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Covering Cover Pebbling Number for Even Cycle Lollipop

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Abstract

In a graph G with a distribution of pebbles on its vertices, a pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. The covering cover pebbling number, denoted by σ (G), of a graph G, is the smallest number of pebbles, such that, however the pebbles are initially placed on the vertices of the graph, after a sequence pebbling moves, the set of vertices with pebbles forms a covering of G. In this paper we determine the covering cover pebbling number for cycles and even cycle lollipops.

Keywords: *Graph, pebbling, covering, lollipop graph.*

1 Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lagarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of pebbling results[3]. Given a connected graph, distribute certain number of pebbles on its vertices in some configuration. Precisely, a configuration on a graph G is a function from V (G) to N $\cup$ {0} representing a placement of pebbles on G. The size of the configuration is the total number of pebbles placed on the vertices. Support vertices of a configuration C are those on which there is at least one pebble of C. In any configuration, if all the pebbles are placed on a single vertex, it is called a simple configuration. A pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. In (regular) pebbling, the target vertex is selected and the aim is to move a pebble to the target vertex. The minimum number of pebbles, such that regardless of the target vertex, we can pebble that target vertex is called the pebbling number of G. In cover pebbling the aim is to cover all the vertices with pebbles, That is, to move a pebble every vertex of the graph simultaneously. The minimum number of pebbles required such that, regardless of their initial placement on G, there is a sequence of pebbling moves, at the end of which, every vertex has at least one pebble on it, is called the cover pebbling number of G. In [2], the cover pebbling number for complete graphs, paths and trees are determined. The covering cover pebbling number, denoted by σ (G), of a graph G, is the smallest number of pebbles, such that, however the pebbles are initially placed on the vertices of the graph, after a sequence pebbling moves, the set of vertices with pebbles forms a covering of G. The concept of covering cover pebbling number was introduced by A.Lourdusamy and A.Punitha Tharani in [5] and they determined the covering cover pebbling for complete graphs, paths, wheel, star graph,complete r-partite graph and binary trees.

In this paper we determine the covering cover pebbling number for cycles in Section 2. With regard to the covering cover pebbling number of cycles, we find the following theorem in [5].

Theorem 1.1[5] Let P_n be a path on n vertices with V = V(P_n) = {v_1, v_2, v_3,…, v_{n-1}, v_n} and E=E(P_n)={v_1v_2, v_2v_3, v_3v_4,…, $v_{n-1}v_n$}. Then $\sigma(P_n) = \left\lfloor \frac{2^n - 1}{3} \right\rfloor$

In Section 3, we then present the covering cover pebbling number for even cycle lollipop.

2 Covering Cover Pebbling Number for Cycles

We begin by proving that placing all the pebbles on one vertex is a “worst case” configuration that determines the covering cover pebbling number of cycles.

Lemma 2.1 *The value of σ (C_m) is attained when the original configuration consists of placing all the pebbles on a single vertex, where $C_m : v_0v_1v_2 \ldots v_{m-1}v_0$ is a cycle on ‘m’ vertices.*

Proof. Assume first that a worst configuration consists of more than one set of consecutively pebbled vertices (“islands”). The maximum number of vertices in each island is at most two. Suppose if any one of the islands consists of three or more pebbled vertices, one could rearrange all the pebbles to the inner one or two vertices of the same island, thereby causing a larger number of pebbles to be needed to cover the edges-a contradiction. Thus, each “island” consists of at most two vertices.

Now, consider the effect of relocating all the pebbles onto a single island. Once, again, we get a contradiction to the fact that there could be more than one island, because we need more pebbles to cover the edges of the graph, after relocating all the pebbles onto a single island. So, our assumption (that a worst configuration consist of more than one island) is wrong.

Next, assume that, the island consists of exactly two vertices. Clearly, a worst initial configuration of pebbles is obtained by placing $\sigma - 1$ pebbles on one vertex, say v_1 and placing one pebble at an adjacent vertex of v_1, say v_2 , since, we need more pebbles to cover the edges of the cycle. Note that, after the distribution of $(\sigma - 1, 1)$ pebbles to $\{v_1, v_2\}$ respectively, we cover all the edges of the cycle. But if we put all the pebbles on v_1, we cannot cover at least one edge of the cycle. Hence the result follows.

Since placing all the pebbles on a single vertex is a worst case, we now determine the value of σ (C_m).

