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Lower k-Hessenberg Matrices and k-Fibonacci, Fibonacci-p and Pell (p, i) Numbers

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Abstract

In this work, we define a family of sparse Hessenberg matrices whose permanents lead us to k-Fibonacci, Fibonacci-p and Pell (p,i) numbers. Furthermore, we show that it contains some well-known general number sequences in it. We provide a Maple 13 source code describing the contraction steps.

Keywords: *Determinant, Fibonacci-p and Pell (p,i) numbers, Hessenberg matrix, k-Fibonacci numbers, Permanent.*

1 Introduction

Matrix theory combines linear algebra, graph theory, algebra, combinatorics and statistics. Some special type of matrices are very important in these areas. In this paper, we consider lower k-Hessenberg matrices which have the pattern

$$H_n(k) = \begin{pmatrix} \bullet & \bullet & & & & \\ & \bullet & \bullet & & & \\ & & \bullet & \bullet & & \\ \bullet & & & \bullet & \bullet & \\ & \bullet & & & \bullet & \bullet \\ & & \bullet & & & \bullet \end{pmatrix}$$

which will defined more precisely later.

Most of the well-known number sequences are defined as a result of a natural events or a mathematical modelling of an occurrence in nature. Fibonacci numbers are one of the most famous number sequence defined on modelling for proliferating of rabbits. In literature, there is a huge number of papers on Fibonacci numbers and their generalizations. For example, Lee et al. [7] investigated the *k-generalized Fibonacci sequence* $(g_n^{(k)})$ with initial conditions

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0\,, \quad g_{k-1}^{(k)} = g_k^{(k)} = 1\,,$$

and, for $n > k \geqslant 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}\,. \tag{1}$$

Then, Lee [6] introduced *k-Lucas numbers*, which has similar recurrence but for different initial conditions.

Kılıç and Stakhov [3] considered certain generalizations of well-known Fibonacci and Lucas numbers and the generalized Fibonacci and Lucas p-numbers defined by the following recurrence relation for $p = 1, 2, 3, \ldots$, and $n > p + 1$

$$\begin{aligned} F_p(n) &= F_p(n-1) + F_p(n-p-1), \\ L_p(n) &= L_p(n-1) + L_p(n-p-1), \end{aligned}$$

where $F_p(0) = 0, F_p(1) = \cdots = F_p(p) = F_p(p+1) = 1$ and $L_p(0) = p + 1, L_p(1) = \cdots = L_p(p) = L_p(p+1) = 1$, respectively. Furthermore they defined n-square $(0,1)$-matrix as below

$$M(n,p) = \begin{cases} 1, & \text{for } m_{i+1,i} = m_{i,i} = m_{i,i+p} \\ 0, & \text{for } j = i+1 \end{cases} \tag{2}$$

for a fixed integer p, which corresponds to the adjacency matrix of the bipartite graph $G(M(n,p))$. Then they showed that permanents of $M(n,p)$ are the number of 1-factors of $G(M(n,p))$ that is the $(n+1)$th generalized Fibonacci p-number. Moreover Yilmaz et al. [4, 9] considered Hessenberg matrices and the Fibonacci, Lucas, Pell and Perrin numbers. Öcal et al. [8] gave some determinantal and permanental representations for k-generalized Fibonacci and

Lucas numbers. On the other hand, Kılıç [2] studied the generalized Pell (p,i)-numbers for $p=1,2,3,\ldots$, $n>p+1$, and $0\leq i\leq p$

$$P_p^{(i)}(n)=2P_p^{(i)}(n-1)+P_p^{(i)}(n-p-1)$$

with initial conditions $P_p^{(i)}(1)=P_p^{(i)}(2)=\cdots=P_p^{(i)}(i)=0$ and $P_p^{(i)}(i+1)=P_p^{(i)}(i+2)=\cdots=P_p^{(i)}(p+1)=1$. Moreover, the author defined n-square integer matrix $M(n,p)=(m_{ij})$ as below:

$$M(n,p)=\begin{cases}1, & \text{for } m_{i+1,i}=m_{i,i+p}\\ 2, & \text{for } m_{i,i}\\ 0, & \text{for } j=i+1\end{cases} \tag{3}$$

for a fixed integer p, then showed

$$\operatorname{per} M(n,p)=P_p^{(p)}(n+p+1).$$

The *permanent* of an $n\times n$ matrix $A=(a_{ij})$ is given by

$$\operatorname{per}(A)=\sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i\sigma(i)},$$

where S_n represents the symmetric group of degree n.

