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Regular Elements of the Complete Semigroups of Binary Relations of the Class $\sum_8(X,7)$

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Abstract

In this paper, let $Q = \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ *be a subsemilattice of* $X-$ *semilattice of unions* D *where* $T_6 \subset T_4 \subset T_3 \subset T_1 \subset T_0$, $T_6 \subset T_4 \subset T_3 \subset T_2 \subset T_0$, $T_5 \subset T_4 \subset T_3 \subset T_1 \subset T_0$, $T_5 \subset T_4 \subset T_3 \subset T_2 \subset T_0$, $T_2 \backslash T_1 \neq \emptyset$, $T_1 \backslash T_2 \neq \emptyset$, $T_5 \backslash T_6 \neq \emptyset$, $T_6 \backslash T_5 \neq \emptyset$, $T_2 \cup T_1 = T_0$, $T_6 \cup T_5 = T_4$, *then we characterize each element of the class* $\sum_8(X,7)$ *which is isomorphic to* Q *by means of the characteristic family of sets, the characteristic mapping and the generate set of* Q*. Moreover, we describe the construction of regular elements* α *of* $B_X(D)$ *satisfying* $V(D,\alpha) = Q$*. Additionally, we find the number of these regular elements, when* X *is finite.*

Keywords: *Semigroups, Binary relations, Regular elements.*

1 Introduction

Representations of partially ordered semigroups by binary relations were first considered by Zaretskii [1]. In [2] Zareckii proved that a binary relation α is a regular element of B_X if and only if $V(\alpha)$ $(= V(P(X), \alpha))$ is a completely distributive lattice. Further, criteria for regularity were given by Markowsky [3] and Schein [4]. Then, Diasamidze proved that, a binary relation α is a regular element of B_X iff $V(X^*, \alpha) \subseteq V(D, \alpha)$ and $V(D, \alpha)$ is complete $XI-$ semilattice of unions in [5]. So, Diasamidze extend Zaretskii's theorem and give an intrinsic characterization of regularity since if $D = P(X)$ then $B_X(D) = B_X$

and $V(\alpha)$ $(= V(P(X), \alpha))$ is a completely distributive lattice. Therefore, Diasamidze generate systematic rules for understanding the structure of semigroups of binary relations and characterization of regular elements of these semigroups in $[5-9]$. In general, he studied semigroups but, in particular, he investigates complete semigroups of the binary relations.

In this paper, we take in particular, $Q = \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ subsemilattice of $X-$semilattice of unions D where the elements T_i' s, $i = 0, 1, \ldots, 6$ are satisfying the following properties, $T_6 \subset T_4 \subset T_3 \subset T_1 \subset T_0$, $T_6 \subset T_4 \subset T_3 \subset T_2 \subset T_0$, $T_5 \subset T_4 \subset T_3 \subset T_1 \subset T_0$,$T_5 \subset T_4 \subset T_3 \subset T_2 \subset T_0$, $T_2 \backslash T_1 \neq \emptyset$, $T_1 \backslash T_2 \neq \emptyset$, $T_5 \backslash T_6 \neq \emptyset$, $T_6 \backslash T_5 \neq \emptyset$, $T_2 \cup T_1 = T_0$, $T_6 \cup T_5 = T_4$. We will investigate the properties of regular element $\alpha \in B_X(D)$ satisfying $V(D, \alpha) = Q$. Moreover, we will calculate the number of these regular elements of $B_X(D)$ for a finite set X.

As general, we also characterize the elements of the class $\sum_8(X, 7)$. This class is the complete $X-$semilattice of unions every elements of which are isomorphic to Q. So, we characterize the class for each element of which is isomorphic to Q by means of the characteristic family of sets, the characteristic mapping and the generate set of D.

The material in this work forms a part of first author's PH.D. Thesis, under the supervision of the second author Dr. Neşet Aydın.

2 Preliminaries

We recall various concepts and properties from $[5-10]$.

Let X be an arbitrary nonempty set. Recall that the set of all binary relations on X is denoted by B_X. The binary operation "$\circ$" on B_X defined by for $\alpha,\ \beta \in B_X$

$$(x, z) \in \alpha \circ \beta \Leftrightarrow (x, y) \in \alpha \text{ and } (y, z) \in \beta, \text{ for some } y \in X$$

is associative and hence B_X is a semigroup with respect to the operation " $\circ$ ". This semigroup is called the *semigroup of all binary relations* on the set X.

Let D be a nonempty subset of $P(X)$ such that it is closed under the union i.e., $\cup D' \in D$ for any nonempty subset D' of D. In that case, D is called a *complete $X-$ semilattice of unions*. The union of all elements of D is denoted by the symbol $\breve{D}$. Clearly, $\breve{D}$ is the largest element of D.

The set $N(D, D') = \{Z \in D \mid Z \subseteq Z' \text{ for any } Z' \in D'\}$ is all lower bounds of D' in D. Moreover, if $N(D, D') \neq \emptyset$ then $\Lambda(D, D') = \cup N(D, D')$ belongs to D and it is *the greatest lower bound* of D'.

Let $\widetilde{D}$ and D' be some nonempty subsets of the complete $X-$ semilattices of unions. We say that a subset $\widetilde{D}$ generates a set D' if any element from D' is a set-theoretic union of the elements from $\widetilde{D}$.

