Gen. Math. Notes, Vol. 25, No. 2, December 2014, pp.6-18
ISSN 2219-7184; Copyright ©ICSRS Publication, 2014
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On λ-Statistical Limit Inferior and Limit Superior of Order β for Sequences of Fuzzy Numbers

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(Received: 3-3-14 / Accepted: 5-7-14)

Abstract

The aim of present work is to introduce and study the concepts of S_λ^β-limit points and cluster points and S_λ^β-limit inferior and limit superior of fuzzy number sequences.

Keywords: *Fuzzy number sequences, limit inferior and limit superior, statistical convergence.*

1 Introduction

The notion of statistical convergence of sequences of number was introduced by Fast [8] and Schoenberg [26] independently and latter dicussed in [9-12] and [17] *etc.* In past years, statistical convergence has also become an interesting area of research for sequences of fuzzy numbers. The credit goes to Nuray and Savaş [21], who first introduced statistical convergence to sequences of fuzzy numbers. After their pioneer work, many authors have made their contribution to study different generalizations of statistical convergence for sequences of fuzzy numbers(see [1-5],[14-16],[18], [20], [22], [25] *etc.*).

Mursaleen [19] generalized the notion of statistical convergence with the help of a non-decreasing sequence $\lambda = (\lambda_n)$ of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and called respectively λ-statistical convergence. Savaş[24] extented the notion to sequences of fuzzy numbers. Tuncer and Benli [27-28] has introduced λ-statistically Cauchy sequences and λ-statistical limit and cluster point for the sequences of fuzzy numbers. Gadjiev and Orhan [13] introduced the notion of statistical convergence with degree $0 < \beta < 1$ for a number sequence. Then, Çolak [6] studied the notions of statistical convergence and p-Cesàro summability with order α and the notion was further generalized by Çolak and Bektaş in [7].

In the present work, we aim to introduce the concepts of λ-statistical convergence, λ-statistical limit and cluster points and λ-statistical limit inferior and limit superior of order β for sequences of fuzzy numbers.

2 Background and Preliminaries

Given any interval A, we shall denote its end points by A_1, A_2 and by D the set of all closed bounded intervals on real line $\mathbb{R}$, i.e., $D = \{A \subset \mathbb{R}: A = [A_1, A_2]\}$. For $A, B \in D$ we define $A \leq B$ if and only if $A_1 \leq B_1$ and $A_2 \leq B_2$. Furthermore, the distance function d defined by $d(A, B) = \max\{|A_1 - B_1|, |A_2 - B_2|\}$, is a Housdroff metric on D and (D, d) is a complete metric space. Also $\leq$ is a partial order on D.

Definition 2.1 *A fuzzy number is a function X from $\mathbb{R}$ to [0,1] which satisfying the following conditions: (i) X is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$; (ii) X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, $X(\lambda x + (1 - \lambda)y) \geq \min\{X(x), X(y)\}$; (iii) X is upper semi-continuous; (iv) The closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ denoted by X^0 is compact.*

Further, every real number r can be expressed as a fuzzy number $\overline{r}$ as follows:

$$\overline{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{otherwise.} \end{cases}$$

The α-cut of fuzzy real number X is denoted by $[X]_\alpha$, $0 < \alpha \leq 1$, where $[X]_\alpha = \{t \in \mathbb{R} : Xt \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut. A fuzzy real number X is said to be upper semi-continuous if for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}$. If there exists $t \in \mathbb{R}$ such that $X(t) = 1$, then the fuzzy real number X is called normal.

A fuzzy number X is said to be convex, if $X(t) = X(s) \wedge X(r) = \min(X(s), X(r))$ where $s < t < r$. The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$ and throughout the article, by a fuzzy real number we mean that the number belongs $L(\mathbb{R})$. Let $X, Y \in L(\mathbb{R})$ and the α-level sets be

$[X]_\alpha = [X_1^\alpha, X_2^\alpha]$ and $[Y]_\alpha = [Y_1^\alpha, Y_2^\alpha]$ for $\alpha \in [0, 1]$.

