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First Non-Abelian Cohomology of Topological Groups

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Abstract

Let G be a topological group and A a topological G-module (not necessarily abelian). In this paper, we define $H^0(G, A)$ and $H^1(G, A)$ and will find a six terms exact cohomology sequence involving H^0 and H^1. We will extend it to a seven terms exact sequence of cohomology up to dimension two. We find a criterion such that vanishing of $H^1(G, A)$ implies the connectivity of G. We show that if $H^1(G, A) = 1$, then all complements of A in the semidirect product $G \ltimes A$ are conjugate. Also as a result, we prove that if G is a compact Hausdorff group and A is a locally compact almost connected Hausdorff group with the trivial maximal compact subgroup then, $H^1(G, A) = 1$.

Keywords: *Almost connected group, inflation, maximal compact subgroup, non-abelian cohomology of topological groups, restriction.*

1 Introduction

Let G and A be topological groups. It is said that A is a topological G-module, whenever G continuously acts on the left of A. For all $g \in G$ and $a \in A$ we denote the action of g on a by ${}^g a$.

In section 2, We define $H^0(G, A)$ and $H^1(G, A)$.

In section 3, we define the covariant functor $H^i(G, -)$ for $i = 0, 1$ from the category of topological G-modules to the category of pointed sets. Also, we define two connecting maps δ^0 and δ^1.

A classical result of Serre [6], asserts that if G is a topological group and $1 \to A \to B \to C \to 1$ a central short exact sequence of discrete G-modules then, the sequence $1 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \to H^1(G,A) \to H^1(G,B) \to H^1(G,C) \to H^2(G,A)$ is exact.

In section 4, we generalize the above result to the case of arbitrary topological G-modules (not necessarily discrete).

We show that if G is a connected group and A a totally disconnected group then, $H^1(G,A) = 1$.

In section 5, we show that if G has an open component (for example G with the finite number of components) and for every discrete (abelian) G-module A $H^1(G,A) = 1$ then, G is a connected group.

In section 6, we show that vanishing of $H^1(G,A)$ implies that the complements of A in the (topological) semidirect product $G \ltimes A$, are conjugate.

In section 7, we prove that, if G is a compact Hausdorff group and A a locally compact almost connected Hausdorff group then there exists a G-invariant maximal compact subgroup K of A such that the natural map $\iota_1^* : H^1(G,K) \to H^1(G,A)$ is onto. As a result, if G is compact Hausdorff and A is a locally compact almost connected Hausdorff group with trivial maximal compact subgroup then, $H^1(G,A) = 1$.

All topological groups are arbitrary (not necessarily abelian). We assume that G acts on itself by conjugation. The center of a group G and the set of all continuous homomorphisms of G into A are denoted by $Z(G)$ and $Hom_c(G,A)$, respectively. The topological isomorphism is denoted by " $\simeq$ ".

Suppose that A is an abelian topological G-module.

Take $\tilde{C}^0(G,A) = A$ and for every positive integer n, let $\tilde{C}^n(G,A)$ be the set of continuous maps $f : G^n \to A$ with the coboundary map $\tilde{\delta}^n : \tilde{C}^n(G,A) \to \tilde{C}^{n+1}(G,A)$ given by

$$\tilde{\delta}^n f(g_1, ..., g_{n+1}) =$$
$$^{g_1}f(g_2, ..., g_{n+1}) + \sum(-1)^i f(g_1, ..., \underbrace{g_i g_{i+1}}_{ith}, ..., g_{n+1}) + (-1)^{n+1} f(g_1, ..., g_n).$$

The nth cohomology of G with coefficients in A in the sense of Hu [5], is the abelian group

$$H^n(G,A) = Ker\tilde{\delta}^n / Im\tilde{\delta}^{n-1}.$$

2 $H^0(G,A)$ and $H^1(G,A)$

Let G be a topological group and A a topological G-module.

Definition 2.1. *We define $H^0(G,A) = \{a | a \in A, {}^g a = a, \forall g \in G\}$, i.e., $H^0(G,A) = A^G$, the set of G-fixed elements of A.*

Definition 2.2. *A map $\alpha : G \to A$ is called a continuous derivation if α is continuous and*

$$\alpha(gh) = \alpha(g)^g\alpha(h), \forall g, h \in G.$$

The set of all continuous derivations from G into A is denoted by $Der_c(G, A)$. Two continuous derivations α, β are cohomologous, denoted by $\alpha \sim \beta$, if there is $a \in A$ such that

$$\beta(g) = a^{-1}\alpha(g)^g a, \text{ for all } g \in G.$$

It is easy to show that $\sim$ is an equivalence relation. Now we define

$$H^1(G, A) = Der_c(G, A)/\sim .$$

Notice 2.3. *There exists the trivial continuous derivation $\alpha_0 : G \to A$ where $\alpha_0(g) = 1$; Hence, $H^1(G, A)$ is nonempty. In general, $H^1(G, A)$ is not a group. Thus, we will view $H^1(G, A)$ as a pointed set with the basepoint α_0.*

Note that $H^0(G, A)$ is a subgroup of A, so it is a pointed set with the basepoint 1. Also if A is a Hausdorff group, then, $H^0(G, A)$ is a closed subgroup of A.

Remark 2.4. *(i) If A is an abelian group then, $H^1(G, A)$ is the first (abelian) group cohomology in the sense of Hu, i.e., it is the group of all continuous derivations of G into A reduced modulo the inner derivations. [5]*

(ii) If A is a trivial topological G-module then, $H^1(G, A) = Hom_c(G, A)/\sim$. Here $\alpha \sim \beta$ if $\exists\ a \in A$ such that $\beta(g) = a^{-1}\alpha(g)a, \forall g \in G$.