Further, let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X$, $T \in D$, $\emptyset \neq D' \subseteq D$ and $t \in \breve{D}$. We use the notations:

$$\begin{array}{ll} y\alpha = \{x \in X \mid (y,x) \in \alpha\} & , \; Y\alpha = \bigcup_{y \in Y} y\alpha, \\ V(D,\alpha) = \{Y\alpha \mid Y \in D\} & , \; D_t = \{Z' \in D \mid t \in Z'\}, \\ D'_T = \{Z' \in D' \mid T \subseteq Z'\} & , \; \ddot{D}'_T = \{Z' \in D' \mid Z' \subseteq T\}. \end{array}$$

Let $X^* = P(X) \backslash \{\emptyset\}$, $\alpha \in B_X$, $Y_T^\alpha = \{y \in X \mid y\alpha = T\}$ and

$$V[\alpha] = \begin{cases} V(X^*,\alpha), \text{ if } \emptyset \notin D, \\ V(X^*,\alpha), \text{ if } \emptyset \in V(X^*,\alpha), \\ V(X^*,\alpha) \cup \{\emptyset\}, \text{ if } \emptyset \notin V(X^*,\alpha) \text{ and } \emptyset \in D. \end{cases}$$

In general, a representation of a binary relation α of the form

$$\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$$

is called *quasinormal*. Note that, if $\alpha \in B_X$ has a quasinormal representation, then $X = \bigcup_{T \in V(X^*,\alpha)} Y_T^\alpha$ and $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ for $T, T' \in V(X^*,\alpha)$ which $T \neq T'$.

In particular, let f be an arbitrary mapping from X into D then $B_X(D)$ denotes the set of all binary relations of the form

$$\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)).$$

It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a *complete semigroup of binary relations* defined by an $X-$semilattice of unions D. Diasamidze introduced this structure and investigated their properties [6].

If $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$ then a binary relation α is called a *regular element* of $B_X(D)$.

A complete $X-$semilattice of unions D is called "$XI-$ semilattice of unions" [9] if it satisfies the following two conditions

1. $\Lambda(D, D_t) \in D$ for any $t \in \breve{D}$,

2. $Z = \bigcup_{t \in Z} \Lambda(D, D_t)$ for any nonempty element Z of D.

In [9] they show that, β is a regular element of $B_X(D)$ iff $V[\beta] = V(D,\beta)$ is a complete $XI-$semilattice of unions.

Let D' be an arbitrary nonempty subset of the complete $X-$semilattice of unions D. A nonempty element $T \in D'$ is a *nonlimiting element* of D' if $T \backslash l(D', T) = T \backslash \cup (D' \backslash D'_T) \neq \emptyset$. A nonempty element $T \in D'$ is a *limiting element* of D' if $T \backslash l(D', T) = \emptyset$.

The family $C(D)$ of pairwise disjoint subsets of the set $\breve{D} = \cup D$ is the *characteristic family* of sets of D if the followings hold

a) $\cap D \in C(D)$,

b) $\cup C(D) = \breve{D}$,

c) There exists a subset $C_Z(D)$ of the set $C(D)$ such that $Z = \cup C_Z(D)$ for all $Z \in D$.

A mapping $\theta : D \rightarrow C(D)$ is called *characteristic mapping* if $Z = (\cap D) \cup \bigcup\limits_{Z' \in \hat{D}} \theta(Z')$ for all $Z \in D$.

The existence and the uniqueness of characteristic family and characteristic mapping is given in Diasamidze [7]. Moreover, it is shown that every $Z \in D$ can be written as

$$Z = \theta(\breve{Q}) \cup \bigcup_{T \in \hat{Q}(Z)} \theta(T),$$

where $\hat{Q}(Z) = Q \backslash \{T \in Q \mid Z \subseteq T\}$.

A one-to-one mapping φ between two complete $X-$ semilattices of unions D' and D'' is called a *complete isomorphism* if $\varphi(\cup D_1) = \underset{T' \in D_1}{\cup} \varphi(T')$ for each nonempty subset D_1 of the semilattice D'. Also, let $\alpha \in B_X(D)$. A complete isomorphism φ between $XI-$semilattice of unions Q and D is called a *complete $\alpha-$ isomorphism* if $Q = V(D, \alpha)$ and $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D, \alpha)$.

Let Q and D' are respectively some $XI-$ and $X-$ subsemilattices of the complete $X-$ semilattice of unions D. Then

$$R_\varphi(Q, D') = \{\alpha \in B_X(D) \mid \alpha \text{ regular}, \varphi \text{ complete } \alpha - \text{isomorphism}\}$$

where $\varphi : Q \rightarrow D'$ complete isomorphism and $V(D, \alpha) = Q$. Besides, let us denote

$$R(Q, D') = \bigcup_{\varphi \in \Phi(Q, D')} R_\varphi(Q, D') \text{ and } R(D') = \bigcup_{Q' \in \Omega(Q)} R(Q', D').$$

where

$\Phi(Q, D') = \{\varphi \mid \varphi : Q \rightarrow D' \text{ is a complete } \alpha\text{−isomorphism for any } \alpha \in B_X(D)\}$
$\Omega(Q) = \{Q' \mid Q' \text{ is } XI\text{−subsemilattices of } D \text{ which is complete isomorphic to } Q\}$

