h^n - \vec{p}_d, \vec{v}), \tag{32}$$

$$-(d_t z_h^n, w) + (\mathrm{div}\vec{q}_h^{n-1}, w) + (\phi'(y_h^n) z_h^{n-1}, w) = (y_h^n - y_d, w), \tag{33}$$

$$z_h^N(x) = 0,\ \forall x \in \Omega, \tag{34}$$

$$(B^* z_h^n + u_h^n, \tilde{u} - u_h^n) \geq 0, \tag{35}$$

where $\vec{v} \in \vec{V}_h$, $w \in W_h$, $\tilde{u} \in K$.
For $\varphi \in W_h$, we shall write

$$\phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2, \tag{36}$$

where

$$\tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi + s(\rho - \varphi))ds,$$

$$\tilde{\phi}''(\varphi) = \int_0^1 (1 - s)\phi''(\rho + s(\varphi - \rho))ds$$

are bounded functions in $\bar{\Omega}$ [23].

3 A Priori Error Estimates

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution $(\vec{p}(\tilde{u}), y(\tilde{u}), \vec{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\vec{p}(\tilde{u}), \vec{v}) - (y(\tilde{u}), \mathrm{div}\vec{v}) = 0,\ \forall \vec{v} \in \vec{V}, \tag{1}$$

$$(y_t(\tilde{u}), w) + (\mathrm{div}\vec{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + B\tilde{u}, w),\ \ \forall w \in W, \tag{2}$$

$$y(\tilde{u})(x, 0) = y_0(x),\ \forall x \in \Omega, \tag{3}$$

$$(A^{-1}\vec{q}(\tilde{u}), \vec{v}) - (z(\tilde{u}), \mathrm{div}\vec{v}) = -(\vec{p}(\tilde{u}) - \vec{p}_d, \vec{v}),\ \forall \vec{v} \in \vec{V}, \tag{4}$$

$$-(z_t(\tilde{u}), w) + (\mathrm{div}\vec{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w),\ \forall w \in W, \tag{5}$$

$$z(\tilde{u})(x, T) = 0,\ \forall x \in \Omega. \tag{6}$$

Then, we define the discrete time state solution $(\vec{p}^n(\tilde{u}), y^n(\tilde{u}), \vec{q}^{n-1}(\tilde{u}), z^{n-1}(\tilde{u}))$ of the system (1)-(6) associated with $\tilde{u} \in K$ that satisfies

$$(A^{-1}\vec{p}^n(\tilde{u}), \vec{v}) - (y^n(\tilde{u}), \text{div}\vec{v}) = 0, \ \forall \vec{v} \in \vec{V}, \tag{7}$$

$$(y_t^n(\tilde{u}), w) + (\text{div}\vec{p}^n(\tilde{u}), w) + (\phi(y^n(\tilde{u})), w) = (f + B\tilde{u}, w), \ \forall w \in W, \tag{8}$$

$$y^0(\tilde{u})(x) = y_0(x), \ \forall x \in \Omega, \tag{9}$$

$$(A^{-1}\vec{q}^{n-1}(\tilde{u}), \vec{v}) - (z^{n-1}(\tilde{u}), \text{div}\vec{v}) = -(\vec{p}^n(\tilde{u}) - \vec{p}_d, \vec{v}), \ \forall \vec{v} \in \vec{V}, \tag{10}$$

$$-(z_t^n(\tilde{u}), w) + (\text{div}\vec{q}^{n-1}(\tilde{u}), w) + (\phi'(y^n(\tilde{u}))z^{n-1}(\tilde{u}), w) \tag{11}$$

$$= (y^n(\tilde{u}) - y_d, w), \forall w \in W, \ z^N(\tilde{u})(x) = 0, \ \forall x \in \Omega. \tag{12}$$

According to the assumption on the domain Ω, we can easily observe that Ω is 2 regular. The domain Ω is said to be 2 regular if the Dirichlet problem

$$L_\lambda \varphi = -\text{div}(A(x)\nabla\varphi) + \lambda\varphi = F, \ x \in \Omega, \tag{13}$$

$$\varphi = 0, \ x \in \partial\Omega, \tag{14}$$

is uniquely solvable for $F \in L^2(\Omega)$ and if $\|\varphi\|_2 \leq \|F\|_0$ for all $F \in L^2(\Omega)$.