Then the arithmetic operations $L(\mathbb{R})$ are defined as follows:

$$[X \oplus Y]_\alpha = [X_1^\alpha + Y_1^\alpha, X_2^\alpha + Y_2^\alpha],$$
$$[X \ominus Y]_\alpha = [X_1^\alpha - Y_2^\alpha, X_2^\alpha - Y_1^\alpha],$$
$$[X \otimes Y]_\alpha = \left[\min_{i,j\in\{1,2\}}\{X_i^\alpha.Y_j^\alpha\}, \max_{i,j\in\{1,2\}}\{X_i^\alpha.Y_j^\alpha\}\right],$$
$$\left[X^{-1}\right]_\alpha = [(X_2^\alpha)^{-1}, (X_1^\alpha)^{-1}],\ 0 \notin X$$

for each $\alpha \in [0, 1]$. The additive identity and multiplicative identity in $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by $\overline{d}(X, Y) = \sup_{\alpha\in[0,1]} d(X^\alpha, Y^\alpha)$. Puri and Ralescu [23] proved that $(L(\mathbb{R}), \overline{d})$ is a complete metric space. Also the ordered structure on $L(\mathbb{R})$ is defined as follows.

For $X, Y \in L(\mathbb{R})$, we define $X \leq Y$ if and only if $X_1^\alpha \leq Y_1^\alpha$ and $X_2^\alpha \leq Y_2^\alpha$ for each $\alpha \in [0, 1]$. We say that $X < Y$ if $X \leq Y$ and there exist $\alpha_0 \in [0, 1]$ such that $X_1^{\alpha_0} < Y_1^{\alpha_0}$ or $X_2^{\alpha_0} < Y_2^{\alpha_0}$. The fuzzy number X and Y are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$.

Before proceeding further, we recall some definitions and results which form the background of the present work.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and the interval $I_n = [n - \lambda_n + 1,\ n]$. The set of all such sequences will be denoted by Ω.

Definition 2.2 *A sequence $X = (X_k)$ of fuzzy numbers is said to be λ-statistically convergent to a fuzzy number X_0, in symbol: $S_\lambda\text{-}\lim_k X_k = X_0$, if for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Let $S_\lambda(X)$ denotes the set of all statistically convergent sequences of fuzzy numbers.

3 S_{λ^β}-Convergence

Definition 3.1 *Let β be any real number such that $\beta \in (0, 1]$. The λ^β-density of a set $A \subseteq N$ is denoted as $\delta_{\lambda^\beta}(A)$ and is defined by*

$$\delta_{\lambda^\beta}(A) = \lim_{n\to\infty} \frac{1}{\lambda^\beta} |\{k \in I_n : k \in A\}|,$$

if the limit exists.

Definition 3.2 *Let $\lambda \in \Omega$ and $\beta \in (0,\, 1]$ is given. A sequence $X = (X_k)$ of fuzzy numbers is said to be λ-statistically convergent of order β to a fuzzy number X_0, in symbol: $S_{\lambda^\beta}\text{-}\lim_k X_k = X_0$, if for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda^\beta}|\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Let $S_{\lambda^\beta}(X)$ denotes the set of all S_{λ^β}-convergent sequences of fuzzy numbers.

Remark 3.3 *1. For the choice of $\beta = 1$, the λ-statistical convergence of order β reduces to λ-statistical convergence.*
2. The λ-statistical convergence of order β is well defined for $\beta \in (0, 1]$ but is not well defined for $\beta > 1$. This case can be seen in the following example.