Theorem 2.1. *[8, Theorem 10] Let α and σ be binary relations of the semigroup $B_X(D)$ such that $\alpha \circ \sigma \circ \alpha = \alpha$. If $D(\alpha)$ is some generating set of the semilattice $V(D,\alpha)\setminus\{\emptyset\}$ and $\alpha = \bigcup\limits_{T\in V(D,\alpha)} (Y_T^\alpha \times T)$ is a quasinormal representation of the relation α, then $V(D,\alpha)$ is a complete $XI-$ semilattice of unions. Moreover, there exists a complete $\alpha-$isomorphism φ between the semilattice $V(D,\alpha)$ and $D' = \{T\sigma \mid T \in V(D,\alpha)\}$, that satisfies the following conditions:*

a) $\varphi(T) = T\sigma$ and $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$

b) $\bigcup\limits_{T'\in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq \varphi(T)$ for any $T \in D(\alpha)$,

c) $Y_T^\alpha \cap \varphi(T) \neq \emptyset$ for all nonlimiting element T of the set $\ddot{D}(\alpha)_T$,

d) If T is a limiting element of the set $\ddot{D}(\alpha)_T$, then the equality $\cup B(T) = T$ is always holds for the set $B(T) = \left\{Z \in \ddot{D}(\alpha)_T \mid Y_Z^\alpha \cap \varphi(T) \neq \emptyset\right\}$.

On the other hand, if $\alpha \in B_X(D)$ such that $V(D,\alpha)$ is a complete $XI-$ semilattice of unions. If for a complete $\alpha-$isomorphism φ from $V(D,\alpha)$ to a subsemilattice D' of D satisfies the conditions $b) - d)$ of the theorem, then α is a regular element of $B_X(D)$.

Theorem 2.2. *[9, Theorem 1.18.2] Let $D_j = \{T_1, \ldots, T_j\}$, X be finite set and $\emptyset \neq Y \subseteq X$. If f is a mapping of the set X, on the D_j, for which $f(y) = T_j$ for some $y \in Y$, then the numbers of those mappings f of the sets X on the set D_j can be calculated by the formula $s = j^{|X\setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$.*

Theorem 2.3. *[9, Theorem 6.3.5] Let X is a finite set. If φ is a fixed element of the set $\Phi(D,D')$ and $|\Omega(D)| = m_0$ and q is a number of all automorphisms of the semilattice D then $|R(D')| = m_0 \cdot q \cdot |R_\varphi(D,D')|$.*

3 Results

Let X be a finite set, D be a complete $X-$semilattice of unions and $Q = \{T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ be a $X-$subsemilattice of unions of D satisfies the following conditions

$$\begin{array}{l} T_6 \subset T_4 \subset T_3 \subset T_1 \subset T_0, \\ T_6 \subset T_4 \subset T_3 \subset T_2 \subset T_0, \\ T_5 \subset T_4 \subset T_3 \subset T_1 \subset T_0, \\ T_5 \subset T_4 \subset T_3 \subset T_2 \subset T_0, \\ T_2\setminus T_1 \neq \emptyset,\ T_1\setminus T_2 \neq \emptyset, \\ T_5\setminus T_6 \neq \emptyset,\ T_6\setminus T_5 \neq \emptyset, \\ T_2 \cup T_1 = T_0,\ T_6 \cup T_5 = T_4. \end{array}$$

The diagram of the Q is shown in Figure 3.1.

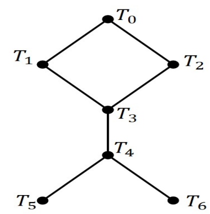


Figure 3.1

Let $C(Q) = \{P_6, P_5, P_4, P_3, P_2, P_1, P_0\}$ be characteristic family of sets of Q and $\theta : Q \to C(Q),\ \theta(T_i) = P_i (i = 0, 1, \ldots, 6)$ be characteristic mapping. Then, by the definition of characteristic family and characteristic mapping for each element $T_i \in Q$ we can write

$$T_i = \theta(\breve{Q}) \cup \bigcup_{T \in \hat{Q}(T_i)} \theta(T), (i = 0, 1, \ldots, 6)$$

where $\hat{Q}(T_i) = Q \setminus \{Z \in Q \mid T_i \subseteq Z\}, \breve{Q} = \cup Q = T_0$ and $\theta(\breve{Q}) = \theta(T_0) = P_0$.

Accordingly, we get

$$\begin{aligned}
T_0 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_0)} \theta(T) = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6,\\
T_1 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_1)} \theta(T) = P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6,\\
T_2 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_2)} \theta(T) = P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6,\\
T_3 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_3)} \theta(T) = P_0 \cup P_4 \cup P_5 \cup P_6,\\
T_4 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_4)} \theta(T) = P_0 \cup P_5 \cup P_6,\\
T_5 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_5)} \theta(T) = P_0 \cup P_6,\\
T_6 &= P_0 \cup \bigcup_{T \in \hat{Q}(T_6)} \theta(T) = P_0 \cup P_5.
\end{aligned} \tag{3.1}$$

