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Modulo Magic Labeling in Digraphs

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Abstract

In this paper we introduce and study a new type of labeling called "Modulo Magic Labeling".

Keywords: *Digraphs, Congruence relation, Cycles, Path, Labeling.*

1 Introduction

A directed graph $\mathrm{G}=(\mathrm{V},\mathrm{E})$ consists of a finite set V of points (vertex) and a collection of ordered pair of distinct points called lines (edges). Any such pair (u,v) is called an arc or directed line (edge) and will usually be denoted by uv. The edge *uv* goes from u to *v* and incident with u and v. An edge connects a vertex to itself is called a loop. Let us consider the set H to be $H=\{0,1,2,3,...,n-1\}$.We consider the directed graph whose vertices are the elements of H and such that there exists exactly one directed edge from a to b if and only if it satisfies the congruence relation. A cycle of length 1 is called a fixed point. Denote by id(a) the number of directed edges coming to a and od(a) the number of directed edges leaving the vertex $a\in H=\{0,1,2,3,...,n-1\}$. . Of course

$\mathrm{id(a)} \geq 0$ and $\mathrm{od(a)} = 1$. For an isolated fixed point, the in-degree and the out-degree are both equal to 1.

In the following sections we show some connection between congruence relation and graph labeling motivated by results of M.Krizek and L.Somer given in [8] and in [9] as well as by the results of S.Bryant[5], G.Chasse[6], T.D.Rogers[11], L.Szalay[12] and Joanna Skowronek-Kaziow[7].

The following definitions are taken from [10] and [13].

Definition 1.1: *Let G be a graph with p vertices and q edges. Assume that the vertices and edges of G are labeled by 1, 2, 3. . . p + q such that each label is used exactly once. We define the valence val(e) of an edge e to the sum of the label of e plus the two labels of the vertices incident with e.*

Definition 1.2: *If a labeling of G is possible such that the valence val(e) for each e is constant, we call the graph G edge-magic. And if it is possible that all the valences of the edge are distinct, we call G edge-antimagic.*

Definition 1.3: *Let the edges of a graph G be labeled by 1, 2,. . ., q, and we call the sum of all the labels of the edges incident with a vertex P the valence val(P) of P. If it is possible that all of the valences of the vertices in G are distinct, the graph G is called antimagic.*

Definition 1.4: *A graph G(p, q) is said to be (1, 1) vertex-magic with the common vertex count k_1 if there exists a bijection $f : V(G) \cup E(G) \to \{1, ..., p + q\}$ such that for each $u \in V(G)$, $f(u) + \sum_e f(e) = k_1$ for all $e = (u, v) \in E(G)$ with $v \in V(G)$. It is said to be (1,1) vertex- antimagic if $f(u) + \sum_e f(e)$ are distinct for all $e = (u, v) \in E(G)$ with $v \in V(G)$. (1, 1) vertex magic labeling can also be said as vertex magic total labeling.*

2 Modulo Magic Labeling

In this section we introduce the concept of modulo magic labeling for some congruence relations with constant valence 2n-1. The out-degree of a vertex *v* is the number of edges going away from *v* in a directed graph and is denoted by $od(v_i)$. The valence of $od(v_i)$ is the sum of the out degree labels of edges from v_i plus the label of v_i.

Definition 2.1: *Let G be a directed graph with $V = \{0,1,2,3,...,n-1\}$ and a bijective function $f : V \cup E \to \{0,1,2,3,...,2n-1\}$ is said to be a modulo magic labeling with common count k, if $f(u) + f(\overrightarrow{uv}) = k \;\forall\; \overrightarrow{uv} \in E$ or val $(od(u))$ is a constant k.*

Theorem 2.2: *If G(V,E) be a directed graph with* $V = \{0,1,2,...,n-1\}$ *and edge set* $\overrightarrow{ab} \in E$ *if and only if aRb when* $R: a+1 \equiv b \bmod n$ *admits modulo magic labeling.*