Example 3.4 *Let $X = (X_K)$ be a sequence of fuzzy number defined as*

$$X_k(x) = \begin{cases} \nu(x), & \text{if } k \text{ is an odd number,} \\ \mu(x) & \text{otherwise,} \end{cases}$$

where

$$\nu(x) = \begin{cases} x-7 & \text{if } x \in [7, 8], \\ 8-x & \text{if } x \in (8, 9], \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\mu(x) = \begin{cases} x-4 & \text{if } x \in [4, 5], \\ 6-x & \text{if } x \in (5, 6] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $\alpha \in (0,\, 1]$, the α-level sets of X_k are

$$[X_k]^\alpha = \begin{cases} [7+\alpha,\, 9-\alpha], & \text{if } k \text{ is odd} \\ [4+\alpha,\, 6-\alpha] & \text{otherwise.} \end{cases}$$

Hence, for $\beta > 1$ the sequence (X_k) is λ-statistical convergence of order β to η_1 and η_2 as well, since

$$\lim_{n\to\infty} \frac{1}{\lambda^\beta}\left|\left\{k \in I_n : \overline{d}(X_k, \eta_1) \geq \epsilon\right\}\right| = \lim_{n\to\infty} \frac{\lambda_n}{2\lambda_n^\beta} = 0$$

$$\lim_{n\to\infty} \frac{1}{\lambda^\beta}\left|\left\{k \in I_n : \overline{d}(X_k, \eta_2) \geq \epsilon\right\}\right| = \lim_{n\to\infty} \frac{\lambda_n}{2\lambda_n^\beta} = 0$$

where $[\eta_1]^\alpha = [7+\alpha,\, 9-\alpha]$ and $[\eta_2]^\alpha = [4+\alpha, 6-\alpha]$. Hence, $S_{\lambda^\beta}\text{-}\lim X_k$ is not unique. ■

Theorem 3.5 *For* $0 < \beta \leq \gamma \leq 1$,

$$S_{\lambda^\beta}(X) \subseteq S_{\lambda^\gamma}(X)$$

and the inclusion is strict.

Proof. Let $0 < \beta \leq \gamma \leq 1$, then obliviously $S_{\lambda^\beta}(X) \subseteq S_{\lambda^\gamma}(X)$ as

$$\frac{1}{\lambda^\gamma}|\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}| \leq \frac{1}{\lambda^\beta}|\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\}|.$$

To prove the strictness of the inclusion, let us consider the following example.

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x-2 & \text{if } x \in [2,3], \\ 4-x & \text{if } x \in (3,4], \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k = n^3 \\ \left.\begin{array}{ll} x-6 & \text{if } x \in [6,7], \\ 8-x & \text{if } x \in (7,8], \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } k \neq n^3. \end{cases}$$

Then, for $\alpha \in (0,\ 1]$,

$$[X_k]^\alpha = \begin{cases} [2+\alpha,\ 4-\alpha], & \text{if } k = n^3 \\ [6+\alpha,\ 8-\alpha] & \text{if } k \neq n^3. \end{cases}$$

For $\gamma \in (\frac{1}{3},\ 1]$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda^\gamma}|\{k \in I_n : \overline{d}(X_k, \zeta) \geq \epsilon\}| = \lim_{n\to\infty} \frac{\sqrt[3]{\lambda_n - 1}}{\lambda_n^\gamma} = 0$$

where $\zeta = [-2+2\alpha,\ 2-2\alpha]$. This means, the sequence (X_k) is S_{λ^β}-convergent of order γ but is not S_{λ^β}-convergent of order β, for $\beta \in (0,\ \frac{1}{3}]$. ■

Theorem 3.6 *Let* $\lambda = (\lambda_n)$ *and* $\mu = (\mu_n)$ *both are in* Ω *such that* $\lambda_n \leq \mu_n$ *for all* $n \geq n_0$ *for some* $n_0 \in \mathbb{N}$. *If, for* $0 < \beta \leq \gamma \leq 1$, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$, *then*

(i) $S_{\mu^\gamma}(X) \subseteq S_{\lambda^\beta}(X)$,
(ii) $S_{\mu^\beta}(X) \subseteq S_{\lambda^\beta}(X)$,
(iii) $S_\mu(X) \subseteq S_{\lambda^\beta}(X)$.

