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Travelling Wave Solutions and Conservation Laws of Fisher-Kolmogorov Equation

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Abstract

Lie symmetry group method is applied to study the Fisher-Kolmogorov equation. The symmetry group is given, and travelling wave solutions are obtained. Finally the conservation laws are determined.

Keywords: *Fisher-Kolmogorov Equation, Lie symmetry, Partial differential equation, Conservation Laws.*

1 Mathematical Formulation

In mathematics, Fisher's equation, also known as the Fisher-Kolmogorov equation and the Fisher-KPP equation, named after R. A. Fisher and A. N. Kolmogorov, is the partial differential equation which describe the spatial spread of an advantageous allele and explored its travelling wave solutions. The aim is to analysis the Lie point symmetry structure of this equation, which is

$$\Delta_{FK}(u) := u_t - u(1-u) - u_{xt} = 0, \tag{1}$$

where u is a smooth function of (x, t).

In this paper we give a method for finding travelling solutions for the Fisher-Kolmogorov equation based on some rational function which is applicable for any kind of partial differential equations, then we determine conservation laws of the Fisher-Kolmogorov equation using Lie point symmetries.

2 Transformed Rational Function Method

As we know a lots of physical phenomena could be discussed with differential equations, specially partial differential equations. Furthermore these phenomena are following from a non-linear structure, such as fluid dynamics, optical fibers, plasma physics, acoustics, solid state physics, mechanics and etc., [2]. In this article we found a special kind of solutions called similarity solutions, but obviously it is a part of solutions space of equation (1). Thus, it is significantly important to investigate for exact solutions of equation (1).

A direct approach to exact solutions of non-linear partial differential equations is recommended by using rational function transformations. This is a systematical method for finding the solutions of non-linear equations, provides a unigeniture between $\tanh -$function type method, the homogeneous balance method, the $\exp -$function method, the mapping method and the $F-$expansion type methods. This method is based on finding rational solutions for ordinary differential equations which generated by reducing of a system of partial differential equations. But we know it is a hard job to find all exact solutions for non-linear partial differential equations, but it a successful idea to generate exact solution of non-linear wave equations by reducing partial differential equation into ordinary differential equations.

There is a lots of literature about above-mentioned method but Ma and Lee, [12], propose a direct and systematical approach to exact solutions of non-linear equations by using rational function transformations, a suitable and effective method for obtaining the exact solutions. Their method carry out the solution process of non-linear wave equation more systematically and conveniently by softwares such as Maple and Mathematica, so it is an encouragement for us for finding exact solutions of equation (1). Finally we can use linear superposition principle, [11], for partial differential equations for this equation to classify a vast line of exact solutions. In the next subsection we will use some transformations mentioned above for finding travelling wave solution for the equation (1).

To describe our solution process, let us focus on a scalar 1+1 dimensional partial differential equation

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{2}$$

though the solution process also works for systems of non-linear equations. We assume that there are exact solutios to the differential equation (2):

$$u(x, t) = u(\zeta), \qquad \zeta = \zeta(x, t). \tag{3}$$

Usually we have travelling wave solution $\zeta(x, t) = ax - \omega t$, [3], where a and ω are arbitrary constants, also in non-constant coefficients we have $\zeta(x, t) = a(t)x - \omega(t)$. Under the transformation (3), the partial differential equation (2) reduced to an ordinary differential equation: $\Gamma(x,t,u',u'',u''',...)=0$, where $u^{(i)}=\frac{d^iu}{d\zeta^i}$. To keep the solution process as simple as possible, the function Γ should not be total $\zeta-$derivative of another function. Otherwise, taking integration with respect to ζ further reduces the transformed equation.