Proof: Let $G = (V,E)$ be a graph with vertex set $V = \{0,1,2,3,...,n-1\}$ and the edge set is defined by the relation $a+1 \equiv b \bmod n$. Consider a bijective function $f: V \cup E \to \{0,1,2,...,n-1,n,...,2n-1\}$ defined by $f(v_i) = i$; $0 \le i \le n-1$ and the edge set is given by $E = \{\overrightarrow{v_i v_{i+1}}; 0 \le i \le n-2\} \cup \{\overrightarrow{v_0 v_{n-1}}\}$. The edge labeling is defined by $f(v_i v_{i+1}) = 2n-1-i; 0 \le i \le n-2$ and $f(\overrightarrow{v_{n-1} v_0}) = n$. The valence of out-degree of the vertex v_i is denoted by val $(od(v_i))$. For every *i*,

$$\text{val}(od(v_i)) = f(v_i) + f(\overrightarrow{v_i v_{i+1}}); 0 \le i \le n-2$$

$$= i + 2n - 1 - i$$

$$= 2n - 1$$

$$\text{val}(od(v_{n-1})) = f(v_{n-1}) + f(\overrightarrow{v_{n-1} v_0})$$

$$= n - 1 + n$$

$$= 2n - 1$$

Hence the directed cycle graph obtained by the relation $a+1 \equiv b \bmod n$ admits modulo magic labeling with constant value 2*n*–1.

Theorem 2.3: *If G(V,E) be a disconnected directed cycle graph with* $V = \{0,1,2,...,n-1\}$ *and edge set* $\overrightarrow{ab} \in E$ *if and only if aRb when* $R: a+2 \equiv b \bmod n$ *admits modulo magic labeling.*

Proof: Let $G = (V,E)$ be a graph with vertex set $V = \{0,1,2,3,...,n-1\}$ and the edge set is defined by the relation $a+2 \equiv b \bmod n$. Consider a function $f: V \cup E \to \{0,1,2,...,n-1,n,...,2n-1\}$ where $f(v_i) = i$; $0 \le i \le n-1$. We now proceed with the following two cases.

Case 1: When $n \equiv 0 \bmod 2$

Let $G = (V,E)$ be a disconnected graph. The two directed cycle graphs obtained from the relation is symmetric. The edge set is defined by $E = E_1 \cup E_2 \cup \{\overrightarrow{v_{n-2} v_0}\} \cup \{\overrightarrow{v_{n-1} v_1}\}$ where $E_1 = \{\overrightarrow{v_{2i} v_{2i+2}}; 0 \le i \le (n-4)/2\}$ and

$E_2 = \{\overrightarrow{v_{2i-1}v_{2i+1}}; 1 \le i \le (n-2)/2\}$.

The edge labeling are given by

$f(\overrightarrow{v_{2i}v_{2i+2}})$ =2*n*-1-2*i*; $0 \le i \le (n-4)/2$
$f(\overrightarrow{v_{2i-1}v_{2i+1}})$=2*n*-2*i*; $1 \le i \le (n-2)/2$
$f(\overrightarrow{v_{n-2}v_0}) = n+1$
$f(\overrightarrow{v_{n-1}v_1}) = n$

$$\begin{aligned}\text{val}(od(v_{2i})) &= f(v_{2i}) + f(\overrightarrow{v_{2i}v_{2i+2}}); 0 \le i \le (n-4)/2\\ &= 2i + 2n - 1 - 2i\\ &= 2n-1\end{aligned}$$

$$\begin{aligned}\text{val}(od(v_{2i+1})) &= f(v_{2i-1}) + f(\overrightarrow{v_{2i-1}v_{2i+1}}); 1 \le i \le (n-2)/2\\ &= 2i - 1 + 2n - 2i\\ &= 2n-1\end{aligned}$$

$$\begin{aligned}\text{val}(od(v_{n-2})) &= f(v_{n-2}) + f(\overrightarrow{v_{n-2}v_0})\\ &= n-2+n+1\\ &= 2n-1\end{aligned}$$

$$\begin{aligned}\text{val}(od(v_{n-1})) &= f(v_{n-1}) + f(\overrightarrow{v_{n-1}v_1})\\ &= n-1+n\\ &= 2n-1\end{aligned}$$