Proof. (i) Suppose $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$ for the sequences $\lambda, \mu \in \Omega$ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Assume that $S_{\mu^\gamma}\text{-}\lim_k X_k = X_0$. Now for $\epsilon > 0$, we have

$$\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\} \subset \left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}$$

where $J_n = [n - \mu_n + 1, n]$. Now, we can write

$$\frac{1}{\mu_n^\gamma}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \leq \frac{1}{\mu_n^\gamma}\left|\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right|$$

or

$$\frac{\lambda_n^\beta}{\mu_n^\gamma}.\frac{1}{\lambda_n^\beta}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \leq \frac{1}{\mu_n^\gamma}\left|\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right|$$

Since S_{μ^γ}- $\lim_k X_k = X_0$ and $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$, taking the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\beta}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| = 0$$

which completes the proof.

The results (ii) and (iii) are the immediate consequences of (i). ■

Theorem 3.7 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If, for $0 < \beta \leq \gamma \leq 1$, $\lim_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$, then*

(i) $S_{\lambda^\beta}(X) = S_{\mu^\gamma}(X)$,

(ii) $S_{\lambda^\beta}(X) = S_{\mu^\beta}(X)$,

(iii) $S_{\lambda^\beta}(X) = S_{\mu}(X)$.

Proof. (i) Suppose $\lim_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$ for the sequences $\lambda, \mu \in \Omega$ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Assume that $(X_k) \in S_{\lambda^\beta}(X)$. Since $I_n \subset J_n$ then, for $\epsilon > 0$, we may write

$$\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\} \supset \left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}$$

Thus, we obtain

$$\begin{aligned}
\frac{1}{\mu_n^\gamma}\left|\left\{k \in J_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| &= \frac{1}{\mu_n^\gamma}\left|\left\{n - \mu_n + 1 \leq k \leq n - \lambda_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\quad + \frac{1}{\mu_n^\gamma}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\gamma}\right) + \frac{1}{\mu_n^\gamma}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\leq \left(\frac{\mu_n - \lambda_n^\beta}{\mu_n^\gamma}\right) + \frac{1}{\mu_n^\gamma}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right| \\
&\leq \left(\frac{\mu_n}{\mu_n^\gamma} - \frac{\lambda_n^\beta}{\mu_n^\gamma}\right) + \frac{1}{\lambda_n^\beta}\left|\left\{k \in I_n : \overline{d}(X_k, X_0) \geq \epsilon\right\}\right|
\end{aligned}$$

for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. The right hand side of the above inequality tends to 0 as $n \to \infty$. Hence, $(X_k) \in S_{\mu^\gamma}(X)$ and therefore $S_{\lambda^\beta}(X) \subseteq S_{\mu^\gamma}(X)$. Now, together with the (i) part of Theorem 2.1, this immediately implies that $S_{\lambda^\beta}(X) = S_{\mu^\gamma}(X)$.

It is very easy to prove (ii) and (iii), hence omitted. ■

4 S_{λ^β}-Limit and Cluster Point

Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. A subsequence $(X)_K$, where $K = \{k(j) : j \in \mathbb{N}\}$, of a fuzzy number sequence $X = (X_k)$ is a λ^β-thin subsequence if

$$\lim_{r\to\infty} \frac{1}{\lambda_n^\beta} |\{k(j) \in I_r : j \in \mathbb{N}\}| = 0,$$

On the other hand, $(X)_K$ is a λ^β-nonthin subsequence of (X_k) if

$$\limsup_{r\to\infty} \frac{1}{\lambda_n^\beta} |\{k(j) \in I_r : j \in \mathbb{N}\}| > 0.$$

.

Theorem 4.1 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. A fuzzy number X_0 is said to be a S_{λ^β}-limit point of the sequence (X) of fuzzy numbers provided that there is a λ^β-nonthin subsequence of (X_k) that is convergent to X_0.*

Let $\Lambda_{S_\lambda^\beta}$ denotes the set of all S_{λ^β}-limit point of the sequence (X_k) of fuzzy numbers.

Theorem 4.2 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. A fuzzy number Y_0 is said to be a S_{λ^β}-cluster point of the sequence (X_k) of fuzzy numbers provided that for each $\epsilon > 0$,*

$$\limsup_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k \in I_n : \overline{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma_{S_\lambda^\beta}$ denotes the set of all S_{λ^β}-cluster point of the sequence (X_k) of fuzzy numbers.