Theorem 2.2 *Let $C_m : v_0v_1v_2 \ldots v_{m-1}v_0$ be a cycle on ‘m’ vertices. Then*

$$\sigma(C_m) = \begin{cases} \left\lceil \dfrac{2^{k+2}-5}{3} \right\rceil, & if\ m=2k\,(k \geq 2) \\ 2^k - 1, & if\ m=2k-1 (k \geq 2) \end{cases}$$

Proof. By Lemma 2.1, we assume all $\sigma(C_m)$ pebbles are on $v_0 \in C_m$. If m = 2k (k ≥ 2), consider the paths P_A and P_B where $P_A = v_0v_1 \ldots v_{k-1}v_k$ and $P_B = v_0v_{2k-1}v_{2k-2} \ldots v_k$. We can cover the edges of the paths P_A and P_B, using $2\sigma(P_{k+1})$

pebbles, since $l(P_A) = l(P_B) = k$ (Figure C_{2k}) and $\sigma(P_n) = \left\lfloor \frac{1}{3}(2^n - 1) \right\rfloor$, where P_n denotes a path on n vertices. Note that, v_0 may be pebbled twice. This happens only when k is odd. Since if k is odd, then the path P_{k+1} is of odd length. That is, both P_A and P_B are of odd length implies v_0 is pebbled twice.

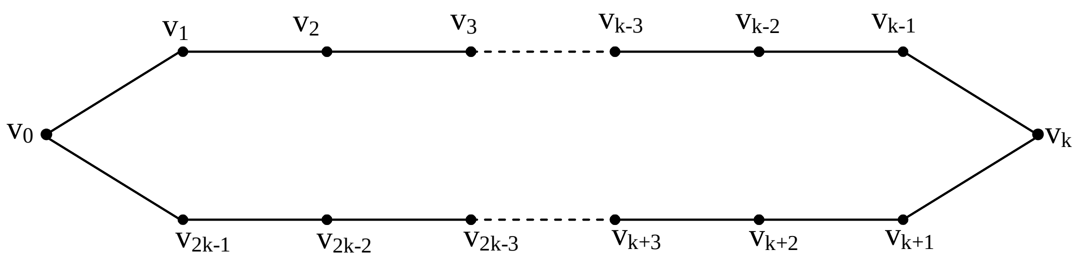
Thus,

$$\sigma(C_{2k}) = \begin{cases} 2\sigma(P_{k+1}), & if\ \ k\ \ is\ \ even \\ 2\sigma(P_{k+1}) - 1, & if\ \ k\ is\ odd \end{cases}$$

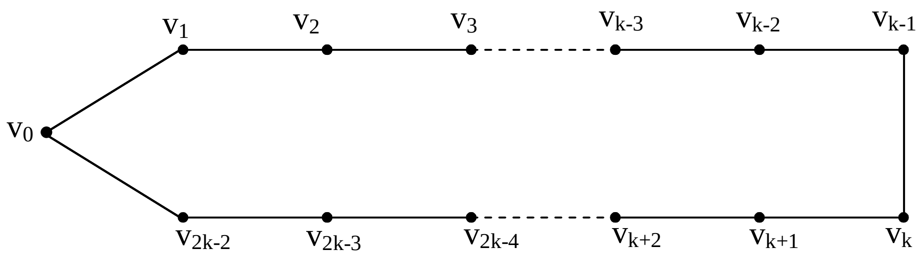
$$= \begin{cases} 2\left(\dfrac{2^{k+1} - 2}{3}\right), & if\ \ k\ \ is\ \ even \\ 2\left(\dfrac{2^{k+1} - 1}{3}\right) - 1, & if\ \ k\ is\ odd \end{cases}$$

$$= \begin{cases} \dfrac{2^{k+2} - 4}{3}, & if\ \ k\ \ is\ \ even \\ \dfrac{2^{k+2} - 5}{3}, & if\ \ k\ is\ odd \end{cases}$$

Therefore, $\sigma(C_{2k}) = \left\lceil \dfrac{2^{k+2} - 5}{3} \right\rceil$



The Cycle C_{2k} (k ≥ 2)



The Cycle C_{2k-1} (k ≥ 2)

Now, consider the case when m = 2k – 1 (k ≥ 2). Also consider the paths P_A and P_B where $P_A = v_0v_1 \ldots v_{k-1}v_k$ and $P_B = v_kv_{k+1} \ldots v_{2k-2}v_0$. We can cover the edges of the paths P_A and P_B using $\sigma(P_{k+1})$ and $\sigma(P_k)$ pebbles respectively, since

$l(P_A) = k$ and
$l(P_B) = k - 1$(Figure C_{2k-1}).