Firstly, let us determine that in which conditions Q is $XI-$ semilattice of unions. Then, we specify the greatest lower bounds of the each semilattice Q_t in Q for $t \in T_0$. Since $T_0 = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$ and $P_i\ (i = 0, 1, \ldots, 6)$

are pairwise disjoint sets, by Equation (3.1) and the definition of Q_t, we have

$$Q_t = \begin{cases} Q & , t \in P_0 \\ \{T_0, T_2\} & , t \in P_1 \\ \{T_0, T_1\} & , t \in P_2 \\ \{T_0, T_1, T_2\} & , t \in P_3 \\ \{T_0, T_1, T_2, T_3\} & , t \in P_4 \\ \{T_0, T_1, T_2, T_3, T_4, T_6\} & , t \in P_5 \\ \{T_0, T_1, T_2, T_3, T_4, T_5\} & , t \in P_6 \end{cases} . \tag{3.2}$$

So, by Equation (3.2) and the definition of $N(Q, Q_t)$,

$$N(Q, Q_t) = \begin{cases} \emptyset & , t \in P_0 \\ \{T_2, T_3, T_4, T_5, T_6\} & , t \in P_1 \\ \{T_1, T_3, T_4, T_5, T_6\} & , t \in P_2 \\ \{T_3, T_4, T_5, T_6\} & , t \in P_3 \\ \{T_3, T_4, T_5, T_6\} & , t \in P_4 \\ \{T_6\} & , t \in P_5 \\ \{T_5\} & , t \in P_6 \end{cases} \tag{3.3}$$

are obtained. From the Equation (3.3) the greatest lower bounds for each semilattice Q_t, we get

$$\cup N(Q, Q_t) = \Lambda(Q, Q_t) = \begin{cases} \emptyset & , t \in P_0 \\ T_2 & , t \in P_1 \\ T_1 & , t \in P_2 \\ T_3 & , t \in P_3 \\ T_3 & , t \in P_4 \\ T_6 & , t \in P_5 \\ T_5 & , t \in P_6 \end{cases} . \tag{3.4}$$

If $t \in P_0$ then $\Lambda(D, D_t) = \emptyset \notin D$. So, it must be $P_0 = \emptyset$. Thus using the

Equations (3.1) and (3.4), we have

$$
\begin{aligned}
&t \in T_6 = P_5 \Rightarrow T_6 = \Lambda(Q, Q_t), \\
&t \in T_5 = P_6 \Rightarrow T_5 = \Lambda(Q, Q_t), \\
&t \in T_4 = P_5 \cup P_6 \Rightarrow \ \Rightarrow \Lambda(Q, Q_t) \in \{T_5, T_6\} \\
&\qquad \Rightarrow T_4 = T_5 \cup T_6 = \bigcup_{t \in T_4} \Lambda(Q, Q_t), \\
&t \in T_3 = P_4 \cup P_5 \cup P_6 \Rightarrow \ \Rightarrow \Lambda(Q, Q_t) \in \{T_3, T_5, T_6\} \\
&\qquad \Rightarrow T_3 = T_3 \cup T_5 \cup T_6 = \bigcup_{t \in T_3} \Lambda(Q, Q_t), \\
&t \in T_2 = P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \Rightarrow \Lambda(Q, Q_t) = \{T_2, T_3, T_5, T_6\} \\
&\qquad \Rightarrow T_2 = T_2 \cup T_3 \cup T_5 \cup T_6 = \bigcup_{t \in T_2} \Lambda(Q, Q_t), \\
&t \in T_1 = P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_3, T_5, T_6\} \\
&\qquad \Rightarrow T_1 = T_1 \cup T_3 \cup T_5 \cup T_6 = \bigcup_{t \in T_1} \Lambda(Q, Q_t), \\
&t \in T_0 = T_2 \cup T_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_5, T_6\} \\
&\qquad \Rightarrow T_0 = T_1 \cup T_2 \cup T_3 \cup T_5 \cup T_6 = \bigcup_{t \in T_0} \Lambda(Q, Q_t).
\end{aligned} \tag{3.5}
$$

Lemma 3.1. *Q is $XI-$ semilattice of unions if and only if $T_6 \cap T_5 = \emptyset$.*

Proof. $\Rightarrow$: Let Q be a $XI-$ semilattice of unions. Then $P_0 = \emptyset$ by Equation (3.4) and $T_6 = P_5$, $T_5 = P_6$ by Equation (3.1), we have $T_6 \cap T_5 = \emptyset$ since P_6 and P_5 are pairwise disjoint sets.

$\Leftarrow$: Let $T_6 \cap T_5 = \emptyset$ holds. From Equation (3.1), we obtain $P_0 = \emptyset$. Using the Equations (3.4) and (3.5), we have Q is $XI-$ semilattice of unions. □

Lemma 3.2. *If Q is $XI-$ semilattice of unions then*

$$\{T_6, T_5, (T_1 \cap T_2) \setminus T_4, T_1 \setminus T_2, T_2 \setminus T_1, X \setminus T_0\}$$

is a partition of the set X.