Theorem 4.3 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If, for $0 < \beta \leq \gamma \leq 1$, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$, then*

(i) $\Lambda_{S_\mu^\gamma} \supseteq \Lambda_{S_\lambda^\beta}$,

(ii) $\Lambda_{S_\mu^\beta} \supseteq \Lambda_{S_\lambda^\beta}$,

(iii) $\Lambda_{S_\mu} \supseteq \Lambda_{S_\lambda^\beta}$.

Proof. (i) Suppose, for $0 < \beta \leq \gamma \leq 1$, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$. Assume that $X_0 \in \Lambda_{S_\lambda^\beta}$, then there is λ^β-nonthin subsequence $(X_{k(j)})$ of (X_k) that is convergent to X_0 and

$$\limsup_{n\to\infty} \frac{1}{\lambda_n^\beta} |\{k(j) \in I_n : j \in \mathbb{N}\}| = d > 0 \tag{1}$$

Now for $\epsilon > 0$, we have

$$\{k(j) \in J_n : j \in \mathbb{N}\} \supset \{k(j) \in I_n : j \in \mathbb{N}\}$$

$$\text{Since, } \frac{1}{\mu_n^\gamma} |\{k(j) \in J_n : j \in \mathbb{N}\}| \geq \frac{\lambda_n^\beta}{\mu_n^\gamma} . \frac{1}{\lambda_n^\beta} |\{k(j) \in I_n : j \in \mathbb{N}\}|$$

it follows by (1) that $\limsup_{n\to\infty} \frac{1}{\mu_n^\gamma} |\{k(j) \in J_n : j \in \mathbb{N}\}| > 0$. Since $(X_{k(j)})$ is already convergent to X_0, so we have $X_0 \in \Lambda_{S_\mu^\gamma}$. Hence $\Lambda_{S_\mu^\gamma} \supseteq \Lambda_{S_\lambda^\beta}$.

The results (ii) and (iii) are the immediate consequences of (i). ■

Theorem 4.4 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If, for $0 < \beta \leq \gamma \leq 1$, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} > 0$, then*

(i) $\Gamma_{S_\mu^\gamma} \supseteq \Gamma_{S_\lambda^\beta}$,

(ii) $\Gamma_{S_\mu^\beta} \supseteq \Gamma_{S_\lambda^\beta}$,

(iii) $\Gamma_{S_\mu} \supseteq \Gamma_{S_\lambda^\beta}$.

Proof. The proof of the theorem can be obtain on the similar lines as that of the above theorem and therefore is omitted here.

Theorem 4.5 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If, for $0 < \beta \leq \gamma \leq 1$, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\liminf_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$, then*

(i) $\Lambda_{S_\mu^\gamma} = \Lambda_{S_\lambda^\beta}$,

(ii) $\Lambda_{S_\mu^\beta} = \Lambda_{S_\lambda^\beta}$,

(iii) $\Lambda_{S_\mu} = \Lambda_{S_\lambda^\beta}$.

Proof. (i) Suppose $\lim_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\lim_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$ for the sequences $\lambda, \mu \in \Omega$ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Assume that $X_0 \in \Lambda_{S_\mu^\gamma}$, then there is μ^γ-nonthin subsequence $(X_{k(j)})$ of (X_k) that is convergent to X_0 and

$$\limsup_{n\to\infty} \frac{1}{\mu_n^\gamma} |\{k(j) \in I_n : j \in \mathbb{N}\}| = d > 0 \tag{2}$$

Since $I_n \subset J_n$ then, for $\epsilon > 0$, we may write

$$\{k(j) \in J_n : j \in \mathbb{N}\} \supset \{k(j) \in I_n : j \in \mathbb{N}\}$$

Thus, as in the proof of Theorem 3.3 (i), we obtain

$$\frac{1}{\mu_n^\gamma} |\{k \in J_n : j \in \mathbb{N}\}| \leq \left(\frac{\mu_n}{\mu_n^\gamma} - \frac{\lambda_n^\beta}{\mu_n^\gamma}\right) + \frac{1}{\lambda_n^\beta} |\{k \in I_n : j \in \mathbb{N}\}|$$

for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. The right hand side of the above inequality tends to 0 as $n \to \infty$ and therefore $(X_k) \in S_{\mu^\gamma}(X)$. Hence, $\Lambda_{S_\mu^\gamma} \subseteq \Lambda_{S_\lambda^\beta}$. Now, together with the (i) part of Theorem 4.1, this immediately implies that $\Lambda_{S_\mu^\gamma} = \Lambda_{S_\lambda^\beta}$.