It is easy to see that we do not pebble v_0 twice. Therefore,

$$\sigma(C_{2k-1}) = \sigma(P_{k+1}) + \sigma(P_k)$$

$$= \begin{cases} \dfrac{2^{k+1}-2}{3} + \dfrac{2^k-1}{3}, & if\ \ k\ \ is\ \ even \\ \dfrac{2^{k+1}-1}{3} + \dfrac{2^k-2}{3}, & if\ \ k\ is\ odd \end{cases}$$

$$= \begin{cases} \dfrac{2^{k+1}+2^k-3}{3}, & if\ \ k\ \ is\ \ even \\ \dfrac{2^{k+1}+2^k-3}{3}, & if\ \ k\ is\ odd \end{cases}$$

$$= \frac{3(2^k-1)}{3}, \ k \geq 2$$

Thus, $\sigma(C_{2k-1}) = 2^k - 1$.

Therefore,

$$\sigma(C_m) = \begin{cases} \left\lceil \dfrac{2^{k+2}-5}{3} \right\rceil, \ if \ m = 2k \ (k \geq 2) \\ 2^k - 1, \ if \ m = 2k-1 (k \geq 2) \end{cases}$$

3 Covering Cover Pebbling Number for Even Cycle Lollipops

We proceed to determine the covering cover pebbling number for a class of unicyclic graphs, called a class of even cycle lollipop graphs.

Definition 3.1 *[4] For a pair of integers m ≥ 3 and n ≥ 2, let L(m, n) be the Lollipop graph of order m + n - 1 obtained from a cycle* C_m *by attaching a path of length n – 1 to a vertex of the cycle.*

If the cycle C_m *in L(m,n) is even, then we call L(m,n) an even cycle lollipop. We will use the following labeling for the graphs* C_m *and* P_n*:*

Let $\mathbf{C_m} = \mathbf{v_0 v_1 v_2 \ldots v_{m-1} v_0}$ and $\mathbf{P_n} = v_0 v_{p_1} v_{p_2} \ldots v_{p_{n-1}}$ be the cycle and the path available in L(m,n).

Theorem 3.2 *Let L(m, 2) be a lollipop graph, where m = 2k (k≥2).*

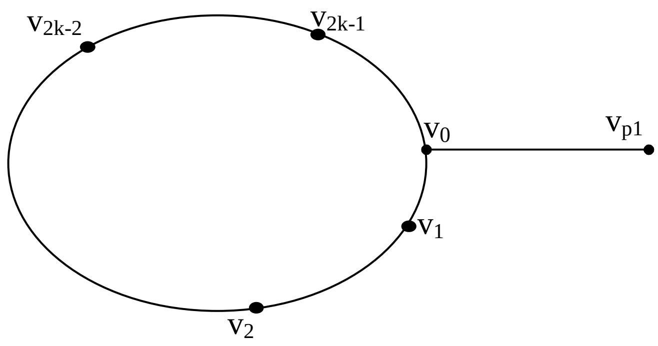
Then, σ(L(m, 2)) =

$$\begin{cases} 2\sigma(C_m)+1, \ if \ m = 2k \ with \ k \geq 2 \ is \ even \\ 2\sigma(C_m), \ otherwise \end{cases}$$

Proof. Consider the Lollipop graph L(m,2).

Case1 Let m = 2k, where k ≥2 is even

Consider the distribution of 2σ(C_m) pebbles on v_{p_1}. Clearly, we cannot cover at least one of the edges of L(m, 2). Thus, σ(L(m, 2)) ≥ 2σ(C_m) + 1.



Now, consider the distribution of $2\sigma(C_m) + 1$ pebbles on L(m, 2).

Case(1A) C_m contains at least $\sigma(C_m)$ pebbles.

If either v_{p_1} or v_0 contains a pebbles on it, then we are done (by our assumption). So, assume both v_0 and v_{p_1} have zero pebbles on it. Now, all the $2\sigma(C_m) + 1$ pebbles are on $V(C_m) - \{v_o\}$. We can cover the edges of C_m using at most $\sigma(C_m)$ pebbles. So, we have $\sigma(C_m) + 1$ remaining pebbles to pebble the vertex v_0. Since v_0 has no pebbles on it, both v_{2k-1} and v_1 contain at least one pebble each while using $\sigma(C_m) + 1$ remaining pebbles. But, if any one of the vertices of $\{v_{2k-1}, v_1\}$ contains two or more pebbles, then we are done. So, both v_{2k-1} and v_1 have exactly one pebble each. Now, consider the paths $P_A = v_2v_3 \ldots v_k$ and $P_B = v_kv_{k+1} \ldots v_{2k-2}$ which are of length $k-2$ each. Then, any one of the paths contains at

least $\left\lceil \frac{\sigma(C_m)-2}{2} \right\rceil = \sigma(P_{k+1}) - 1 =$

$\frac{2^{k+1}-2}{3} - 1 = \frac{2^{k+1}-5}{3} \geq 2^{k-1}$, where the first inequality follows

since $\sigma(C_m) = 2\sigma(P_{k+1})$ and the last inequality follows since $k \geq 4$ is even.