(iii) Let G be a connected group and A a totally disconnected group then, $H^1(G, A) = 1$.

Proof. (i) and (ii) are obtained from the definition of $H^1(G, A)$.
(iii): If $\alpha \in Der_c(G, A)$ then, $\alpha(1) = 1$. On the other hand G is a connected group and A is totally disconnected. So, $\alpha = \alpha_0$. Thus, $H^1(G, A) = 1$.

3 $H^i(G, -)$ as a Functor and the Connecting Map δ^i for $i = 0, 1$

In this section we define two covariant functors $H^0(G, -)$ and $H^1(G, -)$ from the category of topological G-modules ${}_G\mathcal{M}$ to the category of pointed sets $\mathcal{PS}$. Furthermore, We will define the connecting maps δ^0 and δ^1.

Let A, B be topological G-modules and $f : A \to B$ a continuous G-homomorphism. We define $H^i(G, f) = f_i^* : H^i(G, A) \to H^i(G, B)$, $i = 0, 1$, as follows:

For $i = 0$, take $f_0^* = f|_{A^G}$. This gives a homomorphism from $H^0(G, A)$ to $H^0(G, B)$, since f is a homomorphism of G-modules . So if $a \in A^G$, then, ${}^g f(a) = f({}^g a) = f(a)$, for each $g \in G$. Hence, $f(a) \in B^G$, i.e., f_0^* is well-defined.

For $i = 1$, we define f_1^* as follows:
For simplicity, we write α instead of $[\alpha] \in H^1(G, A)$.
If $\alpha \in H^1(G, A)$, then, take $f_1^*(\alpha) = f \circ \alpha$. Now if $g, h \in G$, then,

$$f_1^*(\alpha)(gh) = f(\alpha(gh)) = f(\alpha(g)^g\alpha(h)) = f(\alpha(g))f({}^g\alpha(h)) = f_1^*(\alpha)(g)^g f_1^*(\alpha)(h).$$

So, $f_1^*(\alpha)$ is a continuous derivation.

Moreover, if $\alpha, \beta \in H^1(G, A)$ are cohomologous then, there is $a \in A$ such that $\beta(g) = a^{-1}\alpha(g)^g a$. Hence, $f(\beta(g)) = f(a)^{-1} f(\alpha(g))^g f(a)$. So, $f_1^*(\alpha) \sim f_1^*(\beta)$.
The fact that $H^i(G, -)$ is a functor follows from the definition of f_i^*, $(i = 0, 1)$. Also $H^0(G, -)$ is a covariant functor from ${}_G\mathcal{M}$ to the category of topological groups $\mathcal{TG}$.

Suppose that $1 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ is an exact sequence of topological G-modules and continuous G-homomorphisms such that ι is an embedding. Thus, we can identify A with $\iota(A)$.

Now we define a coboundary map $\delta^0 : H^0(G, C) \to H^1(G, A)$.
Let $c \in H^0(G, C)$, $b \in B$ with $\pi(b) = c$. Then, we define $\delta^0(c)$ by $\delta^0(c)(g) = b^{-1g}b, \forall g \in G$. It is obvious that $\delta^0(c)$ is a continuous derivation. Let $b' \in B$, $\pi(b') = c$. Then, $b' = ba$ for some $a \in A$. So,

$$(b')^{-1g}b' = a^{-1}b^{-1g}b^g a = a^{-1}\delta^0(c)(g)^g a.$$

Thus, the derivation obtained from b' is cohomologous in A to the one obtained from b, i.e., δ^0 is well-defined.

Now, suppose that $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ is a central exact sequence of G-modules and continuous G-homomorphisms such that ι is a homeomorphic embedding and in addition π has a continuous section $s : C \to B$, i.e., $\pi s = Id_C$.

We construct a coboundary map $H^1(G, C) \xrightarrow{\delta^1} H^2(G, A)$. Here $H^2(G, A)$ is defined in the sense of Hu [5]. By assumption $\iota(A) \subset Z(B)$, so, A is an abelian topological G-module.
Let $\alpha \in H^1(G, C)$ and $s : C \to B$ be a continuous section for π. Define $\delta^1(\alpha)$ via $\delta^1(\alpha)(g, h) = s\alpha(g)\ {}^g(s\alpha(h))(s\alpha(gh))^{-1}$. It is clear that $\delta^1(\alpha)$ is a continuous map.

We show that $\delta^1(\alpha)$ is a factor set with values in A, and independent of the choice of the continuous section s. Also δ^1 is well-defined.
Since α is a derivation, we have:

$$\pi(\delta^1(\alpha)(g,h)) = \pi(s\alpha(g)\ {}^g(s\alpha(h))(s\alpha(gh))^{-1}) = \alpha(g)^g\alpha(h)(\alpha(gh))^{-1} = 1.$$

Thus, $\delta^1(\alpha)$ has values in A.

Next, we show that $\delta^1(\alpha)$ is a factor set, i.e.,

$$^g\delta^1(\alpha)(h,k)\delta^1(\alpha)(g,hk) = \delta^1(\alpha)(gh,k)\delta^1(\alpha)(g,h),\ \forall g,h,k \in G. \tag{3.1}$$

First we calculate the left hand side of (3.1). For simplicity, take $b_g = s\alpha(g)$, $\forall g \in G$. Since $A \subset Z(B)$, thus,

$$^g\delta^1(\alpha)(h,k)\delta^1(\alpha)(g,hk) = {}^g(b_h{}^hb_kb_{hk}^{-1})(b_g{}^gb_{hk}b_{ghk}^{-1}) = b_g{}^g(b_h{}^hb_kb_{hk}^{-1}){}^gb_{hk}b_{ghk}^{-1}$$

$$= b_g{}^g(b_h{}^hb_k){}^gb_{hk}b_{ghk}^{-1} = b_g{}^gb_h{}^{gh}b_kb_{ghk}^{-1},$$

On the other hand,

$$\delta^1(\alpha)(gh,k)\delta^1(\alpha)(g,h) = (b_{gh}{}^{gh}b_kb_{ghk}^{-1})(b_g{}^gb_hb_{gh}^{-1}) = b_g{}^gb_h{}^{gh}b_kb_{ghk}^{-1}.$$

Therefore, $\delta^1(\alpha)$ is a factor set.