An important step for finding solution is to introduce a new variable $\eta=\eta(\zeta)$ by an integrable oprdinary differential equation, such as:

$$\eta'=\tau=\tau(\zeta,\eta), \tag{4}$$

for a smooth function τ. The prime is the derivative respect to ζ. In case that we have a general second-order differential equation to begin with, we shoul first obtain its first integrals [13], and then use the method of planar dynamical system to solve [6]. Two simple solvable cases of the above function τ are $\tau=\tau(\eta)=\eta$, and $\tau=\tau(\eta)=\alpha+\eta^2$, where α is a constant. The corresponding first-order equations have a particular solution $\eta=e^{\zeta}$ and

$$\eta=\begin{cases} -\frac{1}{\zeta}, & \text{when } \alpha=0,\\ -\sqrt{-\alpha}\tanh\sqrt{-\alpha}\zeta \text{ or } -\sqrt{-\alpha}\coth\sqrt{-\alpha}\zeta, & \text{when } \alpha<0,\\ \sqrt{\alpha}\tan\sqrt{\alpha}\zeta \text{ or } -\sqrt{\alpha}\cot\sqrt{\alpha}\zeta, & \text{when } \alpha>0, \end{cases} \tag{5}$$

respectively. Those two cases corresponds to the $\exp-$method and the extended $\tanh-$function method, respectively.

More general assumption that τ can engenders special function solutions to non-linear wave equation. For instance, taking $(\eta')^2=T(\eta)$ with some fourth-order polynomials $T(\eta)$ in η (or equivalently, $\eta''=S(\eta)$ with some third-order polynomials $S(\eta)$ in η) can yield Jacobi elliptic function solutions; and such assumptions are the bases for the extended $\tanh-$function method, the $F-$expansion method and the extended $F-$expansion method, and work for many particular non-linear wave equations.

To generate travelling wave solution using the solution process described above, consider the solution

$$u(x,t)=u(\zeta), \qquad \zeta=ax-\omega t, \tag{6}$$

where a is the angular wave number and ω is the wave frequency, we only need to solve the reduced Fisher-Kolmogorov equation

$$a^2u''+\omega u'+u(1-u)=0, \tag{7}$$

where the prime denotes the derivatives with respect to ζ. Set $u'=v$, and then, we have the transformed Fisher-Kolmogorov equation

$$a^2\tau v'+\omega v+\eta(1-\eta)=0. \tag{8}$$

2.1 The case $\eta' = \eta$

In this case the transformed Fisher-Kolmogorov equation becomes

$$a^2\eta v' + \omega v + \eta(1-\eta) = 0. \tag{9}$$

A direct computation tells that there is a solution

$$v(\eta) = \eta\Big(\frac{\eta}{2a^2+\omega} - \frac{1}{a^2+\omega}\Big) + c\eta^{-\frac{\omega}{a^2}}, \qquad c = \text{constant}. \tag{10}$$

Accordingly we have the travelling wave solutions to the Fisher-Kolmogorov equation:

$$u(x,t) = \frac{1}{2}\frac{e^{2\zeta}}{2a^2+\omega} + \frac{e^{\zeta}}{a^2+\omega} - \frac{c_1 e^{-\zeta\frac{\omega}{a^2}}(2a^6+3a^4\omega+a^2\omega)}{\omega(2a^2+\omega)(a^2+\omega)} + c_2, \tag{11}$$

where c_1 and c_2 are arbitrary constants and $\zeta = ax - \omega t$.

2.2 The case $\eta' = \alpha + \eta^2$

In this case, the transformed Fisher-Kolmogorov equation becomes

$$a^2(\alpha+\eta^2)v' + \omega v + \eta(1-\eta) = 0. \tag{12}$$

A direct computation tells that there is a solution

$$v(\eta) = \Bigg(\int \frac{\eta(1-\eta)}{a^2(\alpha+\eta^2)} \exp\Bigg\{\frac{\omega \arctan\frac{\eta}{\sqrt{\alpha}}}{a^2\sqrt{\alpha}}\Bigg\}d\eta + c\Bigg) \exp\Bigg\{-\frac{\omega \arctan\frac{\eta}{\sqrt{\alpha}}}{a^2\sqrt{\alpha}}\Bigg\}. \tag{13}$$

For example if $\eta = \sqrt{\alpha}\tan\sqrt{\alpha}\zeta$, a new travelling wave solution for Fisher-Kolmogorov equation is

$$u(x,t) = \int \frac{\tan^2\sqrt{\alpha}\zeta(\sqrt{\alpha}\tan\sqrt{\alpha}\zeta - 1)}{a^2(1+\tan^2\sqrt{\alpha}\zeta)} \exp\Big\{\frac{\omega\zeta}{a^2}\Big\}d\zeta + c\exp\Big\{-\frac{\omega\zeta}{a^2}\Big\}, \tag{14}$$

where $\zeta = ax - \omega t$.