(Note that, if k = 2, v_2 is the only remaining vertex in the cycle C_4 and we are done).

Since, $l(P_A) = l(P_B) = k - 2$, we can move two pebbles to either v_{2k-2} or v_2 from the path P_B or P_A which contains at least 2^{k-1} pebbles. So, we can put a pebble at v_0 and we are done.

Case (1B) C_m contains $x < \sigma(C_m)$ pebbles.

There are at least $2\sigma(C_m) + 1 - x$ pebbles at v_{p_1}. We can move

$\left\lfloor \frac{2\sigma(C_m)-x}{2} \right\rfloor$ pebbles to v_0. That is,, from v_{p_1}, we can move at least

$\sigma(C_m)-\left\lfloor \frac{x}{2} \right\rfloor$ pebbles to v_0. Now we have at least $x+\sigma(C_m)-\left\lfloor \frac{x}{2} \right\rfloor$

pebbles in C_m and we are done. So, $\sigma(L(m, 2) \leq 2\sigma(C_m) + 1$.
Therefore, $\sigma(L(m, 2) = 2\sigma(C_m) + 1$ if $m=2k$ and k is even.

Case2 Let $m=2k$, where $k\geq 3$ is odd.
Consider the distribution of $2\sigma(C_m)-1$ pebbles on v_{p_1}. Clearly we cannot cover at least one of the edges of $L(m,2)$. So, $\sigma(L(m,2))\geq 2\sigma(C_m)$.

Let us now prove that $\sigma(L(m,2))\leq 2\sigma(C_m)$. Consider the distribution of $2\sigma(C_m)$ pebbles on $L(m,2)$.

Case (2A) C_m contains at least $\sigma(C_m)$ pebbles.
If either v_{p_1} or v_0 contains a pebble on it, then we are done (by our assumption). So, assume both v_0 and v_{p_1} have zero pebbles. Now, all the $2\sigma(C_m)$ pebbles are on $V(C_m) - \{v_o\}$. We can cover the edges of C_m using at most $\sigma(C_m)$ pebbles. So, we have $\sigma(C_m)$ remaining pebbles to pebble the vertex v_0. Since v_0 has zero pebbles on it, both v_{2k-1} and v_1 contain at least one pebble each while using $\sigma(C_m)$ remaining pebbles. But, if v_{2k-1} or v_1 contains two or more pebbles, then we are done. So, both v_{2k-1} and v_1 have exactly one pebble each. Now, consider the paths $P_A = v_2v_3 \ldots v_k$ and $P_B = v_kv_{k+1} \ldots v_{2k-2}$. Note that P_A and P_B are of length $k-2$ each. Then, any one of the paths contains at least

$\left\lceil \frac{\sigma(C_m)-2}{2} \right\rceil = \sigma(P_{k+1})-1=$

$\frac{2^{k+1}-1}{3}-1=\frac{2^{k+1}-4}{3}\geq 2^{k-1}$ pebbles, where the last inequality follows since $k\geq 3$ is odd.

Since, $l(P_A) = l(P_B) = k - 2$, we can move two pebbles to either v_{2k-2} or v_2 from the path P_B or P_A which contains at least 2^{k-1} pebbles. So, we can put a pebble at v_0 and we are done.

Case (2B) C_m contains $x < \sigma(C_m)$ pebbles.

There are at least $2\sigma(C_m) - x$ pebbles at v_{p_1}. From these pebbles, we can send $\left\lfloor \frac{2\sigma(C_m)-x-1}{2} \right\rfloor$ pebbles to v_0.

That is, $\left\lfloor \frac{2\sigma(C_m)-(x+1)}{2} \right\rfloor = \sigma(C_m) - \left\lfloor \frac{x+1}{2} \right\rfloor$

Now, C_m has at least $x + \sigma(C_m) - \left\lfloor \frac{x+1}{2} \right\rfloor \geq \sigma(C_m)$ pebbles and we are done.

Thus $\sigma(L(m, 2)) \leq 2\,\sigma(C_m)$.

Therefore, $\sigma(L(m, 2)) = 2\,\sigma(C_m)$, if $m=2k$ and k is odd.

