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A Survey on Hahn Sequence Space

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Abstract

In this work, we investigate the studies related to the Hahn sequence space.

Keywords: *Sequence space, beta- and gamma-duals, $BK-$spaces, matrix transformations, geometric properties of sequence spaces.*

1 Introduction

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences. We write ℓ_∞, c and c_0 for the classical spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the space of all bounded, convergent, absolutely and p-absolutely convergent series, respectively. We define $bv, d\ell, \sigma_\infty, \sigma_c, \sigma s, dE$ and $\int E$ as follows:

$$\begin{aligned}
bv &= \Big\{x=(x_k)\in\omega : \sum_{k=1}^{\infty}|x_k-x_{k-1}|<\infty\Big\},\\
d\ell_1 &= \Big\{x=(x_k)\in\omega : \sum_{k=1}^{\infty}\frac{1}{k}|x_k|<\infty\Big\},\\
\sigma_\infty &= \Big\{x=(x_k)\in\omega : \sup_n \frac{1}{n}\Big|\sum_{k=1}^{n}x_k\Big|<\infty\Big\},\\
\sigma_c &= \Big\{x=(x_k)\in\omega : \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}x_k \ \text{ exists } \Big\},\\
\sigma s &= \Big\{x=(x_k)\in\omega : (C-1)-\sum x_k=\lim_{n\to\infty}\sum_{k=1}^{n}\Big(1-\frac{k-1}{n}\Big)x_k \ \text{ exists } \Big\},\\
dE &= \big\{x=(x_k)\in\omega : (k^{-1}x_k)\in E\big\},\\
\int E &= \big\{x=(x_k)\in\omega : (kx_k)\in E\big\},
\end{aligned}$$

where dE and $\int E$ are called the differentiated and integrated spaces of E, respectively.

A coordinate space (or $K-$space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise.

An $FK-$*space* is a locally convex Fréchet space which is made up of sequences and has the property that coordinate projections are continuous.

A $BK-$ *space* is locally convex Banach space which is made up of sequences and has the property that coordinate projections are continuous.

A $BK-$space X is said to have AK *(or sectional convergence)* if and only if $\|x^{[n]}-x\|\to 0$ as $n\to\infty$.

X has $(C,1)-AK$ *(Cesàro-sectional convergence of order one)* if for all $x\in X$ and

$$(P_n^1(x))_k=\begin{cases}\Big(1-\frac{k-1}{n}\Big)x_k & , \text{ if } \quad k=1,2,...,n\\ 0 & , \text{ if } \quad k=n+1,n+2,...\end{cases}$$

$P_n^1(x)\in X$ and $\|P_n^1(x)-x\|_X\to 0 \quad (n\to\infty)$.

Let X be an $FK-$space. A sequence (x_k) in X is said to be *weakly Cesàro bounded*, if $\big\{[f(x_1)+f(x_2)+...+f(x_k)]/k\big\}$ is bounded for each $f\in X'$, the dual space of X.

Let Φ stand, for the set of all finite sequences. The space X is said to have AD (or) be an AD space if Φ is dense in X. We note that $AK \Rightarrow AD$ [3].

An $FK-$space $X \supset \Phi$ is said to have AB if $(x^{[n]})$ is bounded set in X for each $x \in X$.

Let X be a $BK-$space. Then X is said to have *monotone norm* if $\|x^{[m]}\| \geq \|x^{[n]}\|$ for $m > n$ and $\|x\| = sup_{[n]}\|x^{[n]}\|$.

Let $D = \{x \in \Phi : \|x\| \leq 1\}$ be in a $BK-$space X, that is, D is the intersection of the closed unit sphere(disc) with Φ. A subset E of Φ is called a *determining set* for X if and only if its absolutely convex hull K is identical with D [18].

The normed space X is said to be *rotund* if and only if $\|(x + y)/2\| < 1$, whenever $x \neq y$ and $\|x\| = \|y\| \leq 1$ in X [17].

The set $S(\lambda, \mu)$ defined by

$$S(\lambda, \mu) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \quad \text{for all} \quad x = (x_k) \in \lambda\} \qquad (1)$$

is called the *multiplier space* of the sequence spaces λ and μ. One can eaisly observe for a sequence space υ with $\lambda \supset \upsilon \supset \mu$ that the inclusions

$$S(\lambda, \mu) \subset S(\upsilon, \mu) \quad \text{and} \quad S(\lambda, \mu) \subset S(\lambda, \upsilon)$$

hold. With the notation of (1), the alpha-, beta-, gamma- and sigma-duals of a sequence space λ, which are respectively denoted by λ^{α}, λ^{β}, λ^{γ} and λ^{σ} are defined by

$$\lambda^{\alpha} = S(\lambda, \ell_1), \quad \lambda^{\beta} = S(\lambda, cs) \quad \lambda^{\gamma} = S(\lambda, bs) \quad \text{and} \quad \lambda^{\sigma} = S(\lambda, \sigma s).$$

For each fixed positive integer k, we write $\delta^k = \{0, 0, ..., 1, 0, ...\}$, 1 in the $k - th$ place and zeros elsewhere. Given an $FK-$space X containing Φ, its conjugate is denoted by X' and its $f-$dual or sequential dual is denoted by X^f and is given by $X^f = \{$ all sequences $(f(\delta^k)) : f \in X'\}$. An $FK-$space X containing Φ is said to be semi replete if $X^f \subset \sigma(\ell_{\infty})$. The space bv is semi replete, because $bv = bs$ [13].

Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$. The sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk}x_k \text{ for each } n \in \mathbb{N}. \tag{2}$$

Throughout the text, for short we suppose that the summation without limits runs from 1 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (2) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x.

Lemma 1.1. *([18], Theorem 7.2.7) Let X be an $FK-$space with $X \supset \Phi$. Then,*

(i) $X^\beta \subset X^\gamma \subset X^f$.

(ii) If X has AK, $X^\beta = X^f$.

(iii) If X has AD, $X^\beta = X^\gamma$.

2 Hahn Sequence Space

Hahn [7] introduced the $BK-$space h of all sequences $x = (x_k)$ such that

$$h = \left\{ x : \sum_{k=1}^{\infty} k|\Delta x_k| < \infty \text{ and } \lim_{k\to\infty} x_k = 0 \right\},$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The following norm

$$\|x\|_h = \sum_k k|\Delta x_k| + \sup_k |x_k|$$

was defined on the space h by Hahn [7] (and also [6]). Rao ([12], Proposition 2.1) defined a new norm on h as $\|x\| = \sum_k k|\Delta x_k|$.

