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$\delta\hat{g}$-Closed Sets in Topological Spaces

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Abstract

In this paper a new class of sets, namely $\delta\hat{g}$-closed sets is introduced in topological spaces. We prove that this class lies between the class of δ-closed sets and the class of δg-closed sets. Also we find some basic properties and applications of $\delta\hat{g}$-closed sets. We also introduce and study a new class of space namely $\hat{T}_{3/4}$-space.

Keywords: *generalized closed sets , δg-closed sets, δ-closure, $\hat{g}$-open sets and $\hat{T}_{3/4}$-space.*

AMS **subject classification :** 54C55.

1 Introduction

Levine [4], Mashhour et al.[8], Njastad[10] and Velicko[13] introduced semi - open sets, pre-open sets, α-open sets and δ-closed sets respectively.Levine[5] introduced generalized closed (briefly g-closed) sets and studied their basic properties.Bhattacharya and Lahiri[2], Arya and Nour[1], Maki et a [6,7], Dontchev and Ganster[3] introduced semi-generalized closed (briefly sg-closed) sets, generalized semi-closed (briefly gs-closed) sets, generalized α-closed (briefly gα-closed) sets, α-generalized closed (briefly αg-closed) sets and δ-generalized closed (briefly δg-closed) sets respectively. Veera Kumar [12] introduced $\hat{g}$-closed sets in topological spaces.The purpose of this present paper is to define a new class of closed sets called $\delta\hat{g}$-closed sets and also we obtain some basic properties of $\delta\hat{g}$-closed sets in topological spaces. Applying these sets, we obtain a new space which is called $\hat{T}_{3/4}$-space.

2 Preliminaries

Throughout this paper (X,τ) (or simply X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively. Let us recall the following definitions, which are useful in the sequel.

Definition 2.1 *A subset A of a space (X,τ) is called a*

(i) semi-open set [4] if A $\subseteq$ cl(int(A)).

(ii) pre-open set [8] if A$\subseteq$ int(cl(A)).

(iii) α-open set [10] if A$\subseteq$ int(cl(int(A))).

(iv) regular open set [11] if A= int(cl(A)).

The complement of a semi-open(resp.pre-open,α-open,regular open)set is called semi-closed (resp. semi-closed,α-closed, regular closed).

Definition 2.2 *The δ-interior[13] of a subset A of X is the union of all regular open set of X contained in A and is denoted by $Int_\delta(A)$. The subset A is called δ-open[13] if A =$Int_\delta(A)$, i.e. a set is δ-open if it is the union of regular open sets. the complement of a δ-open is called δ-closed. Alternatively, a set A$\subseteq$(X,τ) is called δ-closed [13] if A= $cl_\delta(A)$, where $cl_\delta(A)$= $\{$ x$\in$X: int(cl(U)) $\cap$ A $\neq\phi$, U $\in\tau$ and x $\in$ U$\}$.*

Definition 2.3 *A subset A of (X,τ) is called*

(i) generalized closed (briefly g-closed) set[5] if cl(A)$\subseteq$U whenever A $\subseteq$U and U is open set in (X,τ).

(ii) semi-generalized closed (briefly sg-closed) set [2] if scl(A)$\subseteq$U whenever A $\subseteq$U and U is a semi-open set in (X,τ).

(iii) generalized semi-closed (briefly gs-closed) set [1] if scl(A)$\subseteq$U whenever A $\subseteq$U and U is open set in (X,τ).

(iv) α- generalized closed (briefly αg-closed) set [7] if αcl(A)$\subseteq$U whenever A $\subseteq$U and U is open set in (X,τ).

(v) generalizedα- closed (briefly gα-closed) set [6] if αcl(A)$\subseteq$U whenever A $\subseteq$U and U is α-open set in (X,τ).

(vi) δ-generalized closed (briefly δg-closed) set [3] if $cl_\delta(A)$$\subseteq$U whenever A $\subseteq$U and U is open set in (X,τ).

(vii) $\hat{g}$-closed set [12] if cl(A)$\subseteq$U whenever A $\subseteq$U and U is a semi-open set in (X,τ).

(viii) α-$\hat{g}$-closed (briefly $\alpha\hat{g}$-closed) set [9] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a $\hat{g}$- open set in (X,τ).

