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Univalence of Generalized an Integral Operators

Hesam Mahzoon[1], Ali Ebadian[2] and Rahim Kargar[3]

[1]Department of Mathematics, Islamic Azad University
Firoozkouh Branch, Firoozkouh, Iran
E-mail: mahzoon.hesam@gmail.com
[2]Department of Mathematics, Payame Noor University, I.R. of Iran
E-mail: ebadian.ali@gmail.com
[3]Department of Mathematics, Payame Noor University, I.R. of Iran
E-mail: rkargar1983@gmail.com

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Abstract

In this paper we define generalized differential operators from some well-known operators on the class $\mathcal{A}(p)$ of analytic functions in the unit disk $\Delta = \{z \in C : |z| < 1\}$. New class containing these operators is investigated. Also univalence of integral operator is considered.

Keywords: *Univalent, Starlike, Convex, Hadamard Product, Multiplier Transformations.*

1 Introduction

Let $\mathcal{A}(p)$ be the class of analytic functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N := \{1, 2, 3, ...\}), \tag{1}$$

defined in the unit disc $\Delta = \{z \in C : |z| < 1\}$ and the satisfying the normalization condition $f(0) = f'(0) - 1 = 0$. Put $\mathcal{A}(1) = \mathcal{A}$. A function $f \in \mathcal{A}$ is said to be starlike of order γ, if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \quad z \in \Delta,$$

for some $0 \leq \gamma < 1$ and it is defined by $\mathcal{S}^*(\gamma)$. Also, the class of convex functions of order γ, denote by $\mathcal{K}(\gamma)$ consists of function $f \in \mathcal{A}$ if and only if $zf'(z) \in \mathcal{S}^*(\gamma)$. For any two functions f and g such that $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ and $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$, the Hadamard product or Convolution of f and g denoted by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

Following [5], we recall the linear operator $\mathcal{I}(f(z)) := \mathcal{I}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)$ as follows:

$$\mathcal{I}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(c)_k (p+1-\mu)_k (p+1-\lambda+\nu)_k (\alpha+p)_k}{(a)_k (p+1)_k (p+1-\mu+\nu)_k k!} a_{p+k} z^{p+k}, \tag{2}$$

where $a, \mu, \nu, \in R$, $c \in R \setminus Z_0^- := \{..., -2, -1, 0\}$, $\alpha > -p$, $0 \leq \lambda < 1$, $\mu - \nu - p < 1$ and $z \in \Delta$. It should be remarked that the linear operator $\mathcal{I}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z)$ is a generalization of many other linear operators considered earlier (see [5]).

Definition 1.1 *Assume that f_j and g_j be in $\mathcal{A}(p)$ where $1 \leq j \leq r$. For $-1 \leq \delta \leq 1$, $\delta \in R$, $p_j > 0$, $p_j \in C$ and $r \in N$, the generalized integral operator $\mathcal{J}_g(f)(z) := \mathcal{J}_g(f_1, ..., f_r)(z) : \mathcal{A}(p) \to \mathcal{A}(p)$, is defined as*

$$\mathcal{J}_g(f)(z) = \int_0^z \left[w^\delta (\mathcal{I}(f_1(w)) * g_1(w))^{(n)}\right]^{p_1} \dots \left[w^\delta (\mathcal{I}(f_r(w)) * g_r(w))^{(n)}\right]^{p_r} dw \tag{3}$$

where $n \in N_0 := N \cup \{0\}$, $z \in \Delta$.

Remark 1.2 *i) For $\alpha = c = n = 1$, $\delta = \lambda = \mu = 0$, $a = p+1$ and $g_j(z) = \frac{z}{1-z}$ for all $1 \leq j \leq r$, the operator $F_{p_1,...,p_r}$ was introduced and studied by Breaz et al. [2].*
ii) If we take $\delta = \alpha = c = n = 1$, $\lambda = \mu = 0$, $a = p+1$ and $g_j(z) = \frac{z}{1-z}$ for all $1 \leq j \leq r$ in equation (3), it reduced to an integral operator $F_{p_1 \dots p_r}$ (see [7]).
iii) Putting $\delta = -1$, $\alpha = c = n = 1$, $\lambda = \mu = 0$, $a = p+1$ and $g_j(z) = g(z)$ for all $1 \leq j \leq r$, in equation (3), we obtain an integral operator $I_g(f_1, ..., f_r)(z)$ defined by Dileep and Latha (see [4]).