Hahn proved following properties of the space h:

Lemma 2.1. *(i) h is a Banach space.*

(ii) $h \subset \ell_1 \cap \int c_0$.

(iii) $h^\beta = \sigma_\infty$.

Clearly, h is a $BK-$space [6].

In [6], Goes and Goes studied functional analytic properties of the $BK-$space $bv_0 \cap d\ell_1$. Additionally, Goes and Goes considered the arithmetic means of sequences in bv_0 and $bv_0 \cap d\ell_1$, and used an important fact which the sequence of arithmetic means $(n^{-1}\sum_{k=1}^{n} x_k)$ of an $x \in bv_0$ is a quasiconvex null sequence.

Rao [12] studied some geometric properties of Hahn sequence space and gave the characterizations of some classes of matrix transformations.

Now we give some additional properties of h which proved by Goes and Goes [6].

Theorem 2.2. *([6], Theorem 3.2)* $h = \ell_1 \cap \int bv = \ell_1 \cap \int bv_0$.

Proof. For $k = 1, 2, ...$

$$k\Delta x_k = x_{k+1} + \Delta(kx_k). \tag{3}$$

Hence $x \in h$ implies

$$\infty > \sum_{k=1}^{\infty} k|\Delta x_k| \geq \sum_{k=1}^{\infty} |\Delta(kx_k)| - \sum_{k=1}^{\infty} |x_{k+1}|.$$

The last series is convergent since $h \subset \ell_1$ by Part of (ii) of Lemma 2.1. Hence also $\sum_{k=1}^{\infty} |\Delta(kx_k)| < \infty$ and therefore $h \subset \ell_1 \cap \int bv$.

Conversely, (3) implies for $x \in \ell_1 \cap \int bv$ that

$$\infty > \sum_{k=1}^{\infty} |x_{k+1}| + \sum_{k=1}^{\infty} |\Delta(kx_k)| \geq \sum_{k=1}^{\infty} k|\Delta x_k| \quad \text{and} \quad \lim_{k\to\infty} x_k = 0.$$

Thus, $\ell_1 \cap \int bv \subset h$. Hence, we have shown that $h = \ell_1 \cap \int bv$. The second equality in the theorem follows now from Lemma 2.1(ii). □

Lemma 2.3. *([6], Lemma 3.3) If X and Y are $\beta-$dual ($\sigma-$dual) Köthe spaces, then $X \cap Y$ is also a $\beta-$dual ($\sigma-$dual) Köthe space.*

Proof. We use the known fact that if $\zeta = \beta$ or $\zeta = \sigma$, then E is a $\zeta-$dual Köthe space if and only if $E = (E^{\zeta})^{\zeta} \equiv E^{\zeta\zeta}$ ([5], p.139, Theorem 3). Hence if X and Y are $\zeta-$dual Köthe spaces, then

$$(X \cap Y)^{\zeta\zeta} = (X^{\zeta} + Y^{\zeta})^{\zeta\zeta\zeta} = (X^{\zeta} + Y^{\zeta})^{\zeta} = X^{\zeta\zeta} \cap Y^{\zeta\zeta} = X \cap Y.$$

□

Theorem 2.4. *([6], Theorem 3.4)* $h = (\sigma_\infty)^\beta$.

Proof. By Part (iii) of Lemma 2.1, $h^\beta = \sigma_\infty$. Hence by the remark in the beginning of the last proof it is enough to show that h is a $\beta-$dual Köthe space. In fact: By Theorem 2.2, $h = \ell_1 \cap \int bv$ and as is well known $\ell_1 = (c_0)^\beta$ and $\int bv = (d(cs))^\beta$ since $bv = (cs)^\beta$. □

Theorem 2.5. *([6], Theorem 3.5) h is a $BK-$space with AK.*

Proof. ℓ_1 and $\int bv_0$ with norms $\|x\| = \sum_k |x_k|$ and $\|x\| = \sum_k |\Delta(kx_k)|$ respectively are $BK-$spaces with AK. Hence by Theorem 2.2 and since the intersection of two $BK-$spaces with AK is again a $BK-$space with AK ([15], p.500), the theorem follows. □

This result is found in [12] with a different norm.

Remark 2.6. *([6], 3.6) Let E be a $BK-$space with AK, and let E' be the conjugate space of E, i.e. the space of linear continuous functionals on E. It is known that E' can be identified with E^β through the isomorphism $\varphi \in E' \Leftrightarrow (\varphi(e^i))_{i=1}^\infty \in E^\beta$, where*

$$(e^i)_k = \begin{cases} 1 & , \text{ if } k = i \\ 0 & , \text{ if } k \neq i \end{cases}$$

This is true because E has AK if and only if every $\varphi \in E'$ can be written in the form

$$\varphi(x) = \sum_{k=1}^\infty x_k y_k, \qquad (x \in E);$$

where $y \in E^\beta$ [21]. As usual we write $E^\beta = E'$ if E is a $BK-$space with AK. Analogously we have $E^\sigma = E'$ if E is a $BK-$space with $(C,1) - AK$.

Theorem 2.7. *([6], Theorem 3.7)*

(i) $h' = \sigma_\infty$.

(ii) $(\sigma_0)' = h$.

Proof. (i) By Theorem 2.5, h is a $BK-$space with AK and by Lemma 2.1(ii) $h^\beta = \sigma_\infty$. Hence by Remark 2.6, $h' = \sigma_\infty$.
(ii) It is known that σ_0 is a $BK-$space with AK([21], p.61) with the norm

$$\|x\| = \sup_n \frac{1}{n} \left| \sum_{k=1}^n x_k \right|.$$

Hence, again by Remark 2.6, $(\sigma_0)^\beta = (\sigma_0)'$. It remains to be shown that $(\sigma_0)^\beta = h$.

By Zeller([21], Theorem 1.1), $\int \sigma_0 = \int c_0 + cs$. Hence $\sigma_0 = c_0 + d(cs)$ and this implies $(\sigma_0)^\beta = (c_0)^\beta \cap [d(cs)]^\beta = h$ by Theorem 2.2, since $(c_0)^\beta = \ell$ and $(d(cs))^\beta = \int bv$. □

Remark 2.8. *([6], p.97) $\sigma_0 \subset \sigma_c \subset \sigma_\infty$ and $(\sigma_0)^\beta = (\sigma_\infty)^\beta = h$ (by Theorem 2.4 and 2.7) imply $\sigma_c^\beta = h$ (see also [19], p.268).*

Proposition 2.9. *([12], Proposition 2) The space h is not rotund.*

Proof. We take $x = (1, 0, 0, ...)$ and $y = (1/2, 1/2, 0, 0, ...)$. Then x and y are in h. Also $\|x\| = \|y\| = 1$. Obviously $x \neq y$. But $x + y = (3/2, 1/2, 0, ...)$ and $\|(x + y)/2\| = 1$. Therefore, h is not rotund.