The complement of a g-closed (resp. sg-closed, gs-closed, αg-closed, $g\alpha$-closed, δg-closed and $\hat{g}$-closed and $\alpha\hat{g}$-closed)) set is called g-open (resp. sg-open, gs-open, αg-open, $g\alpha$-open, δg-open, $\hat{g}$-open and $\alpha\hat{g}$-open).

Theorem 2.4 *Every open set is $\hat{g}$-open.*

Proof: Let A be an open set in X. Then A^c is closed. Therefore, $Cl(A^c) = A^c \subseteq X$ whenever $A^c \subseteq X$ and X is semi-open. This implies A^c is $\hat{g}$-closed. Hence A is $\hat{g}$-open.

Definition 2.5 *A space (X,τ) is called a*

(i) $T_{1/2}$-space [5] if every g-closed set in it is closed.

(ii) $T_{3/4}$-space [3] if every δg-closed set in it is δ-closed.

(iii) $T_{\alpha\hat{g}}$-space [9] if every $\alpha\hat{g}$-closed set in it is α-closed.

3 $\delta\hat{g}$-Closed Sets

We introduce the following definition.

Definition 3.1 *A subset A of a space (X,τ) is called $\delta\hat{g}$-closed if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is a $\hat{g}$- open set in (X,τ).*

Proposition 3.2 *Every δ-closed set is $\delta\hat{g}$-closed set.*

Proof: Let A be an δ-closed set and U be any $\hat{g}$- open set containing A. Since A is δ-closed, $cl_\delta(A) = A$ for every subset A of X. Therefore $cl_\delta(A) \subseteq U$ and hence A is $\delta\hat{g}$-closed set.

Remark 3.3 *The converse of the above theorem is not true as shown in the following example.*

Example 3.4 *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$*
δ-closed $=\{\phi, X, \{b\}, \{a,c\}\}$; $\delta\hat{g}$-closed $=\{\phi, X, \{b\}, \{c\}, \{a,c\}, \{b,c\}\}$
Here $\{b,c\}$ is $\delta\hat{g}$-closed but not δ-closed in (X,τ).

Proposition 3.5 *Every $\delta\hat{g}$-closed set is g-closed.*

Proof: Let A be an $\delta\hat{g}$-closed set and U be an any open set containing A in (X,τ). Since every open set is $\hat{g}$-open and A is $\delta\hat{g}$-closed, $cl_\delta(A) \subseteq U$ for every subset A of X. Since $cl(A) \subseteq cl_\delta(A) \subseteq U$, $cl(A) \subseteq U$ and hence A is g-closed.

Remark 3.6 *An g-closed set need not be $\delta\hat{g}$-closed set as shown in the following example.*

Example 3.7 *Let X =$\{a,\ b,\ c\}$ with topology $\tau = \{\phi, X, \{b\}, \{a, c\}\}$. Then the set $\{a\}$ is g-closed but not $\delta\hat{g}$-closed in (X,τ).*

Proposition 3.8 *Every $\delta\hat{g}$-closed set is gs-closed.*

proof: Let A be an $\delta\hat{g}$-closed and U be any open set containing A in (X,τ). Since every open set is $\hat{g}$-open, cl_δ(A)$\subseteq$U for every subset A of X. Since scl(A)$\subseteq$ $\mathrm{cl}_\delta(A)$$\subseteq$U, scl(A)$\subseteq$U and hence A is gs-closed.

Remark 3.9 *A gs-closed set need not be $\delta\hat{g}$-closed as shown in the following example.*

Example 3.10 *Let X =$\{a,\ b,\ c\}$ with topology $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then the set $\{c\}$ is gs-closed but not $\delta\hat{g}$-closed in (X,τ).*

Proposition 3.11 *Every $\delta\hat{g}$-closed set is αg-closed.*

proof: It is true that αcl(A)$\subseteq$ $\mathrm{cl}_\delta(A)$ for every subset A of X.