Case 2: When $n \equiv 1 \bmod 2$

Let $G = (V, E)$ be a connected cycle graph. The edge set is defined by $E = \{\overrightarrow{v_iv_{i+2}}; 0 \le i \le n-3\} \cup \{\overrightarrow{v_{n-2}v_0}\} \cup \{\overrightarrow{v_{n-1}v_1}\}$. The edge labeling is given by

$f(\overrightarrow{v_iv_{i+2}}) = 2n-1-i; 0 \le i \le n-3$
$f(\overrightarrow{v_{n-2}v_0}) = n+1$
$f(\overrightarrow{v_{n-1}v_1}) = n$

$$\begin{aligned}\text{val}(od(v_i)) &= f(v_i) + f(\overrightarrow{v_iv_{i+2}}); 0 \le i \le n-3\\ &= i + 2n - 1 - i\\ &= 2n-1\end{aligned}$$

$$\begin{aligned}\text{val}(od(v_{n-2})) &= f(v_{n-2}) + f(\overrightarrow{v_{n-2}v_0})\\ &= n-2+n+1\end{aligned}$$

$$= 2n - 1$$

$$\text{val}(od(v_{n-1})) = f(v_{n-1}) + f(\overrightarrow{v_{n-1}v_1})$$
$$= n - 1 + n$$
$$= 2n - 1$$

Hence the directed cycle graph obtained by the relation $a + 2 \equiv b \bmod n$ admits modulo magic labeling with constant valence 2*n*–1

Example 2.4: The following figure shows a directed graph for the congruence relation $a^7 + 2 \equiv b \bmod 10$ which has modulo magic labeling with constant valence 19.

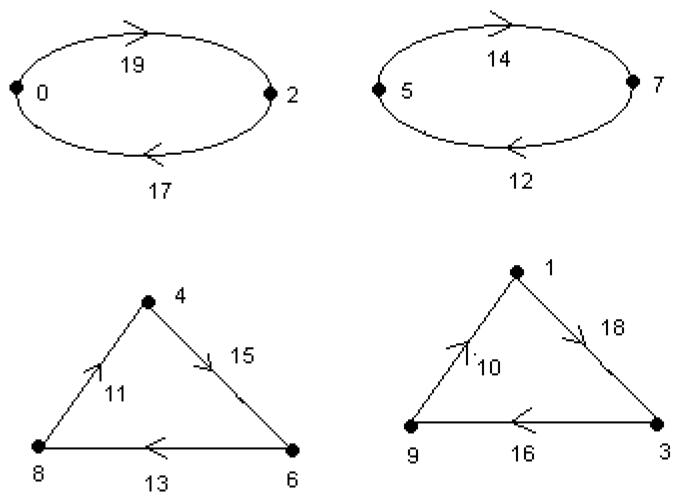

Fig 1: Modulo magic labeling for $a^7 + 2 \equiv b \bmod 10$

Observation 2.5: Let $G = (V, E)$ be a graph with vertex set V ={ 0,1,2,….,n-1 } and the edge set is defined by the relation $a^m + k \equiv b \bmod n$ where m, k and n are positive integers. Let $f : V \cup E \to \{0,1,2,...,n-1, n,...,2n-1\}$ defined by $f(v_i) = i$; $0 \le i \le n-1$. Let us assume that $a^m + k \equiv b \bmod n$ and $a^m + k \equiv c \bmod n$.