2.3 Bäcklund Transformation

Let $u = u(x,t)$ be a solution for the equation (1). Evidently, if a function $v = v(x,t)$ satisfies

$$2uv + \Delta_{FK}(v) = 0, \tag{15}$$

where Δ_{FK} is the Fisher-Kolmogorov equation, then the sum of the two functions, $w = u + v$, gives another solution to the Fisher-Kolmogorov equation. Therefore, once we find a function v satisfying (15), we get a new solution $w = u + v$ from a known function u. This forms a general auto-Bäcklund transformation for the Fisher-Kolmogorov equation. It follows directly from the above Bäcklund transformation if two solutions u and v of the Fisher-Kolmogorov equation satisfy $uv = 0$, then $w = u + v$ is a third solution. For example if we take a travelling wave solution $u = u(x, t) = u(ax - \omega t)$ to the Fisher-Kolmogorov equation, then the function

$$w(x, t) = u(ax - \omega t) + a'x - \omega' t + b, \tag{16}$$

where a' and ω' are constants, presents a new solution to the Fisher-Kolmogorov equation.

3 Conservation Laws

A coservation law of a non-degenerate system of differential equation is a divergence expression that vanishes on all solutions of the given system. In general, any such non-trivial expression that yields a local conservation law of the system arises from a linear combination formed local multipliers (characteristics) with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of the dependent variables of the given system of differential equations. It turns out that a divergence expression depending on independent variables, dependent variables and their derivatives to some finite order is annihilated by the Euler operators associated with each of its dependent variables; conversely, if the Euler operators, associated with each dependent variable in an expression involving independent variables, dependent variables and their derivatives to some finite order, annihilated the expression, then the expression is a divergence expression. From this it follows that a given system of differential equations has a local conservation laws if and only if there exist a set of local multipliers whose scalar product with each differential equation in each differential equation in system is identically annihilated without restricting the dependent variables in the scalar product to solution of the system, i.e., the independent variables, as well as each of their derivatives, are treated as arbitrary functions.

Thus the problem of finding local conservation laws of a system of differential equations reduces to the problem of finding local multipliers whose scalar product with each differential equation in the system is annihilated by the Euler operators associated with each dependent variable where the dependent variables and their derivatives in the given set of local conservation laws multipliers, there is an integral formula to obtain the fluxes of the local conservation laws [4, 10]. Often it straightforward to obtain the conservation law by direct calculation after its multipliers are known [5]. What has been outlined here is the direct method for obtaining local conservation laws of Fisher-Kolmogorov equation.

3.1 The Direct Method

Consider a system $\Delta(x, u^{(n)}) = 0$ of $\ell-$differential equations of order n with $p-$independent variables $x = (x^1, ..., x^p)$ and $q-$dependent variables $u(x) = (u^1(x), ..., u^q(x))$, given by

$$\Delta_\nu[u] = \Delta_\nu(x, u, \partial u, ..., \partial^n u) = 0, \qquad \nu = 1, ..., \ell, \tag{17}$$

a local conservation law of the system (17) is a divergence expression

$$D_i\Phi^i[u] := D_1\Phi^1[u] + \cdots + D_p\Phi^p[u] = 0, \tag{18}$$

holding on all solutions of the system (17). In (18), D_i is the total derivatives respect to x^i and $\Phi^i[u] = \Phi^i(x, u, \partial u, ..., \partial^k u), i = 1, ..., p$, is the fluxes of conservation laws.