Theorem 3.3 *Let L(m,n) be a Lollipop graph where m = 2k ≥4 and n ≥3. Then σ(L(m,n))=*

$$\begin{cases} 2^{n-1}\sigma(C_m)+\sigma(P_n), \text{ if } m=2k \text{ with } k\geq 2 \text{ is even} \\ 2^{n-1}\sigma(C_m)+\sigma(P_{n-1}), \text{ otherwise} \end{cases}$$

Proof. Consider the Lollipop graph $L(m,n)$ where $m = 2k \geq 4$ and $n \geq 3$.

Case1 Let $m = 2k$ with $k \geq 2$ is even and $n \geq 3$.
Consider the distribution of $2^{n-1}\sigma(C_m)+\sigma(P_n)-1$ pebbles on the vertex $v_{p_{n-1}}$. Clearly we cannot cover at least one of the edges of $L(m, n)$. Thus, $\sigma(L(m, n)) \geq 2^{n-1}\sigma(C_m) + \sigma(P_n)$.

Now, consider the distribution of $2^{n-1}\sigma(C_m)+\sigma(P_n)$ pebbles on L(m, n).

Case (1A) C_m contains $\sigma(C_m)$ or more pebbles.

If P_n contains $\sigma(P_n)$ pebbles then we are done. So assume that P_n contains less than $\sigma(P_n)$ pebbles. So, C_m contains at least $2^{n-1}\sigma(C_m) + \sigma(P_n) - (\sigma(P_n) - 1) = 2^{n-1}\sigma(C_m) + 1$ pebbles. From these pebbles we used at most $\sigma(C_m)$ pebbles to cover the edges of C_m. We have at least $(2^{n-1} - 1)\,\sigma\,(C_m) + 1$ pebbles in C_m to cover the edges of P_n. Now, consider the paths $P_A : v_0v_1 \ldots v_{k-2}v_{k-1}$ and $P_B : v_kv_{k+1} \ldots v_{2k-1}$. Now we see that either P_A or P_B contains

$$\left\lceil \frac{(2^{n-1}-1)\sigma(C_m)}{2} \right\rceil + 1 \text{pebbles.}$$

Then we claim that, $\left\lceil \frac{(2^{n-1}-1)\sigma(C_m)}{2} \right\rceil + 1 \geq 2^{k-1}\sigma(P_n)$

Suppose not, then.

$$\left\lceil \frac{(2^{n-1}-1)\sigma(C_m)}{2} \right\rceil + 1 < 2^{k-1}\sigma(P_n)$$

That is, $\left\lceil \frac{(2^{n}-2)\sigma(P_{k+1})}{2} \right\rceil + 1 < 2^{k-1}\sigma(P_n)$

That is, $\left\lceil \frac{(2^{n}-2)}{2}\left(\frac{2^{k+1}-2}{3}\right) \right\rceil + 1 < 2^{k-1}\sigma(P_n)$

That is, $\left\lceil \frac{(2^{n}-2)}{3}\left(\frac{2^{k+1}-2}{2}\right) \right\rceil + 1 < 2^{k-1}\sigma(P_n)$

That is, $\left\lceil \left(\frac{2^n-2}{3}\right)(2^k-1) \right\rceil + 1 < 2^{k-1}\left\lfloor \frac{2^n-1}{3} \right\rfloor$

which is a contradiction, since $k \geq 2$ and $n \geq 3$.

So, either P_A or P_B contains at least $2^{k-1}\sigma(P_n)$ pebbles. If P_A contains $2^{k-1}\sigma(P_n)$ pebbles then we are done (since $l(P_A) = k - 1$ and $v_0 \in P_A$). So, assume that P_B contains at least $2^{k-1}\sigma(P_n)$ pebbles. Also, note, if P_B contains $2^k\sigma(P_n)$ pebbles then we are done. So, assume P_B contains less than $2^k\sigma(P_n)$ pebbles. This implies that P_A contains at least $(2^{n-1} - 1)\sigma(C_m) + 1 - (2^k\sigma(P_n) - 1)$