Remark 3.12 *A αg-closed set need not be $\delta\hat{g}$-closed as shown in the following example.*

Example 3.13 *Let X =$\{a,\ b,\ c\}$ with topology $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then the set $\{b\}$ is αg-closed but not $\delta\hat{g}$-closed in (X,τ)*

Proposition 3.14 *Every $\delta\hat{g}$-closed set is δg-closed.*

proof: Let A be an $\delta\hat{g}$-closed set and U be any open set containing A.Since every open set is $\hat{g}$- open, $\mathrm{cl}_\delta(A)$$\subseteq$U, whenever A$\subseteq$U and U is $\hat{g}$- open. Therefore $\mathrm{cl}_\delta(A)$$\subseteq$U and U is open. Hence A is δg-closed.

Remark 3.15 *A δg-closed set need not be $\delta\hat{g}$-closed as shown in the following example.*

Example 3.16 *Let X =$\{a,\ b,\ c\}$ with topology $\tau = \{\phi, X, \{c\}, \{a, b\}\}$. Then the set $\{a\}$ is δg-closed but not $\delta\hat{g}$-closed in (X,τ).*

Remark 3.17 *The class of $\delta\hat{g}$-closed sets is properly placed between the classes of δ-closed and δg-closed sets.*

Proposition 3.18 *Every $\delta\hat{g}$-closed set is $\alpha\hat{g}$-closed.*

proof: It is true that $\alpha cl(A)$$\subseteq$$\mathrm{cl}_\delta(A)$ for every subset A of (X,τ).

Remark 3.19 *A $\alpha\hat{g}$-closed set need not be $\delta\hat{g}$-closed as shown in the following example.*

Example 3.20 *Let X ={a, b, c} with topology* $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$. *Then the set {a} is $\alpha\hat{g}$-closed but not $\delta\hat{g}$-closed in (X,τ).*

Remark 3.21 *The following examples show that $\delta\hat{g}$-closeness is independent from $\hat{g}$-closeness, sg-closeness, $g\alpha$-closeness and α-closeness.*

Example 3.22 *Let X ={a, b, c} with topology* $\tau = \{\phi, X, \{a\}\}$. *Then the set {a,b} is $\delta\hat{g}$-closed but neither $\hat{g}$-closed nor sg-closed and the set {a,c} is $\delta\hat{g}$-closed but neither $g\alpha$-closed nor α-closed.*

Also the another example Let X ={a, b, c} with topology $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. *Then the set {c} is $\hat{g}$-closed,sg-closed and $g\alpha$-closed but not $\delta\hat{g}$-closed.*

Example 3.23 *Let X ={a, b, c} with topology* $\tau = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$. *Then the set {a} is α-closed but not $\delta\hat{g}$-closed in (X,τ).*

Remark 3.24 *The following diagram shows the relationships of $\delta\hat{g}$-closed sets with other known existing sets.A→B represents A implies B but not conversely.*

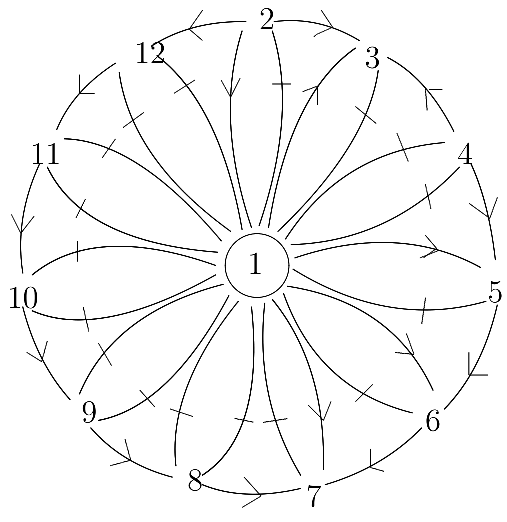

Fig. 1

1. $\delta\hat{g}$-Closed 2.δ-Closed 3.δg-Closed 4. $\hat{g}$-closed 5.g-closed 6.αg-closed 7.gs-closed 8.sg-closed 9.$g\alpha$-closed 10.$\alpha\hat{g}$-closed 11.α-closed 12.closed.