$\Rightarrow a^m + k - b = nx$ and $a^m + k - c = ny$ where x and y are positive integers.

$\Rightarrow a^m + k = b + nx$ and $a^m + k = c + ny$

$\Rightarrow b + nx = c + ny$

$\Rightarrow b - c = n(y - x)$

$\Rightarrow b \equiv c \bmod n.$

Therefore the out-degree for all the vertices in the directed graph is one. Now the edge labels are defined in such a way that the out degree from $v_i; 0 \le i \le n-1$ will receive the edge labels in the descending from 2n-1 to n respectively. Therefore when we sum the vertex with their corresponding edge labels we get a constant value 2n-1 which leads to the modulo magic labeling.

3 Modulo Bimagic Labeling

In this section we introduce the concept of modulo bimagic labeling for some congruence relations. The in-degree of a vertex v is the number of edges coming towards v in a directed graph and is denoted by id(v_i). The valence of id(v_i) is the sum of the in-degree labels of the vertices v_j for some j.

Definition 3.1: *Let* $V = \{0,1,2,3,...,n-1\}$. *A directed graph G(V,E) is said to have modulo bimagic labeling with common count 0 or k, if there exist a bijection* $f : V \to V$ *such that* $\sum_{uv} f(v) = 0$ or $k\ \forall\ \overrightarrow{uv} \in E$.

Example 3.2: The following figure shows a directed graph for the congruence relation $a^2 \equiv b \bmod 13$ which admits modulo bimagic labeling.

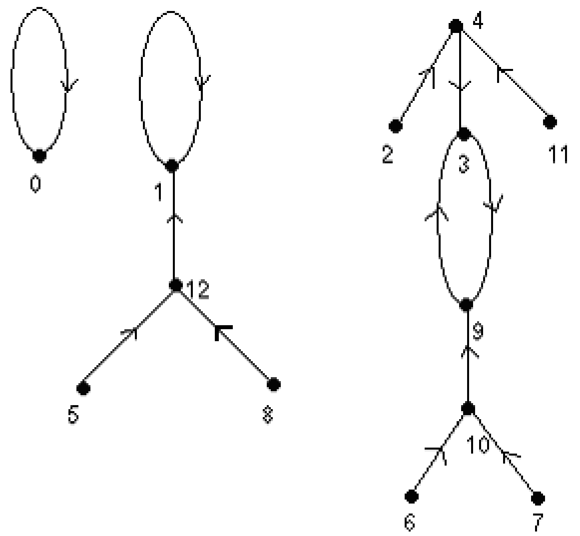

Fig 2: Modulo bimagic labeling for $a^2 \equiv b(\bmod 13)$

Theorem 3.3: *If G (V, E) be a directed graph with* $V = \{0, 1, 2,..., n-1\}$ *and edge set* $\overrightarrow{ab} \in E$ *if and only if aRb when* $R : a^2 \equiv b \bmod n$ *admits modulo bimagic labeling only when n is prime.*

Proof: Let $V = \{0, 1,..., n-1\}$ be the vertices of the digraph where n is any prime greater than two. The edges of the graph is defined by the relation $a^2 \equiv b(\bmod n)$. The graph obtained with this relation is a disconnected graph which is either a path or a cycle, for which the in- degree sum of every vertex is either 0 or n. The pendant vertices have indegree zero and all the internal vertices have in degree 2 and 0 is always an isolated fixed point.

Let us assume that $a^2 \equiv b \bmod n$ and $a^2 \equiv c \bmod n$.

$\Rightarrow a^2 - b = nk$ and $a^2 - c = nm$ where k and m are positive integers.
$\Rightarrow a^2 = b + nk$ and $a^2 = c + nm$
$\Rightarrow b + nk = c + nm$
$\Rightarrow b - c = n(m - k)$
$\Rightarrow b \equiv c \bmod n$

which shows that the out-degree is always one. Consider an arbitrary vertex x whose in degree is 2. Let u and v be the vertices such that $ux, vx \in E$ and $f(u) + f(v) = n$ where $u, v \in V$.