Definition 1.3 *A function $f \in \mathcal{A}(p)$ be in the class $\mathcal{SK}(\delta, \theta, \gamma)$, if it satisfies the following inequality:*

$$Re\left\{e^{i\theta}\left(\delta + \frac{z[\mathcal{I}(f(z)) * g(z)]^{(n+1)}}{[\mathcal{I}(f(z)) * g(z)]^{(n)}}\right)\right\} > \gamma \cos\theta \quad z \in \Delta, \tag{4}$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $0 \leq \gamma < 1$, $\delta \in N_0$, $-1 \leq \delta \leq 1$, $g \in \mathcal{A}(p)$ and $\mathcal{I}(f(z))$ given by (2).

Remark 1.4 *i) For $\theta = \delta = \lambda = \mu = n = 0$, $\alpha = c = 1$, $a = 2$ and $g(z) = \frac{z}{1-z}$, the class $\mathcal{SK}(\delta, \theta, \gamma)$ reduced to the class of starlike functions of order γ.*
ii) Taking $\theta = \lambda = \mu = 0$, $\delta = \alpha = c = n = 1$, $a = 2$ and $g(z) = \frac{z}{1-z}$, the class $\mathcal{SK}(\delta, \theta, \gamma)$ reduced to the class of convex functions of order γ.
iii) If we take $\theta = \delta = \lambda = \mu = n = 0$, $p = \alpha = c = 1$, $a = 2$ and $g(z) = z + \sum_{k=2}^{\infty}[1 + (k-1)\zeta]^n z^k$, $\zeta \geq 0$, the class $\mathcal{SK}(\delta, \theta, \gamma)$ reduced to the class $\mathcal{S}^n(\zeta, \gamma)$ introduced by S. Bulut [3].
iv) Putting $\theta = \delta = \lambda = \mu = n = 0$, $\alpha = c = 1$, $a = 2$ and $f, g \in \mathcal{A}$, we get the class $\mathcal{S}_g(\gamma)$ introduced by Dileep and Latha [4].

To prove our main results we shall need the following lemmas.

Lemma 1.5 *([1]) If $f \in \mathcal{A}$, satisfies the inequality*

$$\left(1 - |\, z \,|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \Delta$, then f is univalent in Δ.

Lemma 1.6 *([6]) If f is regular in $|z| < 1$ and*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M,$$

where M is the root of equation

$$8\sqrt{x(x-2)^3} - 3(4-x)^2 = 12, \quad M \approx 3,05...,$$

then f is univalent in Δ.

In this paper, using Lemma 1.5 and Lemma 1.6, we show that $\mathcal{J}_g(f)(z)$ is univalent. We, also show that $\mathcal{J}_g(f)(z) \in \mathcal{SK}(\delta, \theta, \gamma)$.

2 Main Results

Theorem 2.1 *Let $f_j, g_j \in \mathcal{A}(p)$, $p_j \in C$, $1 \leq j \leq r$, $|\delta| \leq 1$ and $\sum_{j=1}^{r} |p_j| \leq 1$. If*

$$\left| \delta + \frac{z[\mathcal{I}(f_j(z)) * g_j(z)]^{(n+1)}}{[\mathcal{I}(f_j(z)) * g_j(z)]^{(n)}} \right| \leq 1, \quad z \in \Delta, \tag{5}$$

then $\mathcal{J}_g(f)(z)$ given by (3) is univalent.

Proof. From (3) we obtain

$$(\mathcal{J}_g(f)(z))' = \left[z^{\delta}(\mathcal{I}(f_1(z)) * g_1(z))^{(n)}\right]^{p_1} \dots \left[z^{\delta}(\mathcal{I}(f_r(z)) * g_r(z))^{(n)}\right]^{p_r}, \quad (6)$$

which implies that

$$\ln(\mathcal{J}_g(f)(z))' = p_1\left[\delta \ln z + \ln(\mathcal{I}(f_1(z)) * g_1(z))^{(n)}\right] + \cdots$$

$$+p_r\left[\delta \ln z + \ln(\mathcal{I}(f_r(z)) * g_r(z))^{(n)}\right].$$

Taking the derivative for the above equality and by multiplying with z we have

$$\frac{z(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'} = p_1\left[\delta + \frac{z(\mathcal{I}(f_1(z)) * g_1(z))^{(n+1)}}{(\mathcal{I}(f_1(z)) * g_1(z))^{(n)}}\right] + \cdots$$

$$+p_r\left[\delta + \frac{z(\mathcal{I}(f_r(z)) * g_r(z))^{(n+1)}}{(\mathcal{I}(f_r(z)) * g_r(z))^{(n)}}\right]. \quad (7)$$