$= (2^{n-1} - 1)\sigma(C_m) - 2^k\sigma(P_n) + 2$ pebbles

But,

$(2^{n-1} - 1)\sigma(C_m) - 2^k\sigma(P_n) + 2$

$= (2^{n-1} - 1)[2\sigma(P_{k+1})] - 2^k\sigma(P_n) + 2$ since $m = 2k$ and k is even

$$= \begin{cases} (2^n-2)\left(\dfrac{2^{k+1}-2}{3}\right) - 2^k\left(\dfrac{2^n-1}{3}\right) + 2 \text{ if } n \text{ is even} \\ (2^n-2)\left(\dfrac{2^{k+1}-2}{3}\right) - 2^k\left(\dfrac{2^n-2}{3}\right) + 2 \text{ if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \left(\dfrac{2^n-2}{3}\right)(2.2^k - 2 - 2^k) - \dfrac{2^k}{3} + 2\text{ , if } n \text{ is even} \\ \left(\dfrac{2^n-2}{3}\right)(2.2^k - 2 - 2^k) + 2\text{ , if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \left(\frac{2^n-2}{3}\right)(2^k-2) - \frac{2^k}{3} + 2, \textit{if } n \textit{ is even} \\ \left(\frac{2^n-2}{3}\right)(2^k-2) + 2, \textit{if } n \textit{ is odd} \end{cases}$$

$$\geq \left(\frac{2^n-2}{3}\right)(2^k-2) - \frac{2^k}{3}$$

So we can send at least $\left\lfloor \frac{\left(\frac{2^n-2}{3}\right)(2^k-2)}{2^{k-1}} - \frac{2^k}{3.2^{k-1}} \right\rfloor$ pebbles to v_0

That is, we send at least $\left\lfloor \left(\frac{2^n-2}{3}\right)\left(2 - \frac{2}{2^{k-1}}\right) - \frac{2}{3} \right\rfloor$

$\geq \left\lfloor \left(\frac{2^n-2}{3}\right) - \frac{2}{3} \right\rfloor$ pebbles to v_0 where the inequality follows since $k \geq 2$.

So, the minimum number of pebbles that we send to v_0 is

$$\left\lfloor \left(\frac{2^n-4}{3}\right) \right\rfloor = \left\lfloor 4\left(\frac{2^{n-2}-1}{3}\right) \right\rfloor$$

$$\geq 4\left\lfloor \frac{2^{n-2}-1}{3} \right\rfloor$$

$\geq 4\,\sigma(P_{n-2})$

$= 2\sigma(P_{n-2}) + 2\sigma(P_{n-2})$

$\geq 2\,\sigma(P_{n-2}) + 2$

$= \sigma(P_{n-1}) + 1$, where the last inequality follows since $\sigma(P_n) = 2\,\sigma(P_{n-1}) + 1$ or $2\sigma(P_{n-1})$

So, we send $\sigma(P_{n-1}) + 1$ pebbles to v_0 from P_A. Also we send $\left\lfloor \frac{\sigma(P_n)}{2} \right\rfloor \geq \sigma(P_{n-1})$ pebbles to v_0 from P_B. Thus we have $2\,\sigma(P_{n-1}) + 1$ pebbles at v_0 and we are done.

Case (1B) C_m contains $x < \sigma(C_m)$ pebbles.

There are at least $2^{n-1}\,\sigma(C_m) + \sigma(P_n) - x$ pebbles on P_n. From these we use $\sigma(P_n)$ pebbles to cover the edges of P_n. Now we have at least $2^{n-1}\,\sigma(C_m) - x$ pebbles in P_n. We have to use these pebbles to cover the edges of C_m. From these pebbles,

we can send $\dfrac{(2^{n-1})\sigma(C_m) - x}{2^{n-1}}$ pebbles to v_0.

That is, v_0 has at least $\sigma(C_m) - \dfrac{x}{2^{n-1}}$ pebbles.

Now, C_m contains at least $x + \sigma(C_m) - \dfrac{x}{2^{n-1}} \geq \sigma(C_m) + x - \dfrac{x}{4} \geq \sigma(C_m)$ pebbles, since $n \geq 3$ and so we are done.

Thus, $\sigma(L(m, n)) \leq 2^{n-1}\sigma(C_m) + \sigma(P_n)$.

Therefore, $\sigma(L(m, n)) = 2^{n-1}\sigma(C_m) + \sigma(P_n)$, if m=2k and k is even.

Case (2) Let $m = 2k$ with $k \geq 3$ is odd and $n \geq 3$.
Consider the distribution of placing all the $2^{n-1}\,\sigma(C_m) + \sigma(P_{n-1}) - 1$ pebbles on $v_{p_{n-1}}$. Clearly we cannot cover at least one of the edges of L(m, n).

Thus $\sigma(L(m, n)) \geq 2^{n-1}\sigma(C_m) + \sigma(P_{n-1})$

Now, consider the distribution of $2^{n-1}\sigma(C_m) + \sigma(P_{n-1})$ pebbles on L(m, n).