For any $\tilde{u} \in K$, we define an elliptic projection $(\vec{P}^n(\tilde{u}), Y^n(\tilde{u}), \vec{Q}^n(\tilde{u}), Z^n(\tilde{u}))$ of the solution of the differential problem into the finite dimensional space $\vec{V}_h \times W_h$ to be the map $(\vec{P}(\tilde{u}), Y(\tilde{u}), \vec{Q}(\tilde{u}), Z(\tilde{u})) : \{0, t^1, t^2, ..., t^n = T\} \to \vec{V}_h \times W_h$ given by

$$(A^{-1}(\vec{p}^n(\tilde{u}) - \vec{P}^n(\tilde{u})), \vec{v}) - (y^n(\tilde{u}) - Y^n(\tilde{u}), \text{div}\vec{v}) = 0, \ \forall \vec{v} \in \vec{V}_h, \tag{15}$$

$$(\text{div}(\vec{p}^n(\tilde{u}) - \vec{P}^n(\tilde{u})), w) + \lambda(y^n(\tilde{u}) - Y^n(\tilde{u}), w) = 0, \ \forall w \in W_h, \tag{16}$$

$$(A^{-1}(\vec{q}^n(\tilde{u}) - \vec{Q}^n(\tilde{u})), \vec{v}) - (z^n(\tilde{u}) - Z^n(\tilde{u}), \text{div}\vec{v}) = 0, \ \forall \vec{v} \in \vec{V}_h, \tag{17}$$

$$(\text{div}(\vec{q}^n(\tilde{u}) - \vec{Q}^n(\tilde{u})), w) + \lambda(z^n(\tilde{u}) - Z^n(\tilde{u}), w) = 0, \ \forall w \in W_h. \tag{18}$$

Let $\lambda > 0$, such that λ is sufficiently large so that the bilinear form associated with $L_\lambda(\cdot)$ is coercive over $H_0^1(\Omega)$. In fact, let λ be chosen so that [9]:

$$(A^{-1}\xi, \xi) + \lambda(\eta, \eta) \geq C\big(\|\xi\|_0^2 + \|\eta\|_0^2\big), \quad \forall \xi \in \vec{V}, \ \forall \eta \in W. \tag{19}$$

The projection (15)-(18) is associated with the operator L_λ. Let

$$\tau_1^n = y^n(u_h) - Y^n(u_h), \qquad \sigma_1^n = \vec{p}^n(u_h) - \vec{P}^n(u_h), \tag{20}$$

$$\tau_2^n = z^n(u_h) - Z^n(u_h), \qquad \sigma_2^n = \vec{q}^n(u_h) - \vec{Q}^n(u_h). \tag{21}$$

Estimates for τ_1^n, τ_2^n, σ_1^n, and σ_2^n are given in [11]. We state them here without a proof.

Lemma 3.1 *For $t \in J$ and for h sufficiently small, there is a positive constant C independent of h such that*

$$\|\sigma_1^n\|_0 + \|\tau_1^n\|_0 + \|\tau_1^n\|_{0,\infty} \leq Ch, \tag{22}$$

$$\|\sigma_2^n\|_0 + \|\tau_2^n\|_0 + \|\tau_2^n\|_{0,\infty} \leq Ch, \tag{23}$$

$$\|\text{div}\sigma_1^n\|_0 + \|\text{div}\sigma_2^n\|_0 \leq Ch. \tag{24}$$

Estimates for τ_{1t}^n, τ_{2t}^n, σ_{1t}^n, and σ_{2t}^n are given in [12]. We state them here without a proof.

Lemma 3.2 *For $t \in J$ and for h sufficiently small, there is a positive constant C independent of h such that*

$$\|\sigma_{1t}^n\|_0 + \|\tau_{1t}^n\|_0 + \|\tau_{1t}^n\|_{0,\infty} \leq Ch, \tag{25}$$
$$\|\sigma_{2t}^n\|_0 + \|\tau_{2t}^n\|_0 + \|\tau_{2t}^n\|_{0,\infty} \leq Ch, \tag{26}$$
$$\|\mathrm{div}\sigma_{1t}^n\|_0 + \|\mathrm{div}\sigma_{2t}^n\|_0 \leq Ch. \tag{27}$$

With the aid of Lemmas 3.1-3.2, we can also derive the following error estimates:

Theorem 3.3 *There is a positive constant $C > 0$, independent of h, such that*

$$|||\vec{p}(u_h) - \vec{p}_h|||_{L^\infty(J;H(\mathrm{div}))} + |||y(u_h) - y_h|||_{L^\infty(J;L^2(\Omega))} \leq C(\triangle t + h), \tag{28}$$
$$|||\vec{q}(u_h) - \vec{q}_h|||_{L^\infty(J;H(\mathrm{div}))} + |||z(u_h) - z_h|||_{L^\infty(J;L^2(\Omega))} \leq C(\triangle t + h). \tag{29}$$

Set some intermediate errors:

$$e_1^n = \vec{p}^n - \vec{p}^n(u_h), \qquad r_1^n = y^n - y^n(u_h), \tag{30}$$
$$e_2^n = \vec{q}^n - \vec{q}^n(u_h), \qquad r_2^n = z^n - z^n(u_h). \tag{31}$$

From (11)-(16) and (7)-(12), we derive the following error equations:

$$(A^{-1}e_1^n, \vec{v}) - (r_1^n, \mathrm{div}\vec{v}) = 0, \quad \forall \vec{v} \in \vec{V}_h, \tag{32}$$
$$\begin{aligned}(y_t^n - d_t y^n(u_h), w) + (\mathrm{div}e_1^n, w) + (\tilde{\phi}'(y^n) r_1^n, w)\\ = (B(u^n - u_h^n), w),\ \forall w \in W_h,\end{aligned} \tag{33}$$
$$(A^{-1}e_2^{n-1}, \vec{v}) - (r_2^{n-1}, \mathrm{div}\vec{v}) = -(e_1^n, \vec{v}), \quad \forall \vec{v} \in \vec{V}_h, \tag{34}$$
$$\begin{aligned}-(z_t^n - d_t z^n(u_h), w) + (\mathrm{div}e_2^{n-1}, w) + (\phi'(y^n) r_2^{n-1}, w)\\ +(\tilde{\phi}''(y^n) r_1^n z^{n-1}(u_h), w) = (r_1^n, w),\ \forall w \in W_h.\end{aligned} \tag{35}$$

Theorem 3.4 *There is a constant $C > 0$, independent of h and $\triangle t$, such that*

$$\begin{aligned}|||\vec{p} - \vec{p}(u_h)|||_{L^\infty(J;H(\mathrm{div}))} + |||y - y(u_h)|||_{L^\infty(J;L^2(\Omega))}\\ \leq C(\triangle t + h + |||u - u_h|||_{L^2(J;L^2(\Omega))}),\end{aligned} \tag{36}$$
$$\begin{aligned}|||\vec{q} - \vec{q}(u_h)|||_{L^\infty(J;H(\mathrm{div}))} + |||z - z(u_h)|||_{L^\infty(J;L^2(\Omega))}\\ \leq C(\triangle t + h + |||u - u_h|||_{L^2(J;L^2(\Omega))}).\end{aligned} \tag{37}$$

Proof. Part I. Choose $\vec{v} = e_1^n$ and $w = r_1^n$ as the test functions and add the two relations of (32)-(33), then we obtain that

$$(A^{-1}e_1^n, e_1^n) + (\tilde{\phi}'(y^n) r_1^n, r_1^n) = (B(u^n - u_h^n), r_1^n) - (y_t^n - d_t y^n(u_h), r_1^n).$$

By using δ-Cauchy inequality, we can find an estimate as follows

$$\|e_1^n\|_0^2 + \|r_1^n\|_0^2 \le C\left((\triangle t)^2 + h^2 + \|u^n - u_h^n\|_0^2\right) + \delta\|r_1^n\|_0^2, \tag{38}$$

for any small $\delta > 0$. This leads to

$$\|e_1^n\|_0 + \|r_1^n\|_0 \le C(\triangle t + h + \|u^n - u_h^n\|_0). \tag{39}$$

Now, take $w = \mathrm{div} e_1^n$ as a test function in (33), then we get

$$\begin{aligned} &\|\mathrm{div} e_1^n\|_0^2 = (B(u^n - u_h^n), \mathrm{div} e_1^n) \\ &-(y_t^n - d_t y^n(u_h), \mathrm{div} e_1^n) - (\tilde{\phi}'(y_h^n) r_1^n, \mathrm{div} e_1^n) \\ &\le C\|y_t^n - d_t y^n(u_h)\|_0^2 + C\|u^n - u_h^n\|_0^2 + C\|r_1^n\|_0^2 + \delta\|\mathrm{div} e_1^n\|_0^2, \end{aligned} \tag{40}$$

then, using the estimate (39), we have

$$\begin{aligned} \|\mathrm{div} e_1^n\|_0 &\le C\|y_t^n - d_t y^n(u_h)\|_0 + C\|u^n - u_h^n\|_0 + C\|r_1^n\|_0 \\ &\le C(\triangle t + h + \|u^n - u_h^n\|_0). \end{aligned} \tag{41}$$

Then (36) follows from (38) and (41).