Proof. Considering the (3.1) with $P_0 = \emptyset$, straightforward to see that $\{T_6$, T_5, $(T_1 \cap T_2) \setminus T_4$, $T_1 \setminus T_2$, $T_2 \setminus T_1$, $X \setminus T_0\}$ is a partition of the set X. □

Lemma 3.3. *Let $G = \{T_6, T_5, T_4, T_3, T_2, T_1\}$ be a generating set of Q. Then the elements T_6, T_5, T_3, T_2, T_1 are nonlimiting elements of the sets $\ddot{G}_{T_6}$, $\ddot{G}_{T_5}$, $\ddot{G}_{T_3}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_1}$ respectively and T_4 is a limiting element of the set $\ddot{G}_{T_4}$.*

Proof. Definition of $\ddot{D}'_T$, yield the following equations

$$\begin{aligned}
\ddot{G}_{T_6} &= \{T_6\}\,,\\
\ddot{G}_{T_5} &= \{T_5\}\,,\\
\ddot{G}_{T_4} &= \{T_6, T_5, T_4\}\,,\\
\ddot{G}_{T_3} &= \{T_6, T_5, T_4, T_3\}\,,\\
\ddot{G}_{T_2} &= \{T_6, T_5, T_4, T_3, T_2\}\,,\\
\ddot{G}_{T_1} &= \{T_6, T_5, T_4, T_3, T_1\}\,.
\end{aligned} \tag{3.6}$$

Now we get the sets $l(\ddot{G}_{T_i}, T_i)$, $i \in \{1, 2, \ldots, 6\}$,

$$\begin{array}{lcll}
l(\ddot{G}_{T_6}, T_6) & = & \cup(\ddot{G}_{T_6} \setminus \{T_6\}) & = \emptyset,\\
l(\ddot{G}_{T_5}, T_5) & = & \cup(\ddot{G}_{T_5} \setminus \{T_5\}) & = \emptyset,\\
l(\ddot{G}_{T_4}, T_4) & = & \cup(\ddot{G}_{T_4} \setminus \{T_4\}) & = T_4,\\
l(\ddot{G}_{T_3}, T_3) & = & \cup(\ddot{G}_{T_3} \setminus \{T_3\}) & = T_4,\\
l(\ddot{G}_{T_2}, T_2) & = & \cup(\ddot{G}_{T_2} \setminus \{T_2\}) & = T_3,\\
l(\ddot{G}_{T_1}, T_1) & = & \cup(\ddot{G}_{T_1} \setminus \{T_1\}) & = T_3.
\end{array}$$

Then we find nonlimiting and limiting elements of $\ddot{G}_{T_i}$, $i \in \{1, 2, \ldots, 6\}$.

$$\begin{array}{lll}
T_6 \setminus l(\ddot{G}_{T_6}, T_6) & = T_6 \setminus \emptyset & = T_6 \neq \emptyset\\
T_5 \setminus l(\ddot{G}_{T_5}, T_5) & = T_5 \setminus \emptyset & = T_5 \neq \emptyset\\
T_4 \setminus l(\ddot{G}_{T_4}, T_4) & = T_4 \setminus T_4 & = \emptyset\\
T_3 \setminus l(\ddot{G}_{T_3}, T_3) & = T_3 \setminus T_4 & \neq \emptyset\\
T_2 \setminus l(\ddot{G}_{T_2}, T_2) & = T_2 \setminus T_3 & \neq \emptyset\\
T_1 \setminus l(\ddot{G}_{T_1}, T_1) & = T_1 \setminus T_3 & \neq \emptyset
\end{array}$$

So, the elements T_6, T_5, T_3, T_2, T_1 are nonlimiting elements of the sets $\ddot{G}_{T_6}$, $\ddot{G}_{T_5}$, $\ddot{G}_{T_3}$, $\ddot{G}_{T_2}$, $\ddot{G}_{T_1}$ respectively and T_4 is a limiting element of the set $\ddot{G}_{T_4}$. □

Note that, if $\alpha \in B_X(D)$ is regular then from the definition of the set $B_X(D)$ there is a mapping f from X into D such that

$$\alpha = \bigcup_{x \in D} (\{x\} \times f(x))\,.$$

Thus, $f(x) \in D$. Besides, we know that $\alpha \in B_X(D)$ is regular iff $V(D, \alpha)$ is $XI-$ semilattice of unions where $V(D, \alpha) = V[\alpha]$. For this reason, there is a $XI-$ subsemilattice $D' \subset D$ and $V(D, \alpha) = D' = V(D', \alpha)$. So we can write α as,

$$\alpha = \bigcup_{T \in V[\alpha]} (Y_T^{\alpha} \times T) = \bigcup_{T \in D'} (Y_T^{\alpha} \times T)\,.$$

In particular, let us determine the properties of regular elements $\alpha \in B_X(D)$ such that $\alpha = \bigcup_{i=0}^{6} (Y_i^{\alpha} \times T_i)$ where $V(D, \alpha) = Q$.

Theorem 3.4. *Let $\alpha \in B_X(D)$ be a quasinormal representation of the form*

$$\alpha = \bigcup_{i=0}^{6} (Y_i^\alpha \times T_i)$$

such that $V(D,\alpha) = Q$. $\alpha \in B_X(D)$ is a regular iff for some complete α-isomorphism $\varphi : Q \to D' \subseteq D$, the following conditions are satisfied:

$$\begin{aligned}
&Y_6^\alpha \supseteq \varphi(T_6),\\
&Y_5^\alpha \supseteq \varphi(T_5),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \supseteq \varphi(T_1),\\
&Y_1^\alpha \cap \varphi(T_1) \neq \emptyset,\ Y_2^\alpha \cap \varphi(T_2) \neq \emptyset,\\
&Y_3^\alpha \cap \varphi(T_3) \neq \emptyset.
\end{aligned} \tag{3.7}$$

Proof. Let $G = \{T_6, T_5, T_4, T_3, T_2, T_1\}$ be a generating set of Q.