Brualdi and Gibson [1] proposed a method to compute permanent of a matrix. Let $A=(a_{ij})$ be an $m\times n$ matrix with row vectors $r_1,r_2,\ldots,r_m$. We call A is *contractible* on column k, if column k contains exactly two non zero elements. Suppose that A is contractible on column k with $a_{ik}\neq 0, a_{jk}\neq 0$ and $i\neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i+a_{ik}r_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki}\neq 0, a_{kj}\neq 0$ and $i\neq j$, then the matrix $A_{k:ij}=(A^T_{ij:k})^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a integer matrix and B is a contraction of A [1], then

$$\operatorname{per}A=\operatorname{per}B\,. \tag{4}$$

A matrix A is called *convertible* if there exists an n-square $(1,-1)$-matrix H such that $\operatorname{per}A=\det(A\circ H)$, here $\circ$ denotes Hadamard product of A and H. The matrix H is called as *converter* of A. Let H be a $(1,-1)$-matrix such that

$$h_{i,j}=\begin{cases}-1\,, & i+1=j\\ 1\,, & \text{otherwise}\end{cases}\,. \tag{5}$$

Klein [5] established a generalization for Fibonacci numbers for a constant integer $m\geq 2$

$$\begin{aligned} A_n^{(m)}&=A_{n-1}^{(m)}+A_{n-m}^{(m)}\,, && \text{for } n>m+1,\\ A_n^{(m)}&=n-1\,, && \text{for } 1<n\leq m+1.\end{aligned} \tag{6}$$

In particular, $F_n = A_n^{(2)}$ are the standard Fibonacci numbers. Taking into account Klein's generalization, let us consider the sequence $\{u_n\}$ given below:

$$u_n^{(k)} = au_{n-1}^{(k)} + b^k cu_{n-k-1}^{(k)}. \tag{7}$$

Here $k > 1$ and $u_0^{(k)} = 1$, $u_1^{(k)} = d$, $u_2^{(k)} = ad$ and $u_k^{(k)} = a^{k-1}d$. The first few terms of the sequence given in following table:

$k\backslash$n	1	2	3	4	5
$u_n^{(2)}$	d	da	da^2+b^2c	$da^3+ab^2c+cdb^2$	$da^4+a^2cb^2+2cb^2da$
$u_n^{(3)}$	d	da	da^2	da^3+b^3c	$da^4+ab^3c+b^3dc$
$u_n^{(4)}$	d	da	da^2	da^3	da^4+cb^4
$u_n^{(5)}$	d	da	da^2	da^3	da^4

2 Lower k-Hessenberg Matrices and the $\{u_n\}$ Sequence

Let us define the n-square Hessenberg matrix $H_n(k) = (h_{ij})$ as follows:

$$h_{ij} = \begin{cases} a, & \text{for } i = j = 1, 2, \ldots, n-1 \\ b, & \text{for } j = i + 1 \\ c, & \text{for } i = j + k \\ d, & \text{for } i = j = n \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

where $2 \le k \le n-1$ and $a, b, c, d \in \mathbb{R}$.

Example 2.1 *For $k = 3$ and $n = 7$;*

$$H_7(3) = \begin{pmatrix} a & b & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 0 \\ c & 0 & 0 & a & b & 0 & 0 \\ 0 & c & 0 & 0 & a & b & 0 \\ 0 & 0 & c & 0 & 0 & a & b \\ 0 & 0 & 0 & c & 0 & 0 & d \end{pmatrix}.$$

Theorem 2.2 *Let $H_n(k)$ be as in 8, then*

$$\operatorname{per} H_n(k) = u_n^{(k)},$$

for $2 \le k < n$, where $u_n^{(k)}$ is the nth term of the sequence given by 7.

