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Some Results on Monotonic Analysis

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Abstract

The theory of increasing convex along- rays (ICAR)functions and desreasing convex along -rays(DCAR)functions,definded on a convex cone in a locally convex topological vector space X,is well developed.In this article ,we peresent a suitable extension of this theory for ICAR functions, definded on the whole of the space X.

Keywords: *monotonic analysis; abstract convexity*

1 Introduction

The role of convex analysis in optimization is well-knowne.One of the major properties of aconvex function is its representation as the upper envelope of a family of affine functions.A function is said to be abstract convex ,if and only if it can be represented as the upper envelope of a class of function,usually terms as elementary functions.One of the first studies in abstrac convexity concerned the analysis of increasing and positively homogeneous(IPH) functions.And we study increasing and convex along rays(ICAR)functions over topological vector space. We shall use the following notations:

2 Main Results

Let H be a set of arbitrary functions $h : X \to \mathcal{R}_{+\infty}$. A function $f : X \to \mathcal{R}_{+\infty}$ is called abstract convex with respect to H (H-convex) if there exists a set$U \subseteq H$, such that $f(x) = suph(x) : h \in U\}$. Let X be a topological vector space. A set$K \subseteq X$ is called conic, if $\lambda k \subseteq K$ for all $\lambda > 0$ We assume that X is equipped with a closed convex pointed cone K (the letter means that $K \cap -K = 0$). The increasing property of our functions will be understood to be with respect to the orderinginduced on X by K:

$$x \leq y \Longleftrightarrow y - x \in K$$

Definition 2.1. A function$f : X \longrightarrow \mathcal{R}_{+\infty}$ is called convex along rays (shortly CAR) if the function $f_x(\alpha) = f(\alpha x)$ is convex on the ray $[0, +\infty)$ for each $x \in X$.

Definition 2.2. A function $p : X \longrightarrow \mathcal{R}_{+\infty}$ is said to be IPH (Increasing and Positively Homogeneous) if
(i) $x, y \in K$, $x \leq y \Longrightarrow p(x) \leq p(y)$.
(ii)$p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha > 0$.

Also, the function $f : X \to \mathcal{R}_{+\infty}$ is called decreasing if $x \leq y \Longrightarrow f(x) \leq f(y)$. In the sequel, we shall study the ICAR and DCAR functions. Consider the coupling function $\ell : X \times X \to \mathbb{R}_+$ defined by

$$\ell(x, y) := \max\{\lambda \geq 0 \ : \ \lambda y \leq x\}$$

with the conventions $\max \mathbb{R} := +\infty$, $max\emptyset := 0$

3 DCAR Functions

In this section, we shall study DCAR functions defined on X.

Definition 3.1. Function $f : X \to \mathcal{R}_{+\infty}$ to be hyperbolically convex,if the function $\varphi : \mathcal{R}_{++} \to \mathcal{R}_{+\infty}$ definded by $\varphi(t) = f(t^{-1})$ is convex.

Notice that an hyperbolically convex function need not be convex.However, in this article,we will consider only decreasing hyperbolically convex functions,which are necessarily convex:

Theorem 3.2. *Every decreasing hyperbolically convex function $f : X \to \mathcal{R}_{+\infty}$ is convex.*

Proof The equality $\varphi(t) = f(t^{-1})$,$\varphi(t^{-1}) = f(t)$, show that f is the composition of an increasing convex function whit a convex function.

Definition 3.3. A function $f : X \to \mathcal{R}_{+\infty}$ is called hyperbolically DCAR ,if it is desreasing and for every $x \in X$ the unction $f_x(\alpha) = f(\alpha x)$ is hyperbolically convex on the halfline $(0, +\infty)$.In other words, f is hyperbolically DCAR if only if f is desreasing and the function $\alpha \mapsto f((\frac{1}{\alpha})x)$ is convex.

Corollary 3.4. *Every hyperbolically DCAR function $f : X \to \mathcal{R}_{+\infty}$ is DCAR.*

Proof by definition (3.3) f is desreasing and f_x is hyperbolically convex on the halfline $(0, +\infty)$. thus f is DCAR.