It is very easy to prove (ii) and (iii), hence omitted. ■

Theorem 4.6 *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ both are in Ω such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. If, for $0 < \beta \leq \gamma \leq 1$, $\liminf_{n\to\infty} \frac{\lambda_n^\beta}{\mu_n^\gamma} = 1$ and $\liminf_{n\to\infty} \frac{\mu_n}{\mu_n^\gamma} = 1$, then*

(i) $\Gamma_{S_\mu^\gamma} = \Gamma_{S_\lambda^\beta}$,

(ii) $\Gamma_{S_\mu^\beta} = \Gamma_{S_\lambda^\beta}$,

(iii) $\Gamma_{S_\mu} = \Gamma_{S_\lambda^\beta}$.

Proof. The proof of the theorem can be obtain on the similar lines as that of the above theorem and therefore is omitted here.

5 S_{λ^β}-Limit Inferior and Superior

In this section, we introduce the notions of S_{λ^β}-statistical limit inferior and superior for sequences of fuzzy numbers.

Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. For a sequence $X = (X_k)$ of fuzzy numbers, let us define the sets:

$$M = \{\mu \in L(\mathbb{R}) : \delta_{\lambda^\beta}(\{k \in I_n : X_k < \mu\}) \neq 0\};$$
$$N = \{\mu \in L(\mathbb{R}) : \delta_{\lambda^\beta}(\{k \in I_n : X_k > \mu\}) \neq 0\}.$$

Theorem 5.1 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. The S_{λ^β}-limit superior of (X_k) is defined by*

$$S_{\lambda^\beta}\text{-}\limsup X = \begin{cases} \sup N, & \text{if } N \neq \phi, \\ -\overline{\infty}, & \text{otherwise.} \end{cases}$$

Similarly, the S_{λ^β}-limit inferior is defined by

$$S_{\lambda^\beta}\text{- lim inf } X = \begin{cases} \inf M, & \text{if } M \neq \phi, \\ \overline{\infty}, & \text{otherwise.} \end{cases}$$

The concepts defined above can be illustrated with help of the following example.

Example 5.2 *Let $\lambda = (\lambda_n) \in \Omega$. We define a sequence of fuzzy numbers $X = (X_k)$ as follows. For $x \in \mathbb{R}$, define*

$$X_k(x) = \begin{cases} \left.\begin{array}{ll} x - k + 1, & \text{if } k - 1 \leq x \leq k \\ -x + k + 1, & \text{if } k < x \leq k + 1 \\ 0, & \text{otherwise} \end{array}\right\}, & \text{if} \quad k = n^2 \\ \left.\begin{array}{ll} \vartheta_1, & \text{if } k \text{ is even} \\ \vartheta_2, & \text{if } k \text{ is odd} \end{array}\right\}, & \text{if} \quad k \neq n^2. \end{cases}$$

where the fuzzy numbers ϑ_1 and ϑ_2 are given by

$$\vartheta_1 = \begin{cases} x - 5, & \text{if } 5 \leq x \leq 6, \\ 7 - x & \text{if } 6 < x \leq 7, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vartheta_2 = \begin{cases} x + 7, & \text{if } -7 \leq x \leq -6, \\ -5 - x & \text{if } -6 < x \leq -5, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we obtain

$$[X_k]^\alpha = \begin{cases} [k - 1 + \alpha, k + 1 - \alpha], & \text{if } k = n^2 \\ [5 + \alpha, 7 - \alpha], & \text{if } k \neq n^2 \text{ and } k \text{ is even,} \\ [-7 + \alpha, -5 - \alpha], & \text{if } k \neq n^2\text{. and } k \text{ is odd,} \end{cases}$$

Clearly,

$M = (\eta_1, \overline{\infty})$ and $N = (-\overline{\infty}, \eta_2)$

for the sequences (X_k), where

$[\eta_1]^\alpha = [-7 + \alpha, -5 - \alpha]$ and $[\eta_2]^\alpha = [5 + \alpha, 7 - \alpha]$.