□

Proposition 2.10. *([12], Proposition 3) The unit disc in the space h has extreme points.*

Proof. For each fixed $k = 1, 2, ...$, we write

$$s^k = \frac{1}{k}\sum_{i=1}^{k} \delta^i. \tag{4}$$

Let $y = (y_k)$ be any point of h such that $\|s^k + y\| \leq 1$ and $\|s^k - y\| \leq 1$. But

$$\|s^k + y\| = \sum_{\upsilon \geq 1, \upsilon \neq k} \upsilon|\Delta y_\upsilon| + k\big|1/k + y_k - y_{k+1}\big| \tag{5}$$

and

$$\|s^k - y\| = \sum_{\upsilon \geq 1, \upsilon \neq k} \upsilon|\Delta y_\upsilon| + k\big|1/k - y_k + y_{k+1}\big|. \tag{6}$$

Hence $\|s^k + y\| \leq 1$ implies $k|1/k + y_k - y_{k+1}| \leq 1$ and $\|s^k - y\| \leq 1$ implies $k|1/k - y_k + y_{k+1}| \leq 1$. Consequently $y_k = y_{k+1}$. But when this holds we have

$$k|1/k + y_k - y_{k+1}| = 1 = k|1/k - y_k + y_{k+1}|.$$

Thus in order that $\|s^k + y\| \leq 1$ and $\|s^k - y\| \leq 1$ hold, all other terms in the sums of (5) and (6) must be zero. That is, $\Delta y_\upsilon = 0$, $(\upsilon \neq k)$. This together with the equality $y_k = y_{k+1}$ gives $y_k = constant$. For $y = (y_k) \in h$, $y_n = \sum_{k=n}^{\infty}(y_k - y_{k+1})$ so that $n|y_n| \leq \sum_{k=n}^{\infty} k|y_k - y_{k+1}|$ which converges to zero. Thus, $|y_n| = O(1/n)$ and so $y_\upsilon \to 0$ as $\upsilon \to \infty$. Hence $y_\upsilon = 0$, $(\upsilon = 1, 2, ...)$. Thus $\|s^k + y\| \leq 1$ and $\|s^k - y\| \leq 1$ imply that $y = (0, 0, 0, ...)$. Hence, s^k is an extreme point of the unit disc in h (for each fixed $k = 1, 2, ...$). □

Proposition 2.11. *([12], Proposition 4) Let s^k be defined as in (4) for all $k \in \mathbb{N}$. Consider the set $E = \{s^k : k = 1, 2, ...\}$. The E is a determining set for the space h.*

Proof. Let $x \in D$. Then $x \in \Phi$ and $\|x\| \leq 1$. Consequently, $x = \sum_{k=1}^{m} x_k \delta^k = \sum_{k=1}^{m} t_k s^k$ where $t_k = k(x_k - x_{k+1}), (k = 1, 2, ...)$. Also, $\sum_{k=1}^{m} |t_k| \leq \|x\| \leq 1$. Therefore, $x \in K$, the absolutely convex hull of E. Thus

$$D \subset K. \tag{7}$$

On the other hand, let $x \in K$. Then $x = \sum_{k=1}^{m} t_k s^k$ with $\sum_{k=1}^{m} |t_k| \leq 1$. Writing $x = (x_1, x_2, ...)$, we observe that

$$\begin{aligned} x_1 &= t_1 + \frac{t_2}{2} + \cdots + \frac{t_m}{m}, \\ x_2 &= \frac{t_2}{2} + \frac{t_3}{3} + \cdots + \frac{t_m}{m}, \\ &\vdots \quad \vdots \quad \vdots \\ x_m &= \frac{t_m}{m}, \\ x_{m+1} &= x_{m+2} = \cdots = 0. \end{aligned}$$

Hence $\|x\| = \sum_{k=1}^{\infty} k|x_k - x_{k+1}| = \sum_{k=1}^{m} |t_k| \leq 1$. Thus we have

$$K \subset D. \tag{8}$$

Combining (7) and (8) it follows that $K = D$. Therefore, E is a determining set for the space h. □

Proposition 2.12. *([12], Proposition 5) Let X be any $FK-$space which contains Φ. Then X includes h if and only if $E = \{s^k : k = 1, 2, ...\}$ is bounded in X.*

Proof. This is an immediate consequences of our Proposition 2.11 and Theorem 8.2.4 of [18]. □

Lemma 2.13. *([13], Lemma 1) Every semi replete space contains the Hahn space h.*

Proof. Let X be any semi replete space. Then $X^f \subset \sigma(\ell_\infty)$. But $\sigma(\ell_\infty) = h^f$ [12]. Thus $X^f = h^f$. Since h has AD, we have by Theorem 8.6.1 [18], $X^f \subset h^f \Rightarrow h \subset X$. □

Lemma 2.14. *([13], Lemma 2) A sequence $z = (z_k)$ belongs to $\sigma(\ell_\infty)$ if and only if z^β is semi replete.*

Proof. Suppose that $z \in \sigma(\ell_\infty)$. Then $h[\sigma(\ell_\infty)]^\beta \subset z^\beta$ so that $z^{\beta\beta} \subset h^\beta = \sigma(\ell_\infty)$. But z^β has AK under the sequence of norms

$$\begin{aligned} P_0(x) &= \|zx\|_{cs}, \quad (x \in z^\beta), \quad I = (I_k), x = (x_k). \\ P_n(x) &= |x_n|, \qquad (x \in z^\beta), \quad n \geq 1. \end{aligned}$$

Hence $z^{\beta f} \subset \sigma(\ell_\infty)$. Therefore z^β is semi replete.