Next, we prove that $\delta^1(\alpha)$ is independent of the choice of the continuous section. Suppose that s and u are continuous sections for π. Take $b_g = s\alpha(g)$ and $b_g^{'} = u\alpha(g)$, for a fixed $\alpha \in Der_c(G,C)$. Since $\pi(b_g^{'}) = \alpha(g) = \pi(b_g)$, then, $b_g^{'} = b_ga_g$ for some $a_g \in A$. Obviously the function $\kappa : G \to A$, defined by $\kappa(g) = a_g$, is continuous. Thus,

$$(\delta^1)^{'}(\alpha)(g,h) = b_g^{'}{}^gb_h^{'}b_{gh}^{'} = b_g\kappa(g)^gb_h{}^g\kappa(h)(\kappa(gh))^{-1}b_{gh}^{-1})$$

$$=(\kappa(g)^g\kappa(h)(\kappa(gh))^{-1})(b_g{}^gb_hb_{gh}^{-1}) = \tilde{\delta}^1(\kappa)(g,h)\delta^1(\alpha)(g,h),$$

where $\tilde{\delta}^1(\kappa)(g,h) = {}^g\kappa(h)(\kappa(gh))^{-1}\kappa(g)$.
The coboundary map $\tilde{\delta}^1 : \tilde{C}^1(G,A) \to \tilde{C}^2(G,A)$ is defined as in [5]. Consequently, $\delta^1(\kappa)$ and $(\delta^1)^{'}(\kappa)$ are cohomologous.

Suppose that α and β are cohomlologous in $Der_c(G,A)$. Then, there is $c \in C$ such that $\beta(g) = c^{-1}\alpha(g)^gc$, $\forall g \in G$.
Let $s : C \to A$ be a continuous section for π. Since

$$\pi(s(c^{-1}\alpha(g)^gc)) = \pi(s(c)^{-1}s\alpha(g)^gs(c)),$$

then, there exists a unique $\gamma(g) \in ker\pi = A$ such that

$$\gamma(g)(s(c)^{-1}s\alpha(g)^gs(c)) = s(c^{-1}\alpha(g)^gc).$$

It is clear that the map $\gamma : G \to A$, $g \mapsto \gamma(g)$ is continuous. Therefore,

$$\delta^1(\beta)(g,h) = s\beta(g).{}^g s\beta(h).(s\beta(gh))^{-1}$$

$$= s(c^{-1}\alpha(g)^g c).{}^g s(c^{-1}\alpha(h)^h c).(s(c^{-1}\alpha(gh)^{gh} c))^{-1}$$

$$= \gamma(g)[s(c)^{-1}s\alpha(g)^g s(c)].{}^g(\gamma(h)[s(c)^{-1}s\alpha(h)^h s(c)]).(\gamma(gh)[s(c)^{-1}s\alpha(gh)^{gh} s(c)])^{-1}$$

$$= {}^g\gamma(h)\gamma(gh)^{-1}\gamma(g)[s(c)^{-1}s\alpha(g)^g s(c)].{}^g[s(c)^{-1}s\alpha(h)^h s(c)].[s(c)^{-1}s\alpha(gh)^{gh} s(c)]^{-1}$$

$$= \tilde{\delta}^1(\gamma)(g,h)[s(c)^{-1}s\alpha(g)^g s\alpha(h)(a\alpha(gh))^{-1}s(c)]$$

$$= \tilde{\delta}^1(\gamma)(g,h)[s(c)^{-1}\delta^1(\alpha)(g,h)s(c)] = \tilde{\delta}^1(\gamma)(g,h)[\delta^1(\alpha)(g,h)].$$

The last equality is obtained from the fact that $\delta^1(\alpha)(g,h) \in A \subset Z(B)$ and $s(c) \in B$. Now, note that $\delta^1(\alpha)$ is cohomologous to $\delta^1(\beta)$, when α is cohomologous to β. Thus, δ^1 is well-defined.

4 A Cohomology Exact Sequence

Let $(X, x_0), (Y, y_0)$ be pointed sets in $\mathcal{PS}$ and $f : (X, x_0) \to (Y, y_0)$ a pointed map, i.e., $f : X \to Y$ is a map such that $f(x_0) = y_0$. For simplicity, we write $f : X \to Y$ instead of $f : (X, x_0) \to (Y, y_0)$. The kernel of f, denoted by $Ker(f)$, is the set of all points of X that are mapped to the basepoint y_0. A sequence $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ of pointed sets and pointed maps is called an exact sequence if $Ker(g) = Im(f)$.