$\Rightarrow$: Since $\alpha \in B_X(D)$ is regular and $V(D,\alpha) = Q$, Q is $XI-$semilattice of unions. From Theorem 2.1, there exits a complete isomorphism $\varphi : Q \to D'$. Considering Theorem 2.1 (a), $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$. So, φ is complete α-isomorphism. Applying the Theorem 2.1 (b) we have

$$\begin{aligned}
&Y_6^\alpha \supseteq \varphi(T_6),\ Y_5^\alpha \supseteq \varphi(T_5),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \supseteq \varphi(T_2),\\
&Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \supseteq \varphi(T_1).
\end{aligned} \tag{3.8}$$

By using φ is complete α-isomorphism, $Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \supseteq \varphi(T_6) \cup \varphi(T_5) = \varphi(T_4)$ always ensured. Moreover, considering that the elements T_6, T_5, T_3, T_2, T_1 are nonlimiting elements of the sets $\ddot{G}_{T_6}, \ddot{G}_{T_5}, \ddot{G}_{T_3}, \ddot{G}_{T_2}, \ddot{G}_{T_1}$ respectively and using the Theorem 2.1 (c), following properties

$$\begin{aligned}
&Y_1^\alpha \cap \varphi(T_1) \neq \emptyset,\ Y_2^\alpha \cap \varphi(T_2) \neq \emptyset,\\
&Y_3^\alpha \cap \varphi(T_3) \neq \emptyset,\ Y_5^\alpha \cap \varphi(T_5) \neq \emptyset,\ Y_6^\alpha \cap \varphi(T_6) \neq \emptyset
\end{aligned} \tag{3.9}$$

are obtained. From $Y_6^\alpha \supseteq \varphi(T_6)$ and $Y_5^\alpha \supseteq \varphi(T_5)$, $Y_6^\alpha \cap \varphi(T_6) \neq \emptyset$ and $Y_5^\alpha \cap \varphi(T_5) \neq \emptyset$ always ensured. Thus there is no need the condition $Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4)$, $Y_5^\alpha \cap \varphi(T_5) \neq \emptyset$ and $Y_6^\alpha \cap \varphi(T_6) \neq \emptyset$. Therefore, there exist an $\alpha-$isomorphism φ which holds given conditions.

$\Leftarrow$: $V(D,\alpha)$ is $XI-$semilattice of unions, because of $V(D,\alpha)$ is equal to Q . Let $\varphi : Q \to D'$ be a complete $\alpha-$isomorphism which holds given conditions. So, by Equation (3.7), satisfying Theorem 2.1 $(a) - (c)$. Remembering that T_4 is a limiting element of the set $\ddot{G}_{T_4}$, we constitute the set

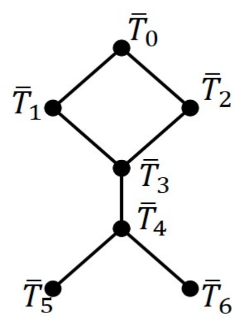
$B(T_4) = \left\{ Z \in \ddot{G}_{T_4} \mid Y_Z^{\alpha} \cap \varphi(T_4) \neq \emptyset \right\}$. If $Y_6^{\alpha} \cap \varphi(T_4) = \emptyset$ we have

$$Y_6^{\alpha} \cup Y_5^{\alpha} \supseteq \varphi(T_6) \cup \varphi(T_5) = \varphi(T_4)$$

So, we get $Y_5^{\alpha} \supseteq \varphi(T_4) \supseteq \varphi(T_6)$, it contradicts with $Y_6^{\alpha} \supseteq \varphi(T_6)$. Therefore, $T_6 \in B(T_4)$. Similarly, if $Y_5^{\alpha} \cap \varphi(T_4) = \emptyset$ then $Y_6^{\alpha} \supseteq \varphi(T_4) \supseteq \varphi(T_5)$. This result in a contradiction since $Y_6^{\alpha} \supseteq \varphi(T_6)$. Therefore, $T_5 \in B(T_4)$. We have $\cup B(T_4) = T_6 \cup T_5 = T_4$. By Theorem 2.1, we conclude that α is the regular element of the $B_X(D)$. □

Now we calculate the number of regular elements α, satisfying the hypothesis of Theorem 3.4.

Let $\alpha \in B_X(D)$ be a regular element which is quasinormal representation of the form $\alpha = \bigcup_{i=0}^{6} (Y_i^{\alpha} \times T_i)$ and $V(D, \alpha) = Q$. Then there exist a complete $\alpha-$ isomorphism $\varphi : Q \to D' = \{\varphi(T_6), \ldots, \varphi(T_1), \varphi(T_0)\}$ satisfying the hypothesis of Theorem 3.4. So, $\alpha \in R_{\varphi}(Q, D')$. We will denote $\varphi(T_i) = \overline{T}_i,\ i = 0, 1, \ldots 6$. Diagram of the $D' = \left\{ \overline{T}_6,\ \overline{T}_5,\ \overline{T}_4,\ \overline{T}_3,\ \overline{T}_2,\ \overline{T}_1,\ \overline{T}_0 \right\}$ is shown in Figure 3.2.