Conversely, suppose that z^β is semi replete. Then $z^{\beta f} \subset \sigma(\ell_\infty)$. Since z^β has AK, we have $z^{\beta\beta} = z^{\beta f}$. So $z \in z^{\beta\beta} \subset \sigma(\ell_\infty)$. This completes the proof. □

Theorem 2.15. *([13], p.45) The intersection of all semi replete spaces in h.*

Proof. Let I denote the intersection of all semi replete spaces. By Lemma 2.13, I contains h. On the other hand, by Lemma 2.14, I is contained in $\bigcap\{z^\beta : z \in \sigma(\ell_\infty)\}$ and so I is contained in $[\sigma(\ell_\infty)]^\beta = h$. Thus $I = h$. This proves the theorem. □

Let X be an $FK-$space with $X \supset \Phi$. Then

$$\begin{aligned} B^+ = X^{f\gamma} = B^+(X) &= \{z \in \omega : (z^{[n]}) \text{ is bounded in } X\} \\ &= \{z \in \omega : (z_n f(\delta^{[n]}) \in bs, \forall f \in X'\}. \end{aligned}$$

Also we write $B = B^+ \cap X$. Let X is an $AB-$space if and only if $B = X$. Any space with monotone norm has AB (see Theorem 10.3.12 of [18]).

Lemma 2.16. *([14], Lemma 2) $h^f = \sigma(\ell_\infty)$.*

Proof. $h^\beta = \sigma(\ell_\infty)$ by Lemma 2.1 ([7] and [6]). Also h has AK ([12] and [6]). We have $h^\beta = h^f$. Therefore $h^f = \sigma(\ell_\infty)$. This completes the proof. □

Theorem 2.17. *([14], Theorem 1) Let Y be any $FK-$space $\supset \Phi$. Then $Y \supset h$ if and only if the sequence $(\delta^{(k)})$ is weakly Cesàro bounded.*

Proof. We know that h has AK. Since every $AK-$space is AD [3], the following two sided implications establish the result.

$$\begin{aligned} Y \supset h \quad &\Leftrightarrow \quad Y^f \subset h^f \ \text{ since } h \text{ has } AD \text{ and hence by using Theorem 8.6.1. in [18]} \\ &\Leftrightarrow \quad Y^f \subset \sigma(\ell_\infty) \ \text{ by Lemma 2.16} \\ &\Leftrightarrow \quad \text{for each } f \in Y', \text{ the topological dual of } Y, f(\delta^{(k)}) \in \sigma(\ell_\infty) \\ &\Leftrightarrow \quad \left\{ \frac{f(\delta^{(1)}) + f(\delta^{(2)}) + ... + f(\delta^{(k)})}{k} \right\} \in \ell_\infty \\ &\Leftrightarrow \quad \left\{ \frac{f(\delta^{(1)}) + f(\delta^{(2)}) + ... + f(\delta^{(k)})}{k} \right\} \ \text{ is bounded.} \\ &\Leftrightarrow \quad \text{The sequence } (\delta^{(k)}) \text{ is weakly Cesàro bounded.} \end{aligned}$$

This completes the proof. □

Theorem 2.18. *([14], Theorem 2) Suppose that h is a closed subspace of an $FK-$space X. Then $B^+(X) \subset h$.*

Proof. Note that c_0 has AK. Hence $\sigma(c_0)$ has AK. Consequently $\sigma(c_0)$ has AD. Therefore by Lemma 1.1, $[\sigma(c_0)]^\beta = [\sigma(c_0)]^\gamma$. By, Theorem 10.3.5 of [18] and Lemma 2.16, we have

$$B^+(X) = B^+(h) = h^{f\gamma} = (h^f)^\gamma = (\sigma(\ell_\infty))^\gamma.$$

But $(\sigma(\ell_\infty))^\gamma \subset (\sigma(c_0))^\gamma = (\sigma(c_0))^\beta$ and $(\sigma(c_0))^\beta = h$ (See p.97 [6]). Hence $B^+(X) \subset h$. This completes the proof. □

Theorem 2.19. *([14], Theorem 3) Let X be an $AK-$space including Φ. Then $X \supset h$ if and only if $B^+(X) \supset h$.*

Proof. (Necessity): Suppose that $X \supset h$. Then the inclusion,

$$B^+(X) \supset B^+(h) \tag{9}$$

holds by monotonicity Theorem 10.2.9 of [18]. By Theorem 10.3.4 of [18], we have

$$B^+(h) = h^{f\gamma} = h \tag{10}$$

From (9) and (10), we obtain $B^+(X) \supset B^+(h) = h$.
(Sufficiency): Suppose that $B^+(X) \supset h$. We have

$$h^\gamma \supset [B^+(X)]^\gamma. \tag{11}$$

But h has AK and so h has AD. Therefore

$$h^\beta = h^\gamma = h^f. \tag{12}$$

But always

$$B^+(X) = X^{f\gamma}. \tag{13}$$

From (11) and (13), $h^\gamma \supset (X^{f\gamma})^\gamma = (X^f)^{\gamma\gamma} \supset X^f$. Thus from (12), $h^f \supset X^f$. Now by Theorem 8.6.1 of [18]; since h has AD we conclude that $X \supset h$. This completes the proof. □

Theorem 2.20. *([14], Theorem 4) The space h has monotone norm.*

Proof. Let $m > n$. It follows from

$$|x_n| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + ... + |x_{m-1} - x_m| + |x_m|$$

that

$$\|x^{(n)}\| \leq \left(\sum_{k=1}^{n-1} k|x_k - x_{k+1}| \right) + n|x_n - x_{n+1}| + ... + (m-1)|x_{m-1} - x_m| + m|x_m| = \|x^{(m)}\|$$

The sequence $(\|x^{(n)}\|)$ being monotone increasing, it thus follows from $x = \lim_{n\to\infty} x^{(n)}$ that

$$\|x\| = \lim_{n\to\infty} \|x^{(n)}\| = \sup_{(n)} \|x^{(n)}\|.$$

□

Rao and Subramanian [14] defined semi-Hahn space as below:
An $FK-$space X is called semi-Hahn if $X^f \subset \sigma(\ell_\infty)$. In other words

$$\begin{aligned} f(\delta^{(k)}) &\in \sigma(\ell_\infty), \forall f \in X' \\ &\Leftrightarrow \{\frac{f(\delta^1) + f(\delta^2) + ... + f(\delta^k)}{k}\} \in \ell_\infty \\ &\Leftrightarrow \{\frac{f(\delta^1) + f(\delta^2) + ... + f(\delta^k)}{k}\} \text{ is bounded for each } f \in X'. \end{aligned}$$

Example 2.21. *([14], p.169) The Hahn space is semi Hahn. Indeed, if h is a Hahn space, then, $h^f = \sigma(\ell_\infty)$ by Lemma 2.16.*

Lemma 2.22. *([18], 4.3.7) Let z be a sequence. Then (z^β, p) is an $AK-$space with $p = (p_k) : k \in \mathbb{N}$, where*

$$p_0(x) = \sup_m \left| \sum_{k=1}^m z_k x_k \right|, \qquad p_n(x) = |x_n|.$$

For any k such that $z_k \neq 0$, p_k may be omitted. If $z \in \Phi$, p_0 may be omitted.