Part II. Similarly, choose $\vec{v} = e_2^{n-1}$ and $w = r_2^{n-1}$ as the test functions and add the two relations of (34)-(35), then we obtain that

$$\begin{aligned} (A^{-1} e_2^{n-1}, e_2^{n-1}) + (\phi'(y^n) r_2^{n-1}, r_2^{n-1}) &= (r_1^n, r_2^{n-1}) + (z_t^n - d_t z^n(u_h), r_2^{n-1}) \\ &-(e_1^n, e_2^{n-1}) - (\tilde{\phi}''(y^n) z^{n-1}(u_h) r_1^n, r_2^{n-1}). \end{aligned}$$

Then, using δ-Cauchy inequality, we can find an estimate as follows

$$\begin{aligned} \|e_2^{n-1}\|_0^2 + \|r_2^{n-1}\|_0^2 &\le C\left((\triangle t)^2 + h^2 + \|u^n - u_h^n\|_0^2\right) \\ &+\delta(\|r_2^{n-1}\|_0^2 + \|e_2^{n-1}\|_0^2), \end{aligned} \tag{42}$$

or equivalently,

$$\|e_2^{n-1}\|_0 + \|r_2^{n-1}\|_0 \le C(\triangle t + h + \|u^n - u_h^n\|_0). \tag{43}$$

Taking $w = \mathrm{div} e_2^{n-1}$ as a test function in (35) and using δ-Cauchy inequality, then we get

$$\begin{aligned} &\|\mathrm{div} e_2^{n-1}\|_0^2 = (r_1^n, \mathrm{div} e_2^{n-1}) - (\phi'(y^n) r_2^{n-1}, \mathrm{div} e_2^{n-1}) \\ &+(z_t^n - d_t z^n(u_h), \mathrm{div} e_2^{n-1}) - (\tilde{\phi}''(y^n) z^{n-1}(u_h) r_1^n, \mathrm{div} e_2^{n-1}) \\ &\le C\|z_t^n - d_t z^n(u_h)\|_0^2 + C\|r_1^n\|_0^2 + C\|r_2^{n-1}\|_0^2 + \delta\|\mathrm{div} e_2^{n-1}\|_0^2, \end{aligned} \tag{44}$$

then, using the estimate (39) and (43), we verify that

$$\|\mathrm{div} e_2^{n-1}\|_0 \le C(\triangle t + h + \|u^n - u_h^n\|_0). \tag{45}$$

This implies (37).

Now we combine the bounds given by Theorems 3.3-3.4 to come up with the following main results.

Theorem 3.5 *Let* $(\vec{p}, y, \vec{q}, z, u) \in (\vec{V} \times W)^2 \times K$ *and* $(\vec{p}_h, y_h, \vec{q}_h, z_h, u_h) \in (\vec{V}_h \times W_h)^2 \times K$ *be the solutions of (11)-(17) and (22)-(28), respectively. Assume that* $\forall n = [0, 1, \cdots, N]$, $B^* z^n + u^n \in H^1(\Omega)$. *Then, we have*

$$|||u - u_h|||_{L^2(J;L^2(\Omega))} \leq C(\triangle t + h), \quad (46)$$

$$|||\vec{p} - \vec{p}_h|||_{L^\infty(J;H(\mathrm{div}))} + |||y - y_h|||_{L^\infty(J;L^2(\Omega))} \leq C(\triangle t + h), \quad (47)$$

$$|||\vec{q} - \vec{q}_h|||_{L^\infty(J;H(\mathrm{div}))} + |||z - z_h|||_{L^\infty(J;L^2(\Omega))} \leq C(\triangle t + h). \quad (48)$$

4 Conclusions

In this paper, we derive a priori error estimates of the variational discretization and mixed finite element methods for semilinear parabolic optimal control problem. Our priori error estimates for the optimal control problems governed by semilinear paraboli equations by the variational discretization and fully discrete mixed finite element methods seem to be new.

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