On multiplying the modulus of equation (7) by $\left(1-\mid z\mid^2\right)$, we obtain

$$\begin{aligned}\left(1-\mid z\mid^2\right)\left|\frac{z(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'}\right| &\leq \left(1-\mid z\mid^2\right)(\mid p_1\mid + \cdots + \mid p_r\mid) \\ &\leq 1.\end{aligned}$$

From Lemma 1.5, we get that $\mathcal{J}_g(f)(z)$ is univalent.

Taking $\alpha = c = n = 1$, $\delta = \lambda = \mu = 0$, $a = p+1$ and $g_j(z) = \frac{z}{1-z}$ for all $1 \leq j \leq r$, we have:

Corollary 2.2 *Assume that* $p_j \in C$ *and* $\sum_{j=1}^{r}|p_j| \leq 1$ *where* $1 \leq j \leq r$. *If* $Re\left\{\frac{zf_j''(z)}{f_j'(z)}\right\} \leq 1$, *then* $F_{p_1,\dots,p_r}(z)$ *is defined in [2] is univalent.*

Corollary 2.3 *Putting* $\delta = \alpha = c = n = 1$, $\lambda = \mu = 0$, $a = p+1$ *and* $g_j(z) = \frac{z}{1-z}$ *for all* $1 \leq j \leq r$, *If* $Re\left\{\frac{zf_j''(z)}{f_j'(z)}\right\} \leq 0$, *then* $F_{s_1\dots s_r}(z)$ *is defined in [7] is univalent, where* $\mid p_1\mid + \cdots + \mid p_r\mid \leq 1$ *and* $z \in \Delta$.

Corollary 2.4 *If*

$$Re\left\{\frac{z(f_j * g)''(z)}{(f_j * g)'(z)}\right\} \leq 2 \quad z \in \Delta,$$

then $I_g(f_1, \dots f_r)(z)$ *is defined in [4] is univalent, where* $\sum_{j=1}^{r}|p_j| \leq 1$ *and* $1 \leq j \leq r$.

Theorem 2.5 *Assume that* $\sum_{j=1}^{r}|p_j| \leq 1$ *and* $f_j,\ g_j \in \mathcal{A}(p)$*. If*

$$\left|\frac{[\mathcal{I}(f_j(z)) * g_j(z)]^{(n+1)}}{[\mathcal{I}(f_j(z)) * g_j(z)]^{(n)}}\right| \leq M - 1/\rho, \quad |z| = \rho < 1, M \approx 3,05..., \tag{8}$$

then $\mathcal{J}_g(f)(z)$ *given by (3) is univalent where* $p_j \in C$, $1 \leq j \leq r$ *and* $|\delta| \leq 1$.

Proof. From equation (6) we have

$$\frac{(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'} = p_1\left[\frac{\delta}{z} + \frac{(\mathcal{I}(f_1(z)) * g_1(z))^{(n+1)}}{(\mathcal{I}(f_1(z)) * g_1(z))^{(n)}}\right] + \cdots$$

$$+p_r\left[\frac{\delta}{z} + \frac{(\mathcal{I}(f_r(z)) * g_r(z))^{(n+1)}}{(\mathcal{I}(f_r(z)) * g_r(z))^{(n)}}\right],$$

which applying the inequality (8) implies that

$$\left|\frac{(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'}\right| \leq M.$$

Using Lemma 1.6, the last inequality implies that the integral operator $\mathcal{J}_g(f)(z)$ is univalent.