Let

$$l^x(y) := l(x, y) \qquad x \in X, y \in X$$

Note that the function l^x is positively homogeneous of degree -1 and decreasing. For each $x \in X$,$\gamma \in \mathcal{R}$ consider the function $h_{x,\gamma}(y) := l^x(y) - \gamma$.We can identify $h_{x,\gamma}(y)$ whit the pair $(x, \gamma) \in X \times \mathcal{R}$.Let $\mathcal{H} := \{h_{x,\gamma}(y) : x \in X, \gamma \in \mathcal{R}\}$.

Proposition 3.5. [5]*Let $f : \mathcal{R}_+ \longrightarrow \mathcal{R}_{+\infty}$ be an increasing lsc convex function. Then there exists a closed convex set $V \subset \mathcal{R}_+ \times \mathcal{R}$ such that $f(t) = \sup\limits_{v \in V} \{v_1 t - v_2\}$ for each $t \in \mathcal{R}_+$.*

Theorem 3.6. *A function $g : X' \longrightarrow \mathcal{R}_+\infty$ is $\mathcal{H}$-convex if and only if it is hyperbolically DCAR and lsc-along-rays.*

Proof One can easily check that $\mathcal{H}$ consists of lsc-along-rays hyperbolically DCAR functions; hence every $\mathcal{H}$-convex function is hyperbolically DCAR and lsc-along-rays.

Conversely, in order to prove that g is $mathcal H$-convex ,we need to introduce some ntation. For $x \in X$,define $\varphi_x : \mathcal{R}_{++} \longrightarrow \mathcal{R}_{+\infty}$ by $\varphi_x(t) := tg(tx)$ whit conjugate φ^* .consider the set

$$V := \{(x, \nu) : x = -\varphi^*_{x'}(\nu)x' for\, some x' \in X' and\, \nu \in \mathcal{R} such\, that \varphi^*_{x'}(\nu) \leq 0\}$$

Let $x' \in X'$ and $\nu \in \mathcal{R}$,be as in the definition of V , and let $y \in X'$ and $\alpha > 0$ be such that $\alpha y \leq x := -\varphi^*_{x'}(\nu)x'$.Since this ineguality implies that $x \neq 0$,and hence$\varphi^*_{x'}(\nu) < 0$,one has

$$\begin{aligned}
\alpha + \nu &= \frac{\alpha}{-\varphi^*_{x'}(\nu)} \left(\frac{-\varphi^*_{x'}(\nu)}{\alpha} \nu - -\varphi^*_{x'}(\nu)\right) \\
&\leq \frac{\alpha}{-\varphi^*_{x'}(\nu)} \varphi_{x'}\left(\frac{-\varphi^*_{x'}(\nu)}{\alpha}\right) = g\left(\frac{-\varphi^*_{x'}(\nu)}{\alpha} x'\right) \\
&= g\left(\frac{x}{\alpha}\right) \leq g(y);
\end{aligned}$$

hence

$$h_{x,-\nu}(y) := l(x, y) + \nu \leq g(y)$$

Thus

$$h_{x,-\nu} : (x, \nu) \in V \subset supp(g, \mathcal{H}$$

It follows from [2,proposition 9] that

$$g(y) = \varphi_y(1) = \sup_{v \in dom(\varphi_y)^*} \{v - (\varphi_y)^*(\nu)\} = \sup_{v \in dom(\varphi_y)^*} \{v + l(-(\varphi_y)^*(\nu)y, y\}$$
$$= \sup_{v \in dom(\varphi_y)^*} h_{-(\varphi_y)^*(\nu)y,-\nu}(y).$$

Therfore,

$$g = \sup_{h \in supp(g,\mathcal{H})} h$$

4 ICAR Functions Defined on L'

Let $X' = X \backslash (-K)$ and $L' = \{L_y : y \in X'\}$.Note that L' is a conic set in the vector space PH(X) of all positively homogeneous of degree one functions p defined on X.We consider an arbitrary topology on PH(X) such thatL' is closed (for example,the topology of pointwise convergence).