4 Characterisation

Theorem 4.1 *The finite union of $\delta\hat{g}$-Closed sets is $\delta\hat{g}$-Closed.*

proof: Let $\{A_i/i = 1, 2, ...n\}$ be a finite class of $\delta\hat{g}$-Closed subsets of a space (X,τ). Then for each $\hat{g}$-open set U_i in X containing A_i, $cl_\delta(A_i) \subseteq U_i$ i$\in$\{1,2,...n\}. Hence $\bigcup_i A_i \subseteq \bigcup_i U_i$=V. Since arbitrary union of $\hat{g}$-open sets in (X,τ) is also $\hat{g}$-open set in(X,τ), V is $\hat{g}$-open in (X,τ). Also $\bigcup_i cl_\delta(A_i)$=$cl_\delta(\bigcup_i A_i) \subseteq$V. Therefore $\bigcup_i A_i$ is $\delta\hat{g}$-Closed in (X,τ).

Remark 4.2 *Intersection of any two $\delta\hat{g}$-Closed sets in (X,τ) need not be $\delta\hat{g}$-Closed since, in Example 3.22, \{a,b\} and \{a,c\} are $\delta\hat{g}$-Closed sets but their intersection \{a\} is not $\delta\hat{g}$-Closed.*

Proposition 4.3 *Let A be a $\delta\hat{g}$-Closed set of (X,τ). Then $cl_\delta(A)$-A does not contain a non-empty $\hat{g}$-closed set.*

proof: Suppose that A is $\delta\hat{g}$-Closed, let F be a $\hat{g}$-closed set contained in $cl_\delta(A)$-A. Now F^c is $\hat{g}$-open set of (X,τ) such that $A \subseteq F^c$. Since A is $\delta\hat{g}$-Closed set of (X,τ), then $cl_\delta(A) \subseteq F^c$. Thus $F \subseteq (cl_\delta((A))^c$. Also $F \subseteq cl_\delta(A)$-A. Therefore $F \subseteq (cl_\delta(A))^c \cap (cl_\delta(A)) = \phi$. Hence $F=\phi$.

Proposition 4.4 *If A is $\hat{g}$-open and $\delta\hat{g}$-Closed subset of (X,τ) then A is an δ-closed subset of (X,τ).*

proof: Since A is $\hat{g}$-open and $\delta\hat{g}$-Closed, $cl_\delta(A) \subseteq$A. Hence A is δ-closed.

Theorem 4.5 *The intersection of a $\delta\hat{g}$-Closed set and a δ-closed set is always $\delta\hat{g}$-Closed.*

proof: Let A be $\delta\hat{g}$-Closed and let F be δ-closed. If U is an $\hat{g}$-open set with $A \cap F \subseteq U$, then $A \subseteq U \cup F^c$ and so $cl_\delta(A) \subseteq U \cup F^c$.
Now $cl_\delta(A \cap F) \subseteq cl_\delta(A) \cap F \subseteq U$.Hence A$\cap$F is $\delta\hat{g}$-Closed.

Theorem 4.6 *In a $T_{3/4}$-space every $\delta\hat{g}$-Closed set is δ-closed.*

proof: Let X be $T_{3/4}$-space. Let A be $\delta\hat{g}$-Closed set of X. We know that every $\delta\hat{g}$-Closed set is δg-closed. Since X is $T_{3/4}$-space, A is δ-closed.

Proposition 4.7 *If A is a $\delta\hat{g}$-Closed set in a space (X,τ) and $A \subseteq B \subseteq cl_\delta(A)$, then B is also a $\delta\hat{g}$-Closed set.*

proof: Let U be a $\hat{g}$-open set of (X,τ) such that $B \subseteq U$. Then $A \subseteq U$. Since A is $\delta\hat{g}$-Closed set, $cl_\delta(A) \subseteq U$. Also since $B \subseteq cl_\delta(A)$, $cl_\delta(B) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A)$. Hence $cl_\delta(B) \subseteq U$. Therefore B is also a $\delta\hat{g}$-Closed set.

Theorem 4.8 *Let A be $\delta\hat{g}$-Closed of (X,τ). Then A is δ-closed iff $cl_\delta(A)$-A is $\hat{g}$-closed.*

proof: Necessity. Let A be a δ-closed subset of X.Then $cl_\delta(A)$=A and so $cl_\delta(A)$-A=ϕ which is $\hat{g}$-closed.
Sufficiency. Since A is $\delta\hat{g}$-Closed, by proposition 4.4, $cl_\delta(A)$-A does not contain a non-empty $\hat{g}$-closed set. But $cl_\delta(A)$-A=ϕ. That is $cl_\delta(A)$=A. Hence A is δ-closed.