Corollary 2.6 *Let* $\sum_{j=1}^{r}|p_j| \leq 1$ *and* $f_j \in \mathcal{A}(p)$ *where* $1 \leq j \leq r$*. If*

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M - 1/\rho \quad z \in \Delta, M \approx 3,05..., \tag{9}$$

then $F_{p_1,\ldots,p_r}(z)$ *is defined in [2] is univalent.*

Remark 2.7 *The least upper bound which obtained by Breaz et al. (see [2]) for* $\left|\frac{f_j''(z)}{f_j'(z)}\right|$ *is* M *until the operator* $F_{p_1,\ldots,p_r}(z)$ *be univalent. But in Corollary 2.6 we obtained the upper bound* $M - 1/\rho$*, therefore the* $M - 1/\rho$ *is best. In particular if* $\rho \to 1^-$ *the* $M - 1/\rho \to 2,05...$ *.*

Theorem 2.8 *Let* $f_j \in \mathcal{SK}(\delta, \theta, \gamma)$, $1 \leq j \leq r$, $p_1, ...p_r$ *be real number with the properties,* $p_j > 0$ *and* $0 \leq \sum_{j=1}^{r} p_j\gamma_j + \delta < 1$*, then the integral operator* $\mathcal{J}_g(f)(z) \in \mathcal{SK}(\delta, \theta, \gamma)$*, where* $\gamma = \sum_{j=1}^{r} p_j\gamma_j + \delta$ *and* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Proof. Using equation (7), we obtain

$$\frac{z(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'} = \sum_{j=1}^{r} p_j\left[\delta + \frac{z(\mathcal{I}(f_j(z)) * g_j(z))^{(n+1)}}{(\mathcal{I}(f_j(z)) * g_j(z))^{(n)}}\right]. \tag{10}$$

The above relation is equivalent to

$$\delta + \frac{z(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'} = \sum_{j=1}^{r} p_j \left[\delta + \frac{z(\mathcal{I}(f_j(z)) * g_j(z))^{(n+1)}}{(\mathcal{I}(f_j(z)) * g_j(z))^{(n)}} \right] + \delta. \tag{11}$$

By multiplying the above relation by $e^{i\theta}$, we get

$$Re\left\{ e^{i\theta} \left(\delta + \frac{z(\mathcal{J}_g(f)(z))''}{(\mathcal{J}_g(f)(z))'} \right) \right\}$$

$$\begin{aligned} &= \sum_{j=1}^{r} p_j Re\left\{ e^{i\theta} \left(\delta + \frac{z(\mathcal{I}(f_j(z)) * g_j(z))^{(n+1)}}{(\mathcal{I}(f_j(z)) * g_j(z))^{(n)}} \right) \right\} + \delta Re\left\{ e^{i\theta} \right\} \\ &> \sum_{j=1}^{r} p_j \gamma_j \cos\theta + \delta \cos\theta = \left(\sum_{j=1}^{r} p_j \gamma_j + \delta \right) \cos\theta. \end{aligned}$$

Sines by hypothesis $0 \leq \sum_{j=1}^{r} p_j \gamma_j + \delta < 1$, we obtain $\mathcal{J}_g(f)(z)) \in \mathcal{SK}(\delta, \theta, \gamma)$, where $\gamma = \sum_{j=1}^{r} p_j \gamma_j + \delta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

References

[1] J. Becker, Löewnersche differential gleichung und quasi-konform fortsetzbare-schlichte Funktionen, *J. Reine Angew. Math.*, 255(1972), 23-43.

[2] D. Breaz, S. Owa and N. Breaz, A new integral univalent operator, *Acta Univ. Apulensis*, 16(2008), 11-16.

[3] S. Bulut, Some properties for an integral operator defined by Al-Oboudi differential operator, *J. Inequal. Pure and Appl. Math.*, 9(4) (Art. 115) (2008), 5.

[4] L. Dileep and S. Latha, A note on generalized integral operator, *Acta Universitatis Apulensis*, 25(2011), 113-117.

[5] R. Kargar, Some properties of an integral operators defined by a new linear operator, *Acta Univ. Apulensis*, 38(2014), 229-237.

[6] S.N. Kudryashov, On some criteria for the univalence of analytic functions, *Mathematical Notes*, 13(3) (March) (1973), 219-223.

[7] S. Ramadan and M. Darus, Univalence of an integral operator defined by generalized operators, *International Journal of Computational and Mathematical Sciences*, 48(2010), 392-394.