In general, for a given non-degenerate differential equation system (17), non-trivial local conservation laws arise from seeking scalar products that involve linear combinations of the equations of the differential equation system (17) with multipliers (factors) that yield nontrivial divergence expressions. In seeking such expressions, the dependent variables and each of their derivatives that appear in the differential equation system (17) or in the multipliers, are replaced by arbitrary functions. Such divergence expressions vanish on all solutions of the differential equation system (17) provided the multipliers are non-singular.

Definition 3.1 *The Euler operator with respect to U^μ is the operator defined by*

$$\mathrm{E}_{U^\mu} = \frac{\partial}{\partial U^\mu} - D_i\frac{\partial}{\partial U^\mu} + \cdots + (-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial U^\mu_{i_1 \dots i_s}} + \cdots. \tag{19}$$

By direct calculation, one can show that the Euler operators (19) annihilate any divergence expression $D_i\Phi^i(x, U, \partial U, ..., \partial^k U)$ for any k. In particular the following identities holds for arbitrary $U(x)$,

$$\mathrm{E}_{U^\mu}(D_i\Phi^i(x, U, \partial U, ..., \partial^k U)) \equiv 0, \qquad \mu = 1, ..., q. \tag{20}$$

It is straightforward to show that the converse also holds. Namely, the only scalar expressions annihilated by Euler operators are divergence expressions. This establishes the following theorem.

Theorem 3.2 *A set of non-singular local multipliers* $\{\Lambda_\nu\}_{\nu=1}^{\ell} = \{\Lambda_\nu(x, U, \partial U, ..., \partial^k U)\}$ *yields a divergence expression for a system of differential equations (17) if and only if the set of equations*

$$\mathrm{E}_{U^\mu}(\Lambda_\nu(x, U, \partial U, ..., \partial^k U)\Delta_\nu(x, U, \partial U, ..., \partial^n U)) \equiv 0, \qquad \mu = 1, ..., q, \quad (21)$$

holds for arbitrary functions $U(x)$.

The set of equations (21) yields the set of linear determining equations to find all sets of local conservation laws multipliers of a given differential equation system (17) by letting $k = 1, 2, ...$ in (21). Since the equations (17) holds for arbitrary $U(x)$, it follows that they also hold for each derivative of $U(x)$ replaced by an arbitrary function.

The direct method to obtain local conservation laws is now illustrated through equation (1). Consider the Fisher-Kolmogorov equation (1), we see all local conservation laws multipliers of the form $\Lambda = \Lambda(x, t, u, u_x, u_t)$, of the equation (1). In terms of Euler operators E_U, we have three local conservation multipliers given by

$$\Lambda_1 = 1, \qquad \Lambda_2 = u_t, \qquad \Lambda_3 = tu_t + x^2 - t^2. \quad (22)$$

For each set of local multipliers, it is straightforward to obtain the following two linearly independent local conservation laws of the equation (1):

$$\Phi = -xtuu_t - \exp(x^2 + t^2 + u^2) + x^3uu_t + \frac{1}{2}(x^2 + u^2), \quad (23)$$

$$\Psi = x^3tu^2u_t + u - u_x. \quad (24)$$

3.2 Lie Point Symmetries and Conservation Laws

In this section we show if any system of differential equations such as (17) maps to system of differential equations

$$\Gamma_\nu[u] = \Gamma_\nu(x, u, \partial u, ..., \partial^n u) = 0, \qquad \nu = 1, ..., \ell, \quad (25)$$

by an invertible transformation, then any conservation law of $\Delta_\nu(x, u^{(n)})$ maps to a conservation law of $\Gamma_\nu(x, u^{(n)})$. When this transformation is a symmetry of system Δ then, the corresponding conservation law is a conservation law of Γ.

Consider the system (17), let

$$\Delta_\nu[U] = \Delta_\nu(x, U, \partial U, ..., \partial^n U) = 0, \qquad \nu = 1, ..., \ell, \quad (26)$$

where $U(x) = (U^1(x), ..., U^q(x))$ is a solution of the system (17).