Case (2A) C_m contains at least $\sigma(C_m)$ pebbles.
If P_n contains $\sigma(P_n)$ or more pebbles then we are done. So, assume that P_n contains less than $\sigma(P_n)$ pebbles. This implies that C_m contains at least $2^{n-1}\sigma(C_m) + \sigma(P_{n-1}) - (\sigma(P_n) - 1)$ pebbles.

That is, the minimum number of pebbles that C_m has is,

$2^{n-1}\sigma(C_m) + \sigma(P_{n-1}) - \sigma(P_n) + 1$
=

$$\begin{cases} 2^{n-1}\sigma(C_m)+\sigma(P_{n-1})-(2\sigma(P_{n-1}+1)+1, \text{ if } n \text{ is even} \\ 2^{n-1}\sigma(C_m)+\sigma(P_{n-1})-(2\sigma(P_{n-1})+1, \text{ if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} 2^{n-1}\sigma(C_m)-\sigma(P_{n-1}) \text{ if } n \text{ is even} \\ 2^{n-1}\sigma(C_m)-\sigma(P_{n-1})+1 \text{ if } n \text{ is odd} \end{cases}$$

$\geq 2^{n-1}\sigma(C_m) - \sigma(P_{n-1})$.

From these pebbles, we use $\sigma(C_m)$ pebbles to cover the edges of C_m.
Consider the paths $P_A : v_0v_1v_2 \ldots v_{k-2}v_{k-1}$ and $P_B : v_kv_{k+1} \ldots v_{2k-1}$ of length k – 1 each. Then any one of the paths contains at least

$$\left\lceil \frac{(2^{n-1}-1)\sigma(C_m) - \sigma(P_{n-1})}{2} \right\rceil \text{ pebbles.}$$

That is, either P_A or $P_{\mathbf{B}}$ contains at least,

$$\left\lceil \frac{(2^{n-1}-1)\sigma(C_m) - \sigma(P_{n-1})}{2} \right\rceil \geq 2^{k-1}\sigma(P_n) \text{ pebbles.}$$

Suppose not, then,

$$\left\lceil \frac{(2^{n-1}-1)\sigma(C_m)-\sigma(P_{n-1})}{2} \right\rceil < 2^{k-1}\,\sigma(P_n)$$

That is, $\left\lceil \dfrac{3\left(\dfrac{2^{n-1}-1}{3}\right)\sigma(C_m)-\left(\dfrac{2^{n-1}-1}{3}\right)}{2} \right\rceil < 2^{k-1}\,\sigma(P_n)$

That is, $\left\lceil \left(\dfrac{2^{n-1}-1}{3}\right)\left(\dfrac{3\sigma(C_m)-1}{2}\right) \right\rceil < 2^{k-1}\,\sigma(P_n)$

That is, $\left\lceil \left(\dfrac{2^{n-1}-1}{3}\right)\left(\dfrac{3(2.\sigma(p_{k+1})-1)-1}{2}\right) \right\rceil < 2^{k-1}\,\sigma(P_n)$

(since m = 2k and k is odd)

That is, $\left\lceil \left(\dfrac{2^{n-1}-1}{3}\right)\left(\dfrac{2(3.\sigma(p_{k+1}))-4}{2}\right) \right\rceil < 2^{k-1}\sigma(P_n)$

That is, $\left\lceil \left(\dfrac{2^{n}-2}{3}\right)\left(\dfrac{3(2^{k+1}-1)}{2(3)}-1\right) \right\rceil < 2^{k-1}\sigma(P_n)$ (since k is odd)

That is, $\left\lceil \left(\dfrac{2^{n}-2}{3}\right)\left(2^{k+1}-\dfrac{3}{2}\right) \right\rceil < 2^{k-1}\sigma(P_n) = 2^{k-1}\left\lfloor \dfrac{2^{n}-1}{3} \right\rfloor$

which is a contradiction, since $k \geq 3$ is odd and $n \geq 3$.