Proof. By the definition of $H_n(k)$, it can be contracted on column n. Let $H_n^{(r)}(k)$ be the rth contraction of the matrix $H_n(k)$. For $r=1$,

$$H_n^{(1)}(k)=\begin{pmatrix} a & b & & & & & & & \\ 0 & a & b & & & & & & \\ \vdots & 0 & a & b & & & & & \\ 0 & \ddots & 0 & a & b & & & & \\ 0 & 0 & \cdots & 0 & a & b & & & \\ c & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & 0 & 0 & \cdots & 0 & a & b & \\ \vdots & & c & 0 & 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & dc & bc & 0 & \cdots & 0 & da \end{pmatrix}.$$

Using the consecutive contraction method on the last column, we get,

$$H_n^{(r)}(k)=\begin{pmatrix} a & b & & & & & & & & 0 \\ 0 & a & b & & & & & & & \\ \vdots & 0 & a & b & & & & & & \\ 0 & \cdots & 0 & a & b & & & & & \\ 0 & 0 & \cdots & 0 & a & b & & & & \\ 0 & 0 & 0 & \cdots & \ddots & \ddots & \ddots & & & \\ c & \ddots & \ddots & \ddots & \cdots & 0 & a & b & & \\ 0 & \ddots & 0 & 0 & 0 & \cdots & 0 & a & b & \\ \vdots & & c & 0 & 0 & 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & cu_r^{(k)} & bcu_{r-1}^{(k)} & b^2cu_{r-2}^{(k)} & \cdots & \cdots & b^{k-1}cu_{r-k+1}^{(k)} & u_{r+1}^{(k)} \end{pmatrix}.$$

Here $2\le r\le n-k-1$ and

$$H_n^{(r)}(k)=\begin{pmatrix} a & b & & & & & \\ 0 & a & b & & & & \\ \vdots & \ddots & \ddots & \ddots & & & \\ 0 & \cdots & 0 & a & b & & \\ 0 & 0 & \cdots & 0 & a & b & \\ 0 & 0 & 0 & \cdots & 0 & a & b \\ bcu_{n-k-1}^{(k)} & b^2cu_{n-k-2}^{(k)} & b^3cu_{n-k-3}^{(k)} & \cdots & \cdots & b^{k-1}cu_{r-k+1}^{(k)} & u_{r+1}^{(k)} \end{pmatrix},$$

where $n-k-1<r\le n-3$. Then, continuing with this process, we get

$$H_n^{(n-2)}(k)=\begin{pmatrix} a & b \\ b^{k-1}cu_{n-k-1}^{(k)} & u_{n-1}^{(k)} \end{pmatrix}.$$

By applying (4), we have $\operatorname{per} H_n(k) = \operatorname{per} H_n^{(n-2)}(k) = u_n^{(k)}$, as desired. ■

a	b	c	d	Name
2	1	-1	1	k-Fibonacci numbers
1	1	1	1	Fibonacci-p numbers
2	1	1	2	Pell(p,i) numbers

As it can be seen from the previous table, the matrix $H_n(k)$ is a general form of the matrices given by 2 and 3. Moreover, for $a = 2$, $b = 1$, $c = -1$ and $d = 1$, the permanent of the sequence gives k-Fibonacci numbers.

Theorem 2.3 *Let us consider the matrix $H_n(k) = (h_{ij})$ with $h_{i,i+1} = 1$, $h_{i,i} = 2$, and $h_{i+k,i} = -1$, where $2 \le k \le n$. Then*

$$\operatorname{per} H_n(k) = \sum_{i=1}^{n} g_i^{(k)}.$$

Proof. By the contraction method on column n, one can see that

$$H_n^{(1)}(k) = \begin{pmatrix}
2 & 1 & & & & & & & \\
0 & 2 & 1 & & & & & & \\
\vdots & 0 & 2 & 1 & & & & & \\
0 & & \ddots & \ddots & \ddots & & & & \\
0 & 0 & & \ddots & \ddots & \ddots & & & \\
-1 & \ddots & \ddots & & \ddots & \ddots & \ddots & & \\
0 & \ddots & 0 & 0 & \cdots & 0 & 2 & 1 & \\
\vdots & & -1 & 0 & 0 & \cdots & 0 & 2 & 1 \\
0 & \cdots & 0 & -2 & -1 & 0 & \cdots & 0 & 4
\end{pmatrix}.$$