Theorem 4.1. *The mapping $\psi : X' \to L'$ defined by $\psi(y) := l_y$ is a bijection fromX' ontoL',and*

$$y_1 \leq y_2 \Longleftrightarrow \ell_{y_2} \leq \ell_{y_1} \qquad y_1, y_2 \in X'$$

and antihomogeneous (positively homogeneous of degree 1):

$$l_{\alpha y} = \frac{1}{\alpha} l_y \qquad y \in X', \alpha > 0$$

Proof Since, by the definition of L', $\psi'$is obviously onto, we only have to prove that ψ' is one-to-one. To this aim, assume that$y_1, y_2 \in X'$ are such that$l_{y_1} = l_{y_2}$. Thus$1 = l(y_1, y_1) = l(y_1, y_2)$ .Hence, we get $y_2 \leq y_1$. By symmetry it follows that $y_2 \geq y_1$. Since K is pointed, we conclude that $y_2 = y_1$.

Assume now that $l_{y_2} \leq l_{y_1}$.Then either $y_2 = 0$,whence $\ell_{y_2} = +\infty = \ell_{y_1}$ so that $y_1 = y_2$,or $y_2 \neq 0$ and hence,

$$1 = \ell_{y_2}(y_2) \leq \ell_{y_1}(y_2) = \max\{\lambda \geq 0 \ : \ \lambda y_1 \leq y_2\}$$

Which implies that .$y_1 \leq y_2$,converse follows from definition of l .

For each function $g : X' \to \mathcal{R}_{+\infty}$ consider the function $\widetilde{g} := go\psi^{-1}$ .In other words ,$\widetilde{g}$ is the function defined on$L'$by $\widetilde{g}(\ell_y) := g(y)$.It follows from (4.1) that g is decreasing if and only if $\widetilde{g}$ is inreasing and g is an hyperbolically DCAR function if and only if $\widetilde{g}$ is an ICAR function defined on L. Clearly g is lsc-along-rays if and only if $\widetilde{g}$ is lsc-along-rays.

For each $x \in X$ defined $\widetilde{x}$ by

$$\widetilde{x}(\ell) := \ell(x) \quad \forall l \in L'$$

Thus if $\ell = \ell_y$

$$\widetilde{x}(\ell_y) = \ell_y(x) = \ell(x, y).$$

Note that $\widetilde{x}$ is a positively homogeneous of degree one function.Indeed,if $l = l_y$ and $\lambda > 0$, then

$$\widetilde{x}(\lambda\ell_y) = \widetilde{x}(\ell_{\frac{y}{\lambda}}) = \ell(x, \frac{y}{\lambda}) = \lambda\ell(x, y) = \lambda\widetilde{x}(\ell_y)$$

We now show that $\widetilde{x}$ is increasing.Let $l_{y_1}, l_{y_2} \in L'$ and $l_{y_1} \geq l_{y_2}$, then

$$\ell(x, y_1) \geq \ell(x, y_2)$$

hence

$$\widetilde{x}(\ell_{y_1}) \geq \widetilde{x}(\ell_{y_2}).$$

Thus $\widetilde{x}$ is an IPH function.

Denote by $\mathcal{H}$ the set of all functions of the form$h_{x,\gamma}$ whit $x \in X'$ and $\gamma \in \mathcal{R}$ defined on L' by $h_{x,\gamma}(l) := \widetilde{x}(l) - \gamma$.

The proof of the following proposition is similar to (3.6),and therefore we omit it.

Proposition 4.2. *A function $\widetilde{g} : L' \longrightarrow \mathcal{R}_{+\infty}$ is $\widetilde{\mathcal{H}}$-convex if and only if $\widetilde{g}$ is ICAR and lsc-along-rays.*

From proposition (4.2) *it follow that a function $\widetilde{g} : L' \longrightarrow \mathcal{R}_{+\infty}$ is conjugate of a function $f : X \longrightarrow \mathcal{R}_{+\infty}$ if and only if it is ICAR and lsc-along -rays.*

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