Theorem 4.1. *(i) Let* $1 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ *be a short exact sequence of topological G-modules and continuous G-homomorphisms, where ι is homeomorphic embedding. Then, the following is an exact sequence of pointed sets,*

$$0 \longrightarrow H^0(G,A) \xrightarrow{\iota_0^*} H^0(G,B) \xrightarrow{\pi_0^*} H^0(G,C) \tag{4.1}$$

$$\xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\iota_1^*} H^1(G,B) \xrightarrow{\pi_1^*} H^1(G,C).$$

(ii) In addition, if $\iota(A) \subset Z(B)$, and π has a continuous section, then

$$0 \longrightarrow H^0(G,A) \xrightarrow{\iota_0^*} H^0(G,B) \xrightarrow{\pi_0^*} H^0(G,C) \tag{4.2}$$

$$\xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\iota_1^*} H^1(G,B) \xrightarrow{\pi_1^*} H^1(G,C) \xrightarrow{\delta^1} H^2(G,A)$$

is an exact sequence of pointed sets.

Proof. (i): We prove the exactness term by term.

1. Exactness at $H^0(G,A)$: This is clear, since ι is one to one.

2. Exactness at $H^0(G,B)$: Since $\pi_0^*\iota_0^* = (\pi\iota)_0^* = 1$, then, $Im(\iota_0^*) \subset Ker(\pi_0^*)$. Now we show that $Ker\pi_0^* \subset Im\iota_0^*$. If $b \in Ker\pi_0^*$, then, $\pi(b) = 1$ and $b \in H^0(G,B)$. There is an $a \in A$ such that $\iota(a) = b$. Moreover, $\iota({}^ga) =^g \iota(a) = \iota(a)$, $\forall g \in G$. So, ${}^ga = a$, $\forall g \in G$, since ι is one to one. Thus, $a \in H^0(G,A)$. Hence, $b \in Im(\iota_0^*)$.

3. Exactness at $H^0(G,C)$: Take $c \in Im(\pi_0^*)$. So, $c = \pi(b)$ for some $b \in H^0(G,B)$. Thus, $\delta^0(c)(g) = b^{-1g}b$. Hence, $\delta^0(c) \sim \alpha_0$. Conversely, if $\delta^0(c) \sim \alpha_0$, then, there is $a_1 \in A$ such that $\delta^0(c)(g) = a_1^{-1g}a_1$, $\forall g \in G$. Let $c = \pi(b)$ for some $b \in B$. Then, by definition of $\delta^0(c)(g)$, there is $a_2 \in A$ such that $b^{-1g}b = a_2^{-1}\delta^0(c)(g)^g a_2$, $\forall g \in G$. So, $b(a_1a_2)^{-1} \in H^0(G,B)$. Since $\pi_0^*(b(a_1a_2)^{-1}) = c$, then, $c \in Im\pi_0^*$.

4. Exactness at $H^0(G,A)$: Let $c \in H^0(G,C)$. Then, there is $b \in B$ such that $\pi(b) = c$. So,

$$\pi_1^*\delta_0^*(c)(g) = \pi(\delta_0^*(c)(g)) = \pi(b^{-1g}b) = \pi(b)^{-1g}\pi(b).$$

Consequently, $\pi_1^*\delta_0^*(c) \sim \beta_0$, where $\beta_0(g) = 1, \forall g \in G$. Conversely, let $\alpha \in Ker\iota_1^*$. Then, there is $b \in B$ such that $\iota\alpha(g) = b^{-1g}b, \forall g \in G$. So, $\pi(b^{-1g}b) = 1, \forall g \in G$. Take $c = \pi(b)$. Hence, $c \in H^0(G,C)$. Thus, $\delta^0(c) \sim \iota(\alpha) = \alpha$.

5. Exactness at $H^1(G,B)$: Since $\pi_1^*\iota_1^* = (\pi\iota)_1^* = 1$, then, $Im\iota_1^* \subset Ker\pi_1^*$. Conversely, let $\beta \in ker\pi_1^*$. Then, there is $c \in C$ such that $\pi\beta(g) = c^{-1g}c$, for all $g \in G$. Let $b \in B$ and $c = \pi(b)$. Therefore, $\pi(\beta(g)) = \pi(b^{-1g}b)$, $\forall g \in G$. On the other hand, the map $\tau : A \to A$, $a \mapsto b^{-1}ab$, is a topological isomorphism, because A is a normal subgroup of B. So, for every $g \in G$ there is a unique element $a_g \in G$ such that, $\beta(g) = (b^{-1}a_gb)(b^{-1g}b)$. Thus, $\beta(g) = b^{-1}a_g{}^gb$, $\forall g \in G$. Hence, $a_g = b\beta(g)^gb^{-1}$, $\forall g \in G$. Obviously, the map $\alpha : G \to A$ via $\alpha(g) = a_g$ is a continuous derivation, and $\iota_1^*(\alpha) \sim \iota_1^*(\beta) = \beta$.

(ii): It is enough to show the exactness at $H^1(G,C)$. Let $[\beta] \in H^1(G,B)$ and s be a continuous section for π. Then,

$$\delta^1(\pi_1^*(\beta))(g,h) = s(\pi\beta(g))^g s(\pi\beta(h))(s(\pi\beta(gh)))^{-1} = \beta(g)^g\beta(h)\beta(gh)^{-1} = 1.$$

So, $Im\pi_1^* \subset Ker\delta^1$. Conversely, let $[\gamma] \in ker\delta^1$. Then, there is a continuous function $\alpha \in \tilde{C}^1(G,A)$ such that $\delta^1(\gamma) = \tilde{\delta}^1(\alpha)$. Thus,

$$s\gamma(g)^g s\gamma(h)(s\gamma(gh))^{-1} = {}^g\alpha(h)\alpha(gh)^{-1}\alpha(g), \forall g,h \in G.$$

Assume $\beta(g) = s\gamma(g)\alpha(g)^{-1}, \forall g \in G$. Since $A \subset Z(B)$, then, β is a continuous derivation from G to B. Also $\pi\beta = \gamma$. Hence, $\pi_1^*([\beta]) = [\gamma]$.