Then the Equation (3.7) reduced to below equation.

$$\begin{aligned}
&Y_6^{\alpha} \supseteq \overline{T}_6, \\
&Y_5^{\alpha} \supseteq \overline{T}_5, \\
&Y_6^{\alpha} \cup Y_5^{\alpha} \cup Y_4^{\alpha} \cup Y_3^{\alpha} \supseteq \overline{T}_3, \\
&Y_6^{\alpha} \cup Y_5^{\alpha} \cup Y_4^{\alpha} \cup Y_3^{\alpha} \cup Y_2^{\alpha} \supseteq \overline{T}_2, \\
&Y_6^{\alpha} \cup Y_5^{\alpha} \cup Y_4^{\alpha} \cup Y_3^{\alpha} \cup Y_1^{\alpha} \supseteq \overline{T}_1, \\
&Y_1^{\alpha} \cap \overline{T}_1 \neq \emptyset,\ Y_2^{\alpha} \cap \overline{T}_2 \neq \emptyset, \\
&Y_3^{\alpha} \cap \overline{T}_3 \neq \emptyset.
\end{aligned} \tag{3.10}$$

Moreover, the image of the sets in Lemma 3.2 under the $\alpha-$ isomorphism φ

$$\overline{T}_6, \overline{T}_5, (\overline{T}_1 \cap \overline{T}_2) \backslash \overline{T}_4, \overline{T}_1 \backslash \overline{T}_2, \overline{T}_2 \backslash \overline{T}_1, X \backslash \overline{T}_0$$

are also pairwise disjoint sets and union of these sets equals X.

Lemma 3.5. *For every $\alpha \in R_{\varphi}(Q, D')$, there exists an ordered system of disjoint mappings which is defined $\{\overline{T}_6, \overline{T}_5, (\overline{T}_1 \cap \overline{T}_2)\backslash\overline{T}_4, \overline{T}_2\backslash\overline{T}_1, \overline{T}_1\backslash\overline{T}_2, X\backslash\overline{T}_0\}$. Also, ordered systems are different which correspond to different binary relations.*

Proof. Let $f_\alpha : X \to D$ be a mapping satisfying the condition $f_\alpha(t) = t\alpha$ for all $t \in X$. We consider the restrictions of the mapping f_α as $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$ and $f_{5\alpha}$ on the sets $\overline{T}_6, \overline{T}_5, (\overline{T}_1 \cap \overline{T}_2)\backslash\overline{T}_4, \overline{T}_1\backslash\overline{T}_2, \overline{T}_2\backslash\overline{T}_1, X\backslash\overline{T}_0$ respectively.

Now, considering the definition of the sets Y_i^α, $(i = 0, 1, \ldots, 6)$ together with the Equation (3.10) we have

$$t \in \overline{T}_6 \Rightarrow t \in Y_6^\alpha \Rightarrow t\alpha = T_6 \Rightarrow f_{0\alpha}(t) = T_6,\ \forall t \in \overline{T}_6.$$
$$t \in \overline{T}_5 \Rightarrow t \in Y_5^\alpha \Rightarrow t\alpha = T_5 \Rightarrow f_{1\alpha}(t) = T_5,\ \forall t \in \overline{T}_5.$$

$$\begin{aligned} t \in (\overline{T}_1 \cap \overline{T}_2)\backslash\overline{T}_4 &\Rightarrow t \in \overline{T}_1 \cap \overline{T}_2 \subseteq Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \\ &\Rightarrow t\alpha \in \{T_6, T_5, T_4, T_3\} \\ &\Rightarrow f_{2\alpha}(t) \in \{T_6, T_5, T_4, T_3\}\,,\ \forall t \in (\overline{T}_1 \cap \overline{T}_2)\backslash\overline{T}_4. \end{aligned}$$

Since $Y_3^\alpha \cap \overline{T}_3 \neq \emptyset$, there is an element $t_1 \in Y_3^\alpha \cap \overline{T}_3$. Then $t_1\alpha = T_3$ and $t_1 \in \overline{T}_3$. If $t_1 \in \overline{T}_4$ then $t_1 \in \overline{T}_4 = \overline{T}_5 \cup \overline{T}_6 \subseteq Y_5^\alpha \cup Y_6^\alpha$. Therefore, $t_1\alpha = \{T_6, T_5\}$ which is in contradiction with the equality $t_1\alpha = T_3$. So $f_{2\alpha}(t_1) = T_3$ for some $t_1 \in \overline{T}_3\backslash\overline{T}_4$.

$$\begin{aligned} t \in \overline{T}_2\backslash\overline{T}_1 &\Rightarrow t \in \overline{T}_2 \subseteq Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \\ &\Rightarrow t\alpha \in \{T_6, T_5, T_4, T_3, T_2\} \\ &\Rightarrow f_{3\alpha}(t) \in \{T_6, T_5, T_4, T_3, T_2\}\,,\ \forall t \in \overline{T}_2\backslash\overline{T}_1. \end{aligned}$$

Also, since $Y_2^\alpha \cap \overline{T}_2 \neq \emptyset$ there is an element t_2, $t_2\alpha = T_2$ and $t_2 \in \overline{T}_2$. If $t_2 \in \overline{T}_1$ then $t_2 \in \overline{T}_1 \subseteq Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha$. Therefore, $t_2\alpha \in \{T_6, T_5, T_4, T_3, T_1\}$ which is in contradiction with the equality $t_2\alpha = T_2$. So $f_{3\alpha}(t_2) = T_2$ for some $t_2 \in \overline{T}_2\backslash\overline{T}_1$.