5 Applications

We introduce the following definition.

Definition 5.1 *A space (X,τ) is called $\hat{T}_{3/4}$-space if every $\delta\hat{g}$-Closed set in it is an δ-closed.*

Theorem 5.2 *For a topological space (X,τ), the following conditions are equivalent.*

(i) (X,τ) is a $\hat{T}_{3/4}$-space.

(ii) Every singleton $\{x\}$ is either $\hat{g}$-closed or δ-open.

proof: (i)$\Rightarrow$ *(ii)* Let x$\in$X. Suppose {x} is not a $\hat{g}$-closed set of (X,τ). Then X-{x} is not a $\hat{g}$-open set. Thus X-{x} is an $\delta\hat{g}$-Closed set of (X,τ). Since (X,τ) is $\hat{T}_{3/4}$-space, X-{x} is an δ-closed set of (X,τ), i.e. {x} is δ-open set of (X,τ).
(ii)$\Rightarrow$ *(i)* Let A be an $\delta\hat{g}$-Closed set of (X,τ). Let x$\in cl_\delta(A)$. By (ii), {x} is either $\hat{g}$-closed or δ-open.
Case(i). Let {x} be $\hat{g}$-closed. If we assume that x$\notin$A, then we would have x$\in cl_\delta(A)$-A, which cannot happen according to proposition 4.4. Hence x$\in$A.
Case(ii) Let {x} be δ-open. Since x$\in cl_\delta(A)$, then {x}$\cap$A$\neq\phi$. This shows that x$\in$A.
So in both cases we have $cl_\delta(A)\subseteq$A. Trivially A$\subseteq cl_\delta(A)$. Therefore A=$cl_\delta(A)$ or equivalently A is δ-closed. Hence (X,τ) is a $\hat{T}_{3/4}$-space.

Theorem 5.3 *Every $T_{3/4}$-space is a $\hat{T}_{3/4}$-space.*

proof: The proof is straight forward since every $\delta\hat{g}$-Closed set is δg-closed set.

Remark 5.4 *The converse of the above theorem is not true as it can be seen from the following example.*

Example 5.5 *Let X ={a, b, c}and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$.(X,$\tau$) is a $\hat{T}_{3/4}$-space but not a $T_{3/4}$-space.*

Theorem 5.6 *Every $\hat{T}_{3/4}$-space is a $T_{\alpha\hat{g}}$-space.*

proof: Let (X,τ) be a $\hat{T}_{3/4}$-space, then every singleton is either $\hat{g}$-closed or δ-open. Since every δ-open is α-open, then every singleton is either $\hat{g}$-closed or α-open. Hence (X,τ) is a $T_{\alpha\hat{g}}$-space.

Remark 5.7 *The following example supports that the converse of the above theorem is not true.*

Example 5.8 *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. (X,$\tau$) is a $T_{\alpha\hat{g}}$-space but not a $\hat{T}_{3/4}$-space.*

Remark 5.9 *$\hat{T}_{3/4}$-space and $T_{1/2}$-space are independent of one another as the following examples show.*

Example 5.10 *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. (X,$\tau$) is a $\hat{T}_{3/4}$-space but is not a $T_{1/2}$-space.*

Example 5.11 *Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$. (X,$\tau$) is a $T_{1/2}$-space but not a $\hat{T}_{3/4}$-space.*

Remark 5.12 *The following diagram shows the relationships $\hat{T}_{3/4}$-space with other known existing spaces. A→B represents A implies B but not conversely*

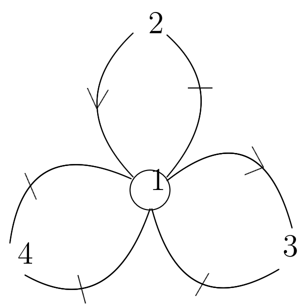

Fig. 2

1. $\hat{T}_{3/4}$-space 2.$T_{3/4}$-space 3.$T_{\alpha\hat{g}}$-space 4.$T_{1/2}$-space

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