The following two corollaries are immediate consequences of Theorem 4.1.

Corollary 4.2. *Let* $1 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ *be a short exact sequence of discrete G-modules, and G-homomorphisms then, there is the exact sequence (i) of pointed sets.*

Corollary 4.3. *Let* $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 1$ *be a central short exact sequence of discrete G-modules and G-homomorphisms then, there is the exact sequence (ii) of pointed sets.*

Remark 4.4. *If we restrict ourselves to the discrete coefficients then, Corollary 4.2 and Corollary 4.3 are the same as Proposition 36 and Proposition 43 in [6, Chapter I], respectively.*

Lemma 4.5. *Let* G *be a connected group, and* A *a totally disconnected abelian topological G-module. Then,* $H^n(G, A) = 0$ *for every* $n \geq 1$.

Proof. Consider the coboundary maps $\tilde{\delta}^n : \tilde{C}^n(G, A) \to \tilde{C}^{n+1}(G, A)$. Since G is connected and A is totally disconnected then, G acts trivially on A, and the continuous maps from G^n into A are constant. If n is an even positive integer then, one can see that $Ker\tilde{\delta}^n = \tilde{C}^n(G, A)$ and $Im\tilde{\delta}^{n-1} = \tilde{C}^n(G, A)$. Thus, $H^n(G, A) = \frac{Ker\tilde{\delta}^n}{Im\tilde{\delta}^{n-1}} = \frac{\tilde{C}^n(G,A)}{\tilde{C}^n(G,A)} = 0$. Now suppose that n is odd. It is easy to check that $Ker\tilde{\delta}^n = 0$. Consequently, $H^n(G, A) = 0$.

Remark 4.6. *The existence of continuous section in theorem 4.1 is essential.*

For example, consider the central short exact sequence of trivial $\mathbb{S}^1$-modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\imath} \mathbb{R} \xrightarrow{\pi} \mathbb{S}^1 \longrightarrow 1 ,$$

here π is the exponential map, given by $\pi(t) = e^{2\pi it}$ and $\imath$ is the inclusion map. This central exact sequence has no continuous section. For if it has a continuous section then by [1, Lemma 3.5] , $\mathbb{R}$ is homeomorphic to $\mathbb{Z} \times \mathbb{S}^1$. This is a contradiction since $\mathbb{R}$ is connected but $\mathbb{Z} \times \mathbb{S}^1$ is disconnected . Thus, $Hom_c(\mathbb{S}^1, \mathbb{R}) = 0$. Now by Lemma 4.5,

$$H^1(\mathbb{S}^1, \mathbb{Z}) = H^1(\mathbb{S}^1, \mathbb{R}) = H^2(\mathbb{S}^1, \mathbb{Z}) = 0,$$

On the other hand,

$$H^1(\mathbb{S}^1, \mathbb{S}^1) = Hom_c(\mathbb{S}^1, \mathbb{S}^1) \neq 0.$$

Thus, we don't obtain the exact sequence (4.2).

5 Connectivity of Topological Groups

In this section by using the inflation and the restriction maps, we find a necessary and sufficient condition for connectivity of a topological group G.

Definition 5.1. *Let A be a topological G-module and A' a topological G'-module. Suppose that $\phi : G' \to G$, $\psi : A \to A'$ are continuous homomorphisms. Then, we call (ϕ, ψ) a cocompatible pair if*

$$\psi({}^{\phi(g')}a) = {}^{g'}\psi(a), \forall g' \in G', \forall a \in A.$$

For example, if N is a subgroup of G and A a topological G-module then, $(\imath, Id_A)$ is a cocompatible pair, where $\imath : N \to G$ is the inclusion map and Id_A is the identity map. Also, suppose that $\pi : G \to G/N$ is the natural projection and $\jmath : A^N \to A$ is the inclusion map. Then, $(\pi, \jmath)$ is a cocompatible pair.

Note that a cocompatible pair (ϕ, ψ) induces a natural map as follows:

$$Der_c(G, A) \to Der_c(G', A') \text{ by } \alpha \mapsto \psi\alpha\phi,$$

which induces the map:

$$(\phi, \psi)^* : H^1(G, A) \to H^1(G', A') \text{ by } [\alpha] \mapsto [\psi\alpha\phi].$$

Definition 5.2. *Let N be a subgroup of G and A a topological G-module. Suppose that $\imath : N \to G$ is the inclusion map. The induced map $(\imath, Id_A)^*$ is called the* restriction map *and it is denoted by $Res^1 : H^1(G, A) \to H^1(N, A)$.*

Definition 5.3. *Let N be a normal subgroup of G and A a topological G-module. Suppose that $\pi : G \to G/N$ is the natural projection and $\jmath : A^N \to A$ is the inclusion map. The induced map $(\pi, \jmath)^*$ is called the* inflation map *and it is denoted by $Inf^1 : H^1(G/N, A^N) \to H^1(G, A)$.*

Note that if A is an abelian topological G-modules then, Inf^1 and Res^1 are group homomorphisms.

Lemma 5.4. *Let A be a topological G-module, and N a normal subgroup of G. Then,*

(i) $H^1(N, A)$ is a G/N-set. Moreover, if A is an abelian topological G-module then, $H^1(N, A)$ is an abelian G/N-module.

(ii) $ImRes^1 \subset H^1(N, A)^{G/N}$.

Proof. (i) Since N is a normal subgroup of G, then, there is an action of G on $Der_c(N, A)$ as follows:
For every $g \in G$ we define ${}^g\alpha = \tilde{\alpha}, \forall g \in G$, with $\tilde{\alpha}(n) = {}^g\alpha({}^{g^{-1}}n), n \in N$.