Theorem 2.23. *([14], Theorem 5) z^β is semi-Hahn if and only if $z \in \sigma(\ell_\infty)$.*

Proof. **Step 1**. Suppose that z^β is semi-Hahn. z^β has AK by Lemma 2.22. Hence $z^\beta = z^f$. Therefore $z^{\beta\beta} = (z^\beta)^f$ by Theorem 7.2.7 of [18]. So z^β is semi-Hahn if and only if $z^{\beta\beta} \subset \sigma(\ell_\infty)$. But then $z \in z^{\beta\beta} \subset \sigma(\ell_\infty)$.
Step 2. Conversely, let $z \in \sigma(\ell_\infty)$. Then $z^\beta \supset \{\sigma(\ell_\infty)\}^\beta$ and $z^{\beta\beta} \subset \sigma(\ell_\infty)^{\beta\beta} = h^\beta = \sigma(\ell_\infty)$. But $(z^\beta)^f = z^{\beta\beta}$. Hence $(z^\beta)^f \subset \sigma(\ell_\infty)$ which gives that z^β is semi-Hahn. This completes the proof. □

Theorem 2.24. *([14], Theorem 6) Every semi-Hahn space contains h.*

Proof. Let X be any semi-Hahn space. Then, one can see that

$$\begin{aligned}
&\Rightarrow \quad X^f \subset \sigma(\ell_\infty). \\
&\Rightarrow \quad f(\delta^{(k)}) \in \sigma(\ell_\infty), \quad \forall f \in X' \\
&\Rightarrow \quad \delta^{(k)} \text{ is weakly Cesaro bounded w.r. to X} \\
&\Rightarrow \quad X \supset h \text{ by Theorem 2.17.}
\end{aligned}$$

□

Theorem 2.25. *([14], Theorem 7) The intersection of all semi-Hahn spaces is h.*

Proof. Let I be the intersection of all semi-Hahn spaces. Then the intersection

$$I \subset \cap\{z^\beta : z \in \sigma(\ell_\infty)\} = \{\sigma(\ell_\infty)\}^\beta = h \tag{14}$$

By Theorem 2.24,

$$h \subset I \tag{15}$$

From 14 and 14, we get $I = h$. □

Corollary 2.26. *([14], p.170) The smallest semi-Hahn space is h.*

3 Matrix Transformations

Now we give some matrix transformations:

Theorem 3.1. *([12] Proposition 6) $A \in (h : c_0)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = 0, \quad (k = 1, 2, ...) \tag{16}$$

$$\sup_{n,k} \frac{1}{k}\left|\sum_{v=1}^{k} a_{nv}\right| < \infty. \tag{17}$$

Proof. We have by Theorem 2.5 that h is a $BK-$space with AK. Also c_0 is a $BK-$space. So, we invoke Theorem 8.3-4 of [18] and conclude that $A \in (h : c_0)$ if and only if the columns of A are in c_0, and $A(E)$ is a bounded subset of c_0.

Here, we recall that $E = \{s^k : k = 1, 2, ...\}$. But $As^k = \left\{\sum_{v=1}^{k} a_{nv}/k : n = 1, 2, ...\right\}$. So, $A \in (h : c_0)$ if and only if (16) and (17) hold. □

Omitting the proofs, we formulate the following results.

Theorem 3.2. *([12], Proposition 7) $A \in (h : c)$ if and only if (17) holds and*

$$\lim_{n\to\infty} a_{nk} \quad \textit{exists}, \quad (k = 1, 2, ...) \tag{18}$$

Theorem 3.3. *([12], Proposition 8) $A \in (h : \ell_\infty)$ if and only if (17) holds.*

Theorem 3.4. *([12], Proposition 9) $A \in (h : \ell_1)$ if and only if*

$$\sum_{n=1}^{\infty} |a_{nk}| \quad \textit{converges}, \quad (k = 1, 2, ...) \tag{19}$$

$$\sup_k \frac{1}{k} \sum_{n=1}^{\infty} \left| \sum_{v=1}^{k} a_{nv} \right| < \infty. \tag{20}$$

Theorem 3.5. *([12], Proposition 10) $A \in (h : h)$ if and only if (16) holds and*

$$\sum_{n=1}^{\infty} n|a_{nk} - a_{n+1,k}| \quad \textit{converges}, \quad (k = 1, 2, ...) \tag{21}$$

$$\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| < \infty. \tag{22}$$

4 The Hahn Sequence Space of Fuzzy Numbers

In this section, we introduce the sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers [2].

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [20]. Sequences of fuzzy numbers have been discussed by Aytar and Pehlivan [1], Mursaleen and Başarır [9], Nanda [10] and many others.

The study of Hahn sequence space was initiated by Rao [12] with certain specific purpose in Banach space theory. Talo and Başar [16] gave the idea of determining the dual of sequence space of fuzzy numbers by using the concept of convergence of a series of fuzzy numbers.

Definition 4.1. *A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following conditions:*

(i) *u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.*

(ii) *u is fuzzy convex, i.e., $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.*

(iii) *u is upper semi continuous.*

(iv) *The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact [20], where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.*

We denote the set of all fuzzy numbers on $\mathbb{R}$ by E' and called it as the space of fuzzy numbers. The $\lambda-$level set $[u]_\lambda$ of $u \in E'$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\} & , \quad (0 < \lambda \leq 1), \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}} & , \quad (\lambda = 0). \end{cases}$$

The set $[u]_\lambda$ is closed bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. Since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r} = \begin{cases} 1 & , \quad (x = r), \\ 0 & , \quad (x \neq r), \end{cases}$$

$\mathbb{R}$ can be embedded in E'. Let $u, w \in E'$ and $k \in \mathbb{R}$. The operations addition, scalar multiplication and product defined on E' by

$$\begin{aligned} u + v = w \quad &\Leftrightarrow \quad [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1] \\ &\Leftrightarrow \quad [w]^-(\lambda) = [u^-(\lambda), v^-(\lambda)] \text{ and } [w]^+(\lambda) = [u^+(\lambda), v^+(\lambda)] \text{ for all } \lambda \in [0,1] \end{aligned}$$

$[ku]_\lambda = k[u]_\lambda$ for all $\lambda \in [0,1]$ and $uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda$ for all $\lambda \in [0,1]$, where it is immediate that

$$[w]^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

and

$$[w]^+(\lambda) \;=\; \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

for all $\lambda \in [0,1]$.