Consider an invertible point transformation

$$x^i = x^i(z, W),\ i = 1, ..., p; \quad U^\alpha = U^\alpha(z, W),\ \alpha = 1, ..., q, \tag{27}$$

where $U(x) = (U^1(x), ..., U^q(x))$, $z = (z^1, ..., z^q)$ and $W(z) = (W^1(z), ..., W^q(z))$.

Under the transformation (27) and its prolongation, any function $\Delta_\nu[U]$ maps to a function $\Gamma_\nu[W] = \Gamma_\nu(z, W, \partial W, ..., \partial^n W)$. In a special case $\Gamma_\nu[W] = \Delta_\nu[U]$, the components $x, U, \partial U, ..., \partial^n U$ is written in the form of components $z, W, \partial W, ..., \partial^n W$ in (27). If $U(x) = u(x)$ is a solution of the system (17), then, $W(z) = w(z)$ is a solution of the system (25) in the form of

$$\Gamma_\nu[w] = \Gamma_\nu(z, w, \partial w, ..., \partial^n w) = 0, \qquad \nu = 1, ..., \ell, \tag{28}$$

with $p-$independent variables $z = (z^1, ..., z^p)$ and $q-$dependent variables $w = (w^1, ..., w^q)$. Let us consider the invertible transformations (27) is a symmetry of system (26). Then, there are smooth functions $A^\nu_\tau[W]$ such that:

$$\Delta_\nu[U] = \Gamma_\nu[W] = A^\nu_\tau[W]\Delta_\tau[U]. \tag{29}$$

Lemma 3.3 *If a point transformation $(x, u) \mapsto (\tilde{x}(x, y), \tilde{u}(x, u))$ be a symmetry of system (26), then, a conservation law $D_i\Phi^i[u] = 0$ leads to a conservation law $D_i\Psi^i[u] = 0$.*

This lemma shows that the action of a symmetry transformation of system (26) on a conservation law $D_i\Phi^i[u] = 0$ leads us to a new conservation law $D_i\Psi^i[u] = 0$.

Theorem 3.4 *Suppose the point transformation (27) is a symmetry of system (26) . If $\{\Lambda_\nu[U]\}_{\nu=1}^{\ell}$ be a set of conservation laws multipliers with conservation laws $D_i\Phi^i[u]$, then,*

$$\tilde{\Lambda}_\tau[W]\Delta_\tau[W] = \tilde{D}_i\Psi^i[W], \tag{30}$$

where

$$\tilde{\Lambda}_\tau[W] = \mathbf{J}[W]A^\nu_\tau[W]\Lambda_\nu[U(z, W)], \qquad \tau = 1, ..., \ell. \tag{31}$$

Corollary 3.5 *The set of multipliers $\{\tilde{\Lambda}_\nu[U]\}_{\nu=1}^{\ell}$ generates new conservation laws for system (26) if and only if it is a linear independent set on the solutions $U(x) = u(x)$.*

The main result of these section is, we can act point symmetries on the obtained conservation laws for finding new conservation laws. Now according to the basic results of Lie point symmetries [8, 9, 10], we can use **Maple** and obtain the Lie algebra of Lie point symmetry of the equation (1) spanned by the vector fields $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\}$, then we apply these vector fileds for finding new conservation laws for Fisher-Kolmogorov equation. Thus the set of new linear independent multipliers are

$$\Lambda_1 = tuu_t - 2x\exp(x^2+t^2+u^2) + 3x^2uu_t + \frac{1}{2}(2x+u^2), \qquad (32)$$

$$\Lambda_2 = xuu_t - 2t\exp(x^2+t^2+u^2). \qquad (33)$$

Acknowledgements: Lie point symmetries of differential equations is an important object for studying structures of all differential equations. There is a lots of literatures for this but we can use **Maple** and **Mathematica** for finding this kind of symmetries. There are some method to obtain solutions of differential equations by using symmetries [1, 7, 9, 10]. Another symmetries which are called higher order symmetries such as contact symmetries and generalized symmetries [10] could be used for finding new conservation laws foe equation (1).

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