In fact, $\tilde{\alpha}$ is continuous and we have:

$$\tilde{\alpha}(mn) = {}^{g}\alpha({}^{g^{-1}}(mn)) = {}^{g}\alpha({}^{g^{-1}}m{}^{g^{-1}}n) = {}^{g}\alpha({}^{g^{-1}}m)^{mg}\alpha({}^{g^{-1}}n) = \tilde{\alpha}(m)^{m}\tilde{\alpha}(n),$$

whence, $\tilde{\alpha} \in Der_c(N, A)$. It is clear that ${}^{gh}\alpha = {}^{g}({}^{h}\alpha)$. Moreover, if A is an abelian group, it is easy to verify that ${}^{g}(\alpha\beta) = {}^{g}\alpha{}^{g}\beta$. Now suppose that $\alpha \sim \beta$. Then, there is an $a \in A$ with $\beta(n) = a^{-1}\alpha(n)^{n}a, \forall n \in N$. Thus, for every $g \in G$, $n \in N$,

$${}^{g}\beta({}^{g^{-1}}n) = {}^{g}a^{-1}({}^{g}\alpha({}^{g^{-1}}n)){}^{g}({}^{g^{-1}n}a).$$

Therefore,

$$\tilde{\beta}(n) = ({}^{g}a)^{-1}\tilde{\alpha}(n)^{n}({}^{g}a), \text{ i.e., } \tilde{\alpha} \sim \tilde{\beta}.$$

Thus, the action of G on $Der_c(G, A)$ induces an action of G on $H^1(N, A)$. It is sufficient to show for every $m \in N$, ${}^{m}\alpha \sim \alpha$. In fact, for every $n \in N$

$$\begin{gathered} {}^{m}\alpha({}^{m^{-1}}n) = {}^{m}\alpha(m^{-1}nm) = {}^{m}(\alpha(m^{-1})^{m^{-1}}\alpha(n)^{m^{-1}n}\alpha(m)) = \\ {}^{m}\alpha(m^{-1})\alpha(n)^{n}\alpha(m) = \alpha(m)^{-1}\alpha(n)^{n}\alpha(m). \end{gathered}$$

Thus, $\tilde{\alpha} \sim \alpha$.

(ii) By a similar argument as in (i), we have

$${}^{g}\alpha({}^{g^{-1}}n) = \alpha(g)^{-1}\alpha(n)^{n}\alpha(g), \ \forall g \in G, n \in N$$

whence, ${}^{gN}(\alpha\imath) \sim \alpha\imath, \forall gN \in G/N$.

Lemma 5.5. *Let N be a normal subgroup of a topological group G and A a topological G-module. Then, there is an exact sequence*

$$1 \longrightarrow H^1(G/N, A^N) \xrightarrow{Inf^1} H^1(G, A) \xrightarrow{Res^1} H^1(N, A)^{G/N} .$$

Proof. The map Inf^1 is one to one: If $\alpha, \beta \in Der_c(G/N, A^N)$ and $Inf^1[\alpha] = Inf^1[\beta]$, then, $\alpha\pi \sim \beta\pi$. Thus, there is an $a \in A$ such that $\beta\pi(g) = a^{-1}\alpha\pi(g)^{g}a, \forall g \in G$. Hence, $\beta(gN) = a^{-1}\alpha(gN)^{g}a, \forall gN \in G/N$.On the other hand, if $g \in G$, then, $\alpha(gN) = \beta(gN) = 1$, and hence, $a \in A^N$. This implies that ${}^{(gN)}a = {}^{g}a, \forall g \in G$. Consequently, $\alpha \sim \beta$, i.e., Inf^1 is one to one.

Now we show that $KerRes^1 = ImInf^1$. Since $Res^1Inf^1[\alpha] = [\alpha(\pi\imath)] = 1$, then, $ImInf^1 \subset KerRes^1$.

Let $[\alpha] \in KerRes^1$. Then, there is an $a \in A$ such that $\alpha(n) = a^{-1n}a$, $\forall n \in N$. Consider the continuous derivation β with $\beta(g) = a\alpha^{g}a^{-1}$, $\forall g \in G$. Since $\beta(n) = 1, \forall n \in N$ then, β induces the continuous derivation $\gamma : G/N \to A$ via $\gamma(gN) = \beta(g)$. Also $Im\gamma \subset A^N$, since for all $n \in N$,

$${}^{n}\gamma(gN) = {}^{n}\beta(g) = \beta(ng) = \beta(g)^{g}\beta(g^{-1}ng) = \beta(g) = \gamma(gN).$$

Hence, $Inf^1[\gamma] = [\gamma\pi] = [\beta] = [\alpha]$. Consequently, $KerRes^1 \subset ImInf^1$.

Lemma 5.6. *Let G be a topological group and A a topological G-module. Suppose that A is totally disconnected and G_0 the identity component of G. Then, the map*

$$H^1(G/G_0, A) \xrightarrow{Inf^1} H^1(G, A)$$

is bijective.

Proof. Since G_0 acts trivially on A, then, $A^{G_0} = A$. On the other hand, $H^1(G_0, A) = 1$. Thus, by Lemma 5.5, the sequence

$$0 \longrightarrow H^1(G/G_0, A) \xrightarrow{Inf^1} H^1(G, A) \longrightarrow 0$$

is exact.

Theorem 5.7. *Let G be a topological group which has an open component. Then, G is connected iff $H^1(G, A) = 1$ for every discrete abelian G-module A.*

Proof. Assume G is a connected group and A a discrete abelian G-module. Since every discrete G-module A is totally disconnected then, $H^1(G, A) = 1$.