Let W be the set of all closed and bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$ i.e., $A = [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A,B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be observed that d is a metric on W [10] and (W,d) is a complete metric space [4]. Now we can define the metric D on $E^{'}$ by means of a Hausdorff metric d as

$$D(u,v) = \sup_{\lambda\in[0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda\in[0,1]} \Big\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\Big\}.$$

$(E^{'}, D)$ is a complete metric space ([11] Theorem 2.1). One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B}.$$

The partial order relation on $E^{'}$ is defined as follows:

$$u \leq v \Leftrightarrow [u]_\lambda \leq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda) \text{ and } u^+(\lambda) \leq v^+(\lambda) \text{ for all } \lambda \in [0,1].$$

An absolute value $|u|$ of a fuzzy number u is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\} & , \ (t \geq 0), \\ 0 & , \ (t < 0). \end{cases}$$

$\lambda-$level set $[|u|]_\lambda$ of the absolute value of $u \in E^{'}$ is in the form $[|u|]_\lambda$, where $|u|^-(\lambda) = \max\{0, u^-(\lambda), u^+(\lambda)\}$ and $|u|^+(\lambda) = \max\{|u^-(\lambda)|, |u^+(\lambda)|\}$. The absolute value $|uv|$ of $u, v \in E^{'}$ satisfies the following inequalities [16]

$$|uv|^-(\lambda) \leq |uv|^+(\lambda) \leq \max\{|u|^-(\lambda)|v|^-(\lambda), |u|^-(\lambda)|v|^+(\lambda), |u|^+(\lambda)|v|^-(\lambda), |u|^+(\lambda)|v|^+(\lambda)\}$$

In the sequel, we require the following definitions and lemmas.

Definition 4.2. *A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into the set $E^{'}$. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called the kth term of the sequence. Let $w(F)$ denote the set of all sequences of fuzzy numbers.*

Lemma 4.3. *The following statements hold:*

(i) $D(uv,\overline{0}) \leq D(u,\overline{0})D(v,\overline{0})$ *for all* $u,v \in E^{'}$.

(ii) *If* $u_k \to u$ *as* $k \to \infty$ *then* $D(u_k,\overline{0}) \to D(u,0)$ *as* $k \to \infty$*; where* $(u_k) \in w(F)$.

Definition 4.4. *A sequence* $u = (u_k) \in w(F)$ *is called convergent with limit* $u \in E^{'}$ *if and only if for every* $\varepsilon > 0$ *there exists an* $n_0 = n_0(\varepsilon) \in \mathbb{N}$ *such that* $D(u_k,u) < \varepsilon$ *for all* $k \geq n_0$.

If the sequence $(u_k) \in w(F)$ *converges to a fuzzy number* u *then by the definition of* D *the sequences of functions* $\{u_k^-(\lambda)\}$ *and* $\{u_k^+(\lambda)\}$ *are uniformly convergent to* $u^-(\lambda)$ *and* $u^+(\lambda)$ *in* $[0,1]$*, respectively.*

Definition 4.5. *A sequence* $u(u_k) \in w(F)$ *is called bounded if and only if the set of all fuzzy numbers consisting of the terms of the sequence* (u_k) *is a bounded set. That is to say that a sequence* $(u_k) \in w(F)$ *is said to be bounded if and only if there exist two fuzzy numbers* m *and* M *such that* $m \leq u_k \leq M$ *for all* $k \in \mathbb{N}$.

Definition 4.6. *Let* $(u_k) \in w(F)$*. Then the expression* $\sum u_k$ *is called a series of fuzzy numbers. Denote* $S_n = \sum_{k=0}^{n} u_k$ *for all* $n \in \mathbb{N}$*. If the sequences* (S_n) *converges to a fuzzy number* u *then we say that the series* $\sum u_k$ *of fuzzy numbers converges to* u *and write* $\sum_{k=0}^{n} u_k = u$ *which implies as* $n \to \infty$ *that* $\sum_{k=0}^{n} u_k^-(\lambda) \to u_k^-(\lambda)$ *and* $\sum_{k=0}^{n} u_k^+(\lambda) \to u_k^+(\lambda)$ *uniformly in* $\lambda \in [0,1]$*. Conversely, if the fuzzy numbers* $u_k = \{[u_k^-(\lambda), u_k^+(\lambda)] : \lambda \in [0,1]\}$, $\sum u_k^-(\lambda)$ *and* $\sum u_k^+(\lambda)$ *converge uniformly in* λ *then* $u = \{[u^-(\lambda), u^+(\lambda)] : \lambda \in [0,1]\}$ *defines a fuzzy number such that* $u = \sum u_k$.

We say otherwise the series of fuzzy numbers diverges. Additionally if the sequence (S_n) is bounded then we say that the series $\sum u_k$ of fuzzy numbers is bounded. By $cs(F)$ and $bs(F)$, we denote the sets of all convergent and bounded series of fuzzy numbers, respectively.

Lemma 4.7. *Let for the series of functions* $\sum_k u_k(x)$ *and* $\sum_k v_k(x)$ *there exists an* $n_0 \in \mathbb{N}$ *such that* $|u_k(x)| \leq v_k(x)$ *for all* $k \geq n_0$ *and for all* $x \in [a,b]$ *with* $u_k : [a,b] \to \mathbb{R}$ *and* $v_k : [a,b] \to \mathbb{R}$*. If the series converges uniformly in* $[a,b]$ *then the series* $\sum_k |u_k(x)|$ *and* $\sum_k |v_k(x)|$ *are uniformly convergent in* $[a,b]$.

Definition 4.8. ***(Weierstrass M Test)*** *Let* $u_k : [a,b] \to \mathbb{R}$ *are given. If there exists an* $M_k \geq 0$ *such that* $|u_k(x)| \leq M_k$ *for all* $k \in \mathbb{N}$ *and the series* $\sum_k M_k$ *converges then the series* $\sum_k u_k(x)$ *is uniformly and absolutely convergent in* $[a,b]$

Definition 4.9. *A mapping T from X_1 and X_2 is said to be fuzzy isometric if $d_2(Tx, Ty) = d_1(x, y)$ for all $x, y \in X_1$. The space X_1 is said to be fuzzy isometric with the space X_2 if there exists a bijective fuzzy isometry from X_1 onto X_2 and write $X_1 \cong X_2$. The spaces X_1 and X_2 are then called fuzzy isometric spaces.*

The following spaces are needed.