Let $f(u) = a_1$, $f(v) = a_2$ and $f(x) = b$ say, then

$$a_1^2 \equiv b \bmod n \quad (1)$$

$$a_2^2 \equiv b \bmod n. \quad (2)$$

We need to prove that $a_1 + a_2 = n.$ From (1) and (2) we infer that

$$a_1^2 \equiv a_2^2 \bmod n \quad (3)$$

$$a_1^2 - a_2^2 \equiv 0 \bmod n$$

$$a_1^2 - a_2^2 = kn$$

$$(a_1 + a_2)(a_1 - a_2) = kn$$

where k is any integer. From (3) it is clear that $a_1 + a_2 = n$ where $a_1 - a_2 = k$.

The vertex labeled with 0 is an isolated fixed point whose indegree vertex sum is always 0.Therefore the graph obtained by the relation R admits (1, 0) modulo bimagic labeling with two constants 0 and n.

Observation 3.4: A component of the iteration digraph is a subdigraph which is a maximal connected subgraph of the symmetrization of this digraph. The digraph G(n) is called symmetric if its set of components can be split into two sets in such a way that there exists a bijection between these two sets such that the corresponding digraphs are isomorphic[8]. Consider the digraph obtained by the congruence relation $\mathrm{a}^2 + 2 \equiv \mathrm{b} \bmod \mathrm{n}$ for $1 \le \mathrm{n} \le 20.$ when n is even, the digraph obtained by the relation is symmetric except when n=8. When n is odd, we get a connected digraph except for n=15 and 19.

4 Conclusion

The digraph constructed from some congruence relations satisfies the modulo magic labeling and modulo bimagic labeling. In this similar way, one can construct and analyze the properties of the digraph obtained from the congruence relations and also fit the suitable labeling to it.

References

[1] J.B. Babujee, Bimagic labeling in path graphs, *The Mathematics Education*, 38(1) (2004), 12-16.

[2] J.B. Babujee and N.P. Rao, Labelings in digraph, *Journal of the Mathematics Education*, 39(2) (2005), 73-78.

[3] J.B. Babujee and L. Shobana, Digraphs arising from number theory, *Indian Journal of Mathematics and Mathematical Sciences*, 5(2) (Dec) (2009), 145-147.

[4] J.B. Babujee and L. Shobana, Graph from relations and its labeling, *European Journal of Scientific Research*, 64(3) (2011), 463-471.

[5] S. Bryant, Groups, graphs and Fermat's last theorem, *Amer. Math. Monthly*, 74(1967), 152-156.

[6] G. Chasse, Combinatorial cycles of a polynomial map over a commutative field, *Discrete Math.*, 61(1986), 21-26.

[7] J. Skowronek-Kaziow, Some digraphs arising from number theory and remarks on the zero-divisor graph of the ring Z_n, *Information Processing Letters*, 108 (3)(2008), 165-169.

[8] M. Krizek and L. Somer, On a connection of number theory with graph theory, *Czechoslovak Math. J.*, 54(129) (2004), 465-485.

[9] M. Krizek and L. Somer, Sophie Germain little suns, *Math. Slovaca*, 54(2004), 433-442.

[10] G. Ringel, Another tree conjecture, *Utilitas Matematica*, 53(1998), 179-181.

[11] T.D. Rogers, The graph of the square mapping on the prime fields, *Discrete Math.*, 148(1996), 317-324.

[12] L. Szalay, A discrete iteration in number theory, *BDTF Tud. Kozl.*, 8(1992), 71-91 (in Hungarian).

[13] V. Yegnanarayanan, On magic graphs, *Utilitas Matematica*, 59(2001), 181-204.

[14] W.D. Wallis, *Magic Graphs*, Birkhauser, (2000).