By the recursive contraction method on the last column, we get

$$H_n^{(r)}(k) = \begin{pmatrix}
2 & 1 & & & & & & & \\
0 & 2 & 1 & & & & & & \\
\vdots & 0 & 2 & 1 & & & & & \\
0 & \cdots & 0 & 2 & 1 & & & & \\
0 & 0 & 0 & \cdots & \ddots & \ddots & & & \\
-1 & \ddots & \ddots & \ddots & \cdots & 2 & 1 & & \\
0 & \ddots & 0 & 0 & 0 & 0 & 2 & 1 & \\
\vdots & & -1 & 0 & 0 & \cdots & 0 & 2 & 1 \\
0 & \cdots & 0 & -\sum_{i=1}^{r+1} g_i^k & -\sum_{i=1}^{r} g_i^k & \cdots & \cdots & -\sum_{i=1}^{r-k+2} g_i^k & \sum_{i=1}^{r+2} g_i^k
\end{pmatrix}$$

for $2 \leq r \leq n-k-1$ and

$$H_n^{(r)}(k) = \begin{pmatrix} 2 & 1 & & & & \\ 0 & 2 & \ddots & & & \\ \vdots & \ddots & \ddots & 1 & & \\ 0 & \cdots & 0 & 2 & 1 & \\ 0 & 0 & \cdots & 0 & 2 & 1 \\ -\sum_{i=1}^{n-k} g_i^k & -\sum_{i=1}^{n-k-1} g_i^k & \cdots & \cdots & -\sum_{i=1}^{r-k+2} g_i^k & \sum_{i=1}^{r+2} g_i^k \end{pmatrix}$$

for $n-k-1 < r \leq n-3$. Going with this process, one gets

$$H_n^{(n-2)}(k) = \begin{pmatrix} 2 & 1 \\ -\sum_{i=1}^{n-k} g_i^{(k)} & \sum_{i=1}^{n} g_i^{(k)} \end{pmatrix}.$$

By applying 4, we have $\operatorname{per} H_n(k) = \operatorname{per} H_n^{(n-2)}(k) = \sum_{i=1}^{n} g_i^{(k)}$, which is the sum of k-Fibonacci numbers given by 1. ■

Theorem 2.4 *Let us consider the n-square Hessenberg matrix $M_n(k) = (m_{ij})$ as*

$$m_{ij} = \begin{cases} a\,, & \text{for } i=j=1,2,\ldots,n-1 \\ -b\,, & \text{for } j=i+1 \\ c\,, & \text{for } i=j+k \\ d\,, & \text{for } i=j=n \\ 0\,, & \text{otherwise} \end{cases}$$

where $2 \leq k \leq n-1$. Then

$$\det M_n(k) = u_n^{(k)}.$$

Proof. It can be seen by using the converter matrix given with 5. ■

3 Appendix A

Using the following Maple 13 source code, it is possible to get the matrix and the steps of the contraction method. Here n is the order of the matrix and s is the shifting diagonal (i.e, $s = k$).

```
restart:
> a:=..:b:=..:c:=..:d:=..:s:=..:n:=..:with(LinearAlgebra):
> permanent:=proc(n)
> local i,j,k,p,C;
> p:=(i,j)->piecewise(i=j+s+1,c,j=i+1,b,j=n and i=n,d,i=j,a);
> C:=Matrix(n,n,p):
```

```
> for k from 1 to n-1 do
> print(k,C):
> for j from 1 to n+1-k do
> C[n-k,j]:=C[n+1-k,n+1-k]*C[n-k,j]+C[n-k,n+1-k]*C[n+1-k,j]:
> od:
> C:=DeleteRow(DeleteColumn(Matrix(n+1-k,n+1-k,C),n+1-k),n+1-k):
> od:
> print(k,eval(C)):
> end proc:with(LinearAlgebra):
> permanent(n);
```

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