$$\begin{aligned} \ell_\infty(F) &= \left\{(u_k) \in w(F) : \sup_{k \in N} D(u_k, \overline{0}) < \infty\right\}, \\ c(F) &= \left\{(u_k) \in w(F) : \exists \ell \in E' \lim_{k \to \infty} D(u_k, \ell) = 0\right\}, \\ c_0(F) &= \left\{(u_k) \in w(F) : \lim_{k \to \infty} D(u_k, 0) = 0\right\}, \\ \ell_p(F) &= \left\{(u_k) \in w(F) : \sum_k D(u_k, \overline{0}) < \infty\right\}. \end{aligned}$$

Let A denote the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} n(-1)^{n-k} & , \quad n-1 \le k \le n \\ 0 & , \quad 1 \le k \le n-1 \ \text{ or } \ k > n \end{cases}$$

Define the sequence $y = (y_k)$ which will be frequently used as the $A-$transform of a sequence $x = (x_k)$, i.e.,

$$y_k = (Ax)_k = k(x_k - x_{k-1}) \quad k \ge 1 \tag{23}$$

We introduce the sets $h(F)$ and $h_\infty(F)$ as the sets of all sequences such that their $A-$transforms are in $\ell(F)$ and $\ell_\infty(F)$ that is,

$$\begin{aligned} h(F) &= \{u = (u_k) \in \omega(F) : \sum_k D[(Au)_k, \overline{0}] < \infty \ \text{ and } \ \lim_{k \to \infty} D[u_k, \overline{0}] = 0\} \\ h_\infty(F) &= \{u = (u_k) \in \omega(F) : \sup_k D[(Au)_k, \overline{0}] < \infty\}. \end{aligned}$$

Example 4.10. *([2], Example 3.1) Consider the sequence $u = (u_k)$ defined by*

$$u_k = \begin{cases} \overline{1} & , \quad 1 \le k \le n \\ \overline{0} & , \quad k > n \end{cases}$$

$$\sum D[(Au)_k, \overline{0}] = \sum D[k(u_k - u_{k-1}), \overline{0}] = 0$$

which is convergent. Also $\lim_{k \to \infty} D(u_k, \overline{0}) = 0$. *Hence,* $u \in h(F)$.

Theorem 4.11. *([2], Theorem 3.2) $h(F)$ and $h_\infty(F)$ are complete metric spaces with the metrics dh and dh_∞ defined by*

$$\begin{aligned} dh(u,v) &= \sum_k D[(Au)_k, (Av)_k] \\ dh_\infty(u,v) &= \sup_{k\in\mathbb{N}} D[(Au)_k, (Av)_k] \end{aligned}$$

respectively, where $u=(u_k)$ and $v=(v_k)$ are the elements of the spaces $h(F)$ or $h_\infty(F)$.

Proof. Let $\{u^i\}$ be any Cauchy sequence in the space $h(F)$, where $u^i = \{u_0^{(i)}, u_1^{(i)}, u_2^{(i)}, \cdots\}$. Then for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$dh(u^i, u^j) = \sum_n D[(Au)_n^i, (Au)_n^j] < \varepsilon \tag{24}$$

for $i, j \geq n_0(\varepsilon)$. We obtain for each fixed $n \in \mathbb{N}$ from (24) that

$$D[(Au)_n^i, (Au)_n^j] < \varepsilon$$

for every $i, j \geq n_0(\varepsilon)$. We obtain for each fixed $n \in \mathbb{N}$ from (24) that

$$\sum_{k=0}^{m} D[(Au)_n^i, (Au)_n^j] \leq dh(u^i, u^j) < \varepsilon. \tag{25}$$

Take any $i \geq n_0(\varepsilon)$ and takin limit as $j \to \infty$ first and next $m \to \infty$ in (24), we obtain

$$dh(u^i, u) < \varepsilon.$$

Finally, we proceed to prove $u \in h(F)$. Since $\{u^i\}$ is a Cauchy sequence in $h(F)$, we have

$$\sum_k D[(Au)_k^i, \overline{0}] \leq \varepsilon \text{ and } \lim_{k\to\infty} [(Au)_k^i, \overline{0}] = 0.$$

Now

$$D[(Au)_k, \overline{0}] \leq D[(Au)_k, (Au)_k^i] + D[(Au)_k^i, (Au)_k^j] + D[(Au)_k^j, \overline{0}].$$

Hence,

$$D[(Au)_k, \overline{0}] \leq \sum_k D[(Au)_k, (Au)_k^i] + \sum_k D[(Au)_k^i, (Au)_k^j] + \sum_k D[(Au)_k^j, \overline{0}] < \varepsilon.$$

Also from (25), $\lim_{k\to\infty} [(Au)_k^i, \overline{0}] = 0$. Hence, $u \in h(F)$. Since, $\{u^i\}$ is an arbitrary Cauchy sequence, the space $h(F)$ is complete. □

Theorem 4.12. *([2], Definition 3.3) The space $h(F)$ is isomorphic to the space $\ell_1(F)$.*

Proof. Consider the transformation T defined from $h(F)$ to $\ell_1(F)$ by $x \mapsto y = T(x)$. To prove the fact $h(f) \cong \ell_1(F)$, we show the existence of a bijection between the spaces $h(F)$ and $\ell_1(F)$. We can find that only one $x \in h(F)$ with $Tx = y$. This means that T is injective.
Let $y \in \ell_1(F)$. Define the sequence $x = (x_k)$ such that $(Ax)_k = y_k$ for all $k \in \mathbb{N}$.

Then $dh(x, 0) = \sum_k D[(Ax)_k, \overline{0}] = \sum_k D[y_k, \overline{0}] < \infty$. Thus, $x \in h(F)$. Consequently, T is bijective and is isometric. Therefore, $h(F)$ and $\ell_1(F)$ are isomorphic. □

Theorem 4.13. *([2], Theorem 3.4) Let d denote the set of all sequences of fuzzy numbers defined as follows*

$$d = \Big\{x = (x_k) \in w(F) : \sum_k k|x_k - x_{k-1}| < \infty \quad and \quad x \in c_0(F)\Big\}.$$

Then, the set d is identical to the set $h(F)$.