Conversely, Suppose that $H^1(G, A) = 1$, for every discrete abelian G-module A. By Lemma 5.6, $H^1(G/G_0, A) = 1$, for every discrete abelian G-module A. Since G/G_0 is discrete, then, the cohomological dimension of G/G_0 is equal to 0 which implies that $G/G_0 = 1$ [4, Chapter VIII], i.e., $G = G_0$.

6 Complements and First Cohomology

Let G and A be topological groups. Suppose that $\chi : G \times A \to A$ is a continuous map such that $\tau_g : A \to A$, defined by $\tau_g(a) = \chi(g, a)$, is a homeomorphic automorphism of A and the map $g \mapsto \tau_g$ is a homomorphism of G into the group of homeomorphic automorphisms, $Aut_h(A)$, of A. By $G \ltimes_\chi A$ we mean the (topological) semidirect product with the group operation, $(g, a)(h, b) = (gh, \tau_h(a)b)$, and the product topology of $G \times A$. Sometimes for simplicity we denote $G \ltimes_\chi A$ by $G \ltimes A$ and view G and A as topological subgroups of $G \ltimes A$ in a natural way. So every element e in $G \ltimes N$ can be written uniquely as $e = gn$ for some $g \in G$ and $n \in N$.

Let $E = G \ltimes N$. A subgroup X of E such that $E \simeq X \ltimes N$ is called a complement of N in E. Indeed, any conjugate of G is a complement.

We show that the complements of N in E correspond to continuous derivations from G to N. If X is any complement, for every $g \in G$, then, g^{-1} has a unique expression of the form $g^{-1} = xn$ where $x \in X$ and $n \in N$. Define $\alpha_X : G \to N$ by $\alpha_X(g) = n$. Obviously, $\alpha_X(g) = \pi_2|_G(g^{-1})$, where

$\pi_2 : X \ltimes N \to N$ is given by $\pi_2(x,n) = n$. Hence, α_X is continuous. Now if $g_i \in G$ then, $g_i^{-1} = x_i n_i$ for some $x_i \in X$, $n_i \in N$, $i = 1, 2$. We have:

$$(g_1g_2)^{-1} = g_2^{-1}g_1^{-1} = x_2n_2x_1n_1 = x_2x_1^{x_1^{-1}}n_2n_1 = x_2x_1n_1^{(n_1^{-1}x_1^{-1})}n_2 = x_2x_1n_1^{g_1}n_2.$$

By definition of α_X, $\alpha_X(g_1g_2) = \alpha_X(g_1)^{g_1}\alpha(g_2)$, i.e., $\alpha_X \in Der_c(G,N)$.

So, we have associated a continuous derivation with each complement. Conversely, suppose that $\alpha : G \to N$ is a continuous derivation. Then, $X_\alpha = \{\alpha(g)g | g \in G\} \subset E$ is a corresponding complement to α in E. Obviously, the continuous map $\kappa : g \mapsto \alpha(g)g$, is a homomorphism. Suppose that $\pi_1 : G \ltimes N \to G$ is given by $\pi_1(g,n) = g$. Hence, $\pi_1|_{X_\alpha} : X_\alpha \to G$ is the inverse of κ, since $\pi_1|_{X_\alpha}(\alpha(g)g) = \pi_1|_{X_\alpha}(g^{g^{-1}}n) = g, \forall g \in G$. Thus, $X_\alpha \simeq G$.

Define the map $\chi : X_\alpha \times N \to N$ by $\chi(\alpha(g)g) = {}^g n$, for all $g \in G, n \in N$. Clearly, χ is a continuous map. Hence, $X_\alpha \ltimes_\chi N \simeq G \ltimes N = E$.

In fact we have proved the following theorem.

Theorem 6.1. *Let G be a topological group and N a topological G-module. Then, the map $X \mapsto \alpha_X$ is a bijection from the set of all complements of N in $G \ltimes N$ onto $Der_c(G,N)$.*

Theorem 6.2. *If A is a topological G-module then, there is a map from $H^1(G,A)$ onto the set of conjugacy classes of complements of A in $G \ltimes A$. Moreover, if A is an abelian group then, this map is one to one.*

Proof. Suppose that X and Y are the complements of A in $G \ltimes A$ such that $\alpha_X \sim \alpha_Y$. Hence, there is $a \in A$ such that $\alpha_Y(g) = a^{-1}\alpha_X(g)^g a$, $\forall g \in G$. Thus, for each $g \in G$, we have $\alpha_Y(g)g = a^{-1}\alpha_X(g)^g ag = a^{-1}\alpha_X(g)ga$. This implies that $X = {}^{a^{-1}}Y$.

Moreover, suppose that A is an abelian group and X and Y are conjugate complements. So, $X = {}^nY$ for some $n \in N$. If $g \in G$, then, $\alpha(g)g \in X$ where α_X is a continuous derivation arising from X. Hence, $\alpha_X(g)g = {}^ny$ for some $y \in Y$. Now ${}^ny = [n,y]y$, so, $\alpha_X(g)g = [n,y]y$, which shows that

$$[n,g] = ngn^{-1}g^{-1} = n(\alpha_X(g)^ny)n^{-1}(\alpha_X(g)^ny)^{-1} =$$
$$n(\alpha_X(g))nyn^{-1}n^{-1}ny^{-1}n^{-1}(\alpha_X(g))^{-1} =$$
$$(n\alpha_X(g))(nyn^{-1}y^{-1})(n^{-1}\alpha_X(g)^{-1}) = [n,y] = {}^nyy^{-1}$$

because A is an abelian group. Therefore,

$$g^{-1} = y^{-1}[n,y]^{-1}\alpha_X(g) = y^{-1}(y^ny^{-1}\alpha_X(g)) = y^{-1}(y\alpha_X(g)^ny^{-1}).$$

Thus, by definition of α_Y, we get $\alpha_Y(g) = y\alpha_X(g)^ny^{-1}$. Consequently, $\alpha_X \sim \alpha_Y$.