If P_A contains at least $2^{k-1}\sigma(P_n)$ pebbles, then we are done, since $l(P_A) = k-1$.So we assume that P_B contains at least $2^{k-1}\sigma(P_n)$ pebbles. Also, note that if P_B contains $2^k\sigma(P_n)$ pebbles then we are done, since $l(P_B\cup\{v_0\}) = k)$. Assume that P_B contains less than $2^k\sigma(P_n)$ pebbles. Then the minimum number of pebbles that P_A has is,

$$(2^{n-1}-1)\sigma(C_m) - \sigma(P_{n-1}) - (2^k\sigma(P_n) -1)$$

$$= \begin{cases} (2^{n-1}-1)\sigma(C_m)-\sigma(P_{n-1})-2^k(2\sigma(P_{n-1})+1)+1 \text{ if } n \text{ is even} \\ (2^{n-1}-1)\sigma(C_m)-\sigma(P_{n-1})-2^k(2\sigma(P_{n-1})+1 \text{ if } n \text{ is odd} \end{cases}$$

$$\geq (2^{n-1}-1)\sigma(C_m) - \sigma(P_{n-1}) - 2^{k+1}\sigma(P_{n-1})$$

$$= (2^{n-1}-1)\sigma(C_m) - (1+2^{k+1})\sigma(P_{n-1})$$

$$= \left(\frac{2^{n-1}-1}{3}\right)[3(2\sigma(P_{k+1})-1]-(2^{k+1}+1)\left\lfloor\frac{2^{n-1}-1}{3}\right\rfloor$$

$$\geq \left(\frac{2^{n-1}-1}{3}\right)\left[2(2^{k+1}-1)-3\right]-(2^{k+1}+1)$$

$$= \left(\frac{2^{n-1}-1}{3}\right)\left[(2^{k+1}-6)\right] = 2\left(\frac{2^{n-1}-1}{3}\right)\left(2^k-3\right).$$

Thus the minimum number of pebbles that we can send to v_0 is,

$$\left\lfloor\frac{2\left(\frac{2^{n-1}-1}{3}\right)\left(2^k-3\right)}{2^{k-1}}\right\rfloor$$

$$= \left\lfloor 2\left(\frac{2^{n-1}-1}{3}\right)\left(2-\frac{3}{2^{k-1}}\right)\right\rfloor \geq \left\lfloor 2\left(\frac{2^{n-1}-1}{3}\right)\left(\frac{5}{4}\right)\right\rfloor$$

$$= \frac{5}{2}\left\lfloor \frac{2^{n-1}-1}{3}\right\rfloor \quad \geq 2\,\sigma(P_{n-1})$$

$\geq \sigma(P_{n-1}) + 1$ where the second inequality follows since $k \geq 3$.

Also we can send $\left\lfloor \frac{\sigma(P_n)}{2} \right\rfloor \geq \sigma(P_{n-1})$ pebbles to v_0 from the path P_B. So, we have at least $2\sigma(P_{n-1}) + 1$ pebbles at v_0. Thus we have enough pebbles to cover the edges of the path P_n and we are done.

Case (2B) C_m contains $x < \sigma(C_m)$ pebbles.

There are $2^{n-1}\sigma(C_m) + \sigma(P_{n-1}) - x$ pebbles on P_n. we use $\sigma(P_n)$ pebbles to cover the edges of P_n. The number of pebbles remaining on P_n for the purpose of covering the edges of C_m is $2^{n-1}\sigma(C_m) + \sigma(P_{n-1}) - x - \sigma(P_n)$

$$\geq 2^{n-1}\sigma(C_m) + \sigma(P_{n-1}) - x - (2\sigma(P_{n-1}) + 1)$$

$$= 2^{n-1}\sigma(C_m) - \sigma(P_{n-1}) - (x+1)$$

So, the minimum number of pebbles that we can send to v_0 is,

$$\left\lfloor \frac{2^{n-1}\sigma(C_m) - \sigma(P_{n-1}) - (x+1)}{2^{n-1}} \right\rfloor$$

$$\geq \left\lfloor \sigma(C_m) - \frac{2^{n-1}-1}{3.2^{n-1}} - \frac{(x+1)}{2^{n-1}} \right\rfloor$$

$$\geq \left\lfloor \sigma(C_m) - \frac{1}{3} - \frac{1}{3.2^{n-1}} - \frac{x+1}{4} \right\rfloor$$

$$\geq \left\lfloor \sigma(C_m) - \frac{x}{4} - \frac{7}{12} \right\rfloor.$$

Now, the minimum number of pebbles that C_m has is,

$$x + \sigma(C_m) - \left\lfloor \frac{x}{4} + \frac{7}{12} \right\rfloor$$

$$\geq \sigma(C_m) + \left(\frac{3x}{4} - \frac{7}{12} \right)$$

$\geq \sigma(C_m)$, where the last inequality follows since $x > 0$. So, we are done.

Thus, $\sigma(L(m, n)) \leq 2^{n-1}\, \sigma(C_m) + \sigma(P_{n-1})$.

Therefore, $\sigma(L(m, n)) = 2^{n-1}\sigma(C_m) + \sigma(P_{n-1})$.

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