Proof. Let $x \in h(F)$. Then

$$\sum_k D((Ax)_k, \overline{0}) < \infty \quad \text{and} \quad \lim_{k\to\infty} D[x_k, \overline{0}] = 0. \tag{26}$$

Using (23),

$$\sum_k D(y_k, \overline{0}) < \infty \quad \text{and} \quad \lim_{k\to\infty} D[x_k, \overline{0}] = 0.$$

We have, $\sum_k D(y_k, \overline{0}) = \sup_{\lambda\in[0,1]} \max\{|y_k^-(\lambda)|, |y_k^+(\lambda)|\}$. Now, $\max\{|y_k^-(\lambda)|, |y_k^+(\lambda)|\} \leq \sum_k D(y_k, \overline{0}) < \infty$. This implies that $\sum_k |y_k| < \infty$. That is $\sum_k k|x_k - x_{k-1}| < \infty$. Also from (26), $x \in c_0(F)$. Thus, $x \in d$. Then, $\sum_k k|x_k - x_{k-1}| < \infty$. That is $\sum_k |y_k| < \infty$. Therefore $\sum_k k \max\{|y_k^-(\lambda)|, |y_k^+(\lambda)|\}$ converges for $\lambda \in [0, 1]$. This gives for $\lambda = 0$, $\sum_k D(y_k, \overline{0}) < \infty$. Also $(x_k) \in c_0(F)$ implies $\lim_{k\to\infty} D(x_k, \overline{0}) = 0$. This completes the proof. □

Definition 4.14. *([2], Definition 3.5) The $\alpha-$dual, $\beta-$dual and $\gamma-$dual $S(F)^\alpha, S(F)^\beta$ and $S(F)^\gamma$ of a set $S(F) \subset w(F)$ are defined by*

$$\begin{aligned}
\{S(F)\}^\alpha &= \Big\{(u_{k)} \in w(F) : (u_k v_k) \in \ell_1(F) \quad for\ all \quad (v_k) \in S(F)\Big\},\\
\{S(F)\}^\beta &= \Big\{(u_{k)} \in w(F) : (u_k v_k) \in cs(F) \quad for\ all \quad (v_k) \in S(F)\Big\},\\
\{S(F)\}^\gamma &= \Big\{(u_{k)} \in w(F) : (u_k v_k) \in bs(F) \quad for\ all \quad (v_k) \in S(F)\Big\},
\end{aligned}$$

Definition 4.15. *([2], Definition 3.6) Let B denote the matrix $B = (b_{n_k})$ defined by*

$$b_{n_k} = \begin{cases} 1/n & , \quad 1 \leq k \leq n \\ 0 & , \quad otherwise \end{cases}$$

Define the sequence $y = (y_k)$ by the $B-$transform of a sequence $x = (x_k)$, i.e., $y_k = (Bx)_k = \sum_{i=1}^{k} x_i/k$ for all $k \in \mathbb{N}$.

The Cesàro space of $\ell_\infty(F)$ *is the set of all sequences such that their $B-$transforms are in $\ell_\infty(F)$. That is,*

$$\sigma(\ell_\infty(F)) = \{x = (x_k) : \sup_k D[(Bx)_k, \overline{0}] < \infty\}.$$

Theorem 4.16. *([2], Theorem 3.7) $\sigma(\ell_\infty(F))$ is a complete metric space with the metric*

$$d_\sigma(u, v) = \sup_k D[(Bu)_k, (Bv)_k],$$

where $u = (u_k)$ and $v = (v_k)$ are the elements of the space $\sigma(\ell_\infty(F))$.

Theorem 4.17. *([2], Theorem 3.8) The $\beta-$ and $\gamma-$dual of the set $h(F)$ is the set $\sigma(\ell_\infty(F))$.*

Proof. Let $(u_k) \in h(F)$ and $(v_k) \in \sigma(\ell_\infty(F))$. If $(u_k) \in h(F)$, then $\lim_{k\to\infty} D[u_k, \overline{0}] = 0$. Therefore for given $\varepsilon > 0$ there exists n_0 such that $D(u_k, \overline{0}) < \varepsilon$. If $(v_k) \in \sigma(\ell_\infty(F))$, then $\sup_k D[(Bv)_k, \overline{0}] < \infty$. Thus, $D(v_k, \overline{0}) < \infty$ for all k and n. Hence, there exists a $M > 0$ such that $D(v_k, \overline{0}) < M$ for all k and n. Now,

$$|(u_k)^-(\lambda)| \leq D(u_k, \overline{0}) \leq D(u_k, \overline{0}) D(v_k, \overline{0}) < \varepsilon M.$$

Weierstrass Test yields that $\sum_k (u_k)^-(\lambda)$ and $\sum_k (u_k)^+(\lambda)$ converge uniformly and hence $\sum_k u_k$ converges. Thus $\sigma(\ell_\infty(F)) \subset h^\beta(F)$.

Conversely, suppose that $(v_k) \in h^\beta(F)$. Then the series $\sum_k u_k v_k$ converges for all $(u_k) \in h(F)$. This also holds for the sequence (u_k) of fuzzy numbers defined by $u_k = \chi[-1, 1]$ for all $k \in \mathbb{N}$. Since $u_k^-(\lambda) = -1$ and $u_k^+(\lambda) = 1$ for all $\lambda \in [0, 1]$, the series

$$\begin{aligned}
\sum_k (u_k v_k)^+(\lambda) &= \sum_k \max\{u_k^-(\lambda)v_k^-(\lambda), u_k^-(\lambda)v_k^+(\lambda), u_k^+(\lambda)v_k^-(\lambda), u_k^+(\lambda)v_k^+(\lambda)\} \\
&= \sum_k \max\{-v_k^-(\lambda), -v_k^+(\lambda), v_k^-(\lambda), v_k^+(\lambda)\} \\
&= \sum_k \max\{|v_k^-(\lambda)|, |v_k^+(\lambda)|\}
\end{aligned}$$

converges uniformly. Thus $\sup_k D[(Bv)_k, \overline{0}] < \infty$. Hence, $(v_k) \in \sigma(\ell_\infty(F))$ and $h^\beta(F) = \sigma(\ell_\infty(F))$. This completes the proof. □

5 Conclusion

Hahn defined the space h and gave its some general properties. Goes and Goes [6] studied the functional analytic properties of this space. The study of Hahn sequence space was initiated by Rao [12] with certain specific purpose in Banach space theory. Also Rao [12] computed some matrix transformations. Rao and Srinivasalu [13] introduce a new class of sequence spaces called semi replete spaces. Rao and Subramanian [14] defined the semi Hahn space and proved that the intersection of all semi Hahn spaces is the Hahn space. Balasubramanian and Pandiarani[2] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that $\beta-$ and $\gamma-$duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences.

Determine the matrix domain h_A of arbitrary triangles A and compute their $\alpha-$, $\beta-$ and $\gamma-$duals and characterize matrix transformations on them into the classical sequence spaces and almost convergent sequence space may happen new results. Also it may study the some geometric properties of this space.

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