$$\begin{aligned} t \in \overline{T}_1\backslash\overline{T}_2 &\Rightarrow t \in \overline{T}_1 \subseteq Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \\ &\Rightarrow t\alpha \in \{T_6, T_5, T_4, T_3, T_1\} \\ &\Rightarrow f_{4\alpha}(t) \in \{T_6, T_5, T_4, T_3, T_1\}\,,\ \forall t \in \overline{T}_1\backslash\overline{T}_2. \end{aligned}$$

Similarly, $t_3 \in Y_1^\alpha \cap \overline{T}_1$ since $Y_1^\alpha \cap \overline{T}_1 \neq \emptyset$. Then $t_3\alpha = T_1$ and $t_3 \in \overline{T}_1$. If $t_3 \in \overline{T}_2$ then $t_3 \in Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha$. So $t_3\alpha \in \{T_6, T_5, T_4, T_3, T_2\}$. However this contradicts to $t_3\alpha = T_1$. So $f_{4\alpha}(t_3) = T_1$ for some $t_3 \in \overline{T}_1\backslash\overline{T}_2$.

$$t \in X\backslash\overline{T}_0 \Rightarrow t \in X = \bigcup_{i=0}^{6} Y_i^\alpha \Rightarrow t\alpha \in Q \Rightarrow f_{5\alpha}(t) \in Q,\ \forall t \in X\backslash\overline{T}_0.$$

Therefore, for every binary relation $\alpha \in R_{\varphi}(Q, D')$ there exists an ordered system $(f_{0\alpha},\ f_{1\alpha},\ f_{2\alpha},\ f_{3\alpha},\ f_{4\alpha},\ f_{5\alpha})$.

On the other hand, suppose that for $\alpha, \beta \in R_\varphi(Q, D')$ which $\alpha \neq \beta$, be obtained $f_\alpha = (f_{0\alpha},\ f_{1\alpha},\ f_{2\alpha},\ f_{3\alpha},\ f_{4\alpha},\ f_{5\alpha})$ and $f_\beta = (f_{0\beta},\ f_{1\beta},\ f_{2\beta},\ f_{3\beta},\ f_{4\beta},\ f_{5\beta})$. If $f_\alpha = f_\beta$, we get

$$f_\alpha = f_\beta \Rightarrow f_\alpha(t) = f_\beta(t),\ \forall t \in X \Rightarrow t\alpha = t\beta,\ \forall t \in X \Rightarrow \alpha = \beta$$

which contradicts to $\alpha \neq \beta$. Therefore, different binary relations's ordered systems are different. □

Lemma 3.6. *Let $f = (f_0,\ f_1,\ f_2,\ f_3,\ f_4,\ f_5)$ be ordered system from X in the semilattice D such that*

$f_0{:}\overline{T}_6 \to \{T_6\}$,
$f_1{:}\overline{T}_5 \to \{T_5\}$,
f_2: $(\overline{T}_2 \cap \overline{T}_1)\backslash\overline{T}_4 \to \{T_6, T_5, T_4, T_3\}$ *and* $f_2(a) = T_3,\ \exists\ a \in \overline{T}_3\backslash\overline{T}_4$,
f_3: $\overline{T}_2\backslash\overline{T}_1 \to \{T_6, T_5, T_4, T_3, T_2\}$ *and* $f_3(b) = T_2,\ \exists\ b \in \overline{T}_2\backslash\overline{T}_1$,
f_4: $\overline{T}_1\backslash\overline{T}_2 \to \{T_6, T_5, T_4, T_3, T_1\}$ *and* $f_4(c) = T_1,\ \exists\ c \in \overline{T}_1\backslash\overline{T}_2$,
f_5: $X\backslash\overline{T}_0 \to Q$.

Then $\beta = \bigcup\limits_{x \in X}(\{x\}\times f(x)) \in B_X(D)$ *is regular and* φ *is a complete* $\beta-$*isomorphism. So* $\beta \in R_\varphi(Q, D')$.

Proof. First we see that $V(D, \beta) = Q$. Considering $V(D, \beta) = \{Y\beta \mid Y \in D\}$, the properties of f mapping, $\overline{T}_i\beta = \bigcup\limits_{x\in\overline{T}_i} x\beta$ and $D' \subseteq D$, we get

$T_6 \in Q \Rightarrow \overline{T}_6\beta = T_6 \Rightarrow T_6 \in V(D, \beta)$,
$T_5 \in Q \Rightarrow \overline{T}_5\beta = T_5 \Rightarrow T_5 \in V(D, \beta)$,
$T_4 \in Q \Rightarrow \overline{T}_4\beta = \overline{T}_5\beta \cup \overline{T}_6\beta = T_5 \cup T_6 = T_4 \Rightarrow T_4 \in V(D, \beta)$,
$T_3 \in Q \Rightarrow \overline{T}_3\beta = \left(\left(\overline{T}_3\backslash\overline{T}_4\right) \cup \overline{T}_4\right)\beta = T_6 \cup T_5 \cup T_4 \cup T_3 = T_3 \Rightarrow T_3 \in V(D, \beta)$,
$T_2 \in Q \Rightarrow \overline{T}_2\beta = T_6 \cup T_5 \cup T_4 \cup T_3 \cup T_2 = T_2 \Rightarrow T_2 \in V(D, \beta)$,
$T_1 \in Q \Rightarrow \overline{T}