Therefore $S_{\lambda^\beta}\text{- lim inf}\, X = \eta_1$ and $S_{\lambda^\beta}\text{- lim sup}\, X = \eta_2$, only for $\beta \in (\frac{1}{2}, 1]$ ■

Theorem 5.3 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. Then, for a sequence $X = (X_k)$ of fuzzy numbers, $S_{\lambda^\beta}\text{- lim sup}\, X = \eta$ if and only if, for each positive fuzzy number $\overline{\epsilon}$*

$$\delta_{\lambda^\beta}(\{k \in I_n : X_k > \eta \ominus \overline{\epsilon}\} \neq 0 \textit{ and } \delta_{\lambda^\beta}(\{k \in I_n : X_k > \eta \oplus \overline{\epsilon}\}) = 0.$$

Theorem 5.4 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. Then, for a sequence $X = (X_k)$ of fuzzy numbers, S_{λ^β}- $\liminf X = \xi$ if and only if, for each positive fuzzy number $\overline{\epsilon}$*

$$\delta_{\lambda^\beta}(\{k \in I_n : X_k > \xi \oplus \overline{\epsilon}\} \neq 0 \text{ and } \delta_{\lambda^\beta}(\{k \in I_n : X_k > \xi \ominus \overline{\epsilon}\}) = 0.$$

The proofs of the above theorems are routine work so is omitted here.

Theorem 5.5 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. Then, for a sequence $X = (X_k)$ of fuzzy numbers*

$$S_{\lambda^\beta}\text{- }\liminf X \leq S_{\lambda^\beta}\text{- }\limsup X.$$

Proof. If S_{λ^β}- $\limsup X = +\overline{\infty}$, then the result is obvious one but if S_{λ^β}- $\limsup X = -\overline{\infty}$, then $N = \phi$. It means for every $\mu \in L(R)$, $\delta_{\lambda^\beta}(\{k \in I_n : X_k > \mu\}) = 0$. Hence, for every $\nu \in L(R)$, $\delta_{\lambda^\beta}(\{k \in I_n : X_k < \nu\}) \neq 0$. By the definition S_{λ^β}- $\liminf X = -\overline{\infty}$.

Now, let us consider the case when S_{λ^β}- $\limsup X = \eta$ is finite and S_{λ^β}- $\liminf X = \theta$. Since S_{λ^β}- $\limsup X = \eta$, therefore $\delta_{\lambda^\beta}(\{k \in I_n : X_k > \eta \oplus \frac{\overline{\epsilon}}{2}\}) = 0$. This implies that $\delta_{\lambda^\beta}(\{k \in I_n : X_k \leq \eta \oplus \frac{\overline{\epsilon}}{2}\}) \neq 0$, which gives in return that $\delta_{\lambda^\beta}(\{k \in I_n : X_k < \eta \oplus \overline{\epsilon}\}) \neq 0$. Hence $\eta \oplus \overline{\epsilon} \in M$. But, by the definition of $S_{\lambda^\beta} - \liminf$, $\inf M = \theta$ so $\theta \leq \beta \oplus \overline{\epsilon}$. Since $\overline{\epsilon}$ is arbitrary, hence completes the result. ■

Corollary 5.6 *Let $\lambda = (\lambda_n) \in \Omega$ and $\beta \in (0, 1]$. Then, for a sequence $X = (X_k)$ of fuzzy numbers*

$$\liminf X \leq S_{\lambda^\beta}\text{- }\liminf X \leq S_{\lambda^\beta}\text{- }\limsup X \leq \limsup X.$$

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