As an immediate result, we have the following corollary.

Corollary 6.3. *Let A be a topological G-module and $H^1(G, A) = 1$. Then, the complements of A in $G \ltimes A$ are conjugate.*

7 Vanishing of $H^1(G, A)$

Let G be a compact Hausdorff group and A a topological G-module. Suppose that A is an almost connected locally compact Hausdorff group. Then, we prove there exists a G-invariant maximal compact subgroup K of A, and for every such topological submodule K, the natural map $\iota_1^* : H^1(G, K) \to H^1(G, A)$ is onto. In addition, as a result, If A has trivial maximal compact subgroup then, $H^1(G, A) = 1$.
Recall that G is almost connected if G/G_0 is compact where G_0 is the connected component of the identity of G.

Definition 7.1. *An element $g \in G$ is called periodic if it is contained in a compact subgroup of G. The set of all periodic elements of G is denoted by $P(G)$.*

Definition 7.2. *A maximal compact subgroup K of a topological group G is a subgroup K that is a compact space in the subspace topology, and maximal amongst such subgroups.*

If a topological group G has a maximal compact subgroup K, then, clearly gKg^{-1} is a maximal compact subgroup of G for any $g \in G$. There exist topological groups with maximal compact subgroups and compact subgroups which are not contained in any maximal one [3]. Note that if G is almost connected then, $P(G/G_0) = G/G_0$.

Lemma 7.3. *Let G be a locally compact topological group such that $P(G/G_0)$ is a compact subgroup of G/G_0, and K a maximal compact subgroup of G. Then, any compact subgroup of G can be conjugated into K [3, Theorem 1].*

Lemma 7.4. *Let G be a compact group and A a topological G-module such that A is a locally compact almost connected, and let C be a G-invariant compact subgroup of A. Then, there exists a G-invariant maximal compact subgroup K of A which contains C.*

Proof. Let $E = G \ltimes A$, be the semidirect product of A and G with respect to the action of G on A. Note that topologically E is the product of A and G. We first observe that E/E_0 is almost connected. Let A_0, G_0 and E_0 be the components of A, G and E, respectively. It is easily seen that $E_0 = A_0 \times G_0$. Also $E/(A_0 \times G_0)$ is homeomorphic to the compact space $A/A_0 \times G/G_0$. Hence, E/E_0 is compact. Consequently, E is almost connected. Now, by assumption, C is a G-invariant compact subgroup of A. Thus, $G \ltimes C$ is a compact subgroup of E. Since E is almost connected, there exists a maximal compact subgroup L of E which contains $G \ltimes C$. Let $K = L \cap A$. Since K is a closed subspace of L, then, K is compact. Also L contains G. Thus, L is G-invariant. In fact, for every $g \in G$ and every $\ell \in L$, we have ${}^{g}\ell = g\ell g^{-1} \in L$. This immediately implies that K is G-invariant, since A is G-invariant. Let K' be a compact subgroup of G. By Lemma 7.3, there is $e \in E$ such that $eK'e^{-1} \subset L$. Thus, $eK'e^{-1} \subset L \cap A = K$. But there exist $g \in G$ and $a \in A$ such that $e = ga$. Thus, $aK'a^{-1} \subset g^{-1}Kg = {}^{g^{-1}}K = K$. Therefore, K is a G-invariant maximal compact subgroup of A which contains C.

Theorem 7.5. *Let G be a compact Hausdorff group and A a topological G-module. Let A be an almost connected locally compact Hausdorff group. Then, there exists a G-invariant maximal compact subgroup K of A, and for every such topological submodule K, the natural map $\iota_1^* : H^1(G,K) \to H^1(G,A)$ is onto.*

Proof. By Lemma 7.4, there exists a G-invariant maximal compact subgroup K of A. Also $G \ltimes K$ is a maximal compact subgroup of $G \ltimes A$ [2, Theorem 1.1]. Let $\alpha : G \to A$ be a continuous derivation. Then, define the continuous homomorphism $\kappa : G \to G \ltimes A$ via $g \mapsto \alpha(g)g$. Since κ is a continuous homomorphism then, $\kappa(G)$ is a compact subgroup of $G \ltimes A$. By Lemma 7.3 there is $ag \in G \ltimes A$ such that $(ag)\kappa(G)(ag)^{-1} \subset G \ltimes K, \forall x \in G$. This is equivalent to $(ag)\alpha(x)x(ag)^{-1} \subset G \ltimes K, \forall x \in G$. Hence, for all $x \in G$, ${}^{g}[{}^{g^{-1}}a\alpha(x){}^{x}({}^{g^{-1}}a^{-1})]gxg^{-1} \in G \ltimes K$. Since K is G-invariant then, $({}^{g^{-1}}a)\alpha(x){}^{x}({}^{g^{-1}}a^{-1}) \in K, \forall x \in G$. Now define $\beta : G \to K$ by $\beta(x) = ({}^{g^{-1}}a)\alpha(x){}^{x}({}^{g^{-1}}a^{-1}), \forall x \in G$. Hence, $\iota_1^*([\beta]) = [\alpha]$, i.e., ι_1^* is onto map.

Corollary 7.6. *Let G be a compact Hausdorff group and A a topological G-module. Let A be an almost connected locally compact Hausdorff group with the trivial maximal compact subgroup. Then, $H^1(G,A) = 1$.*

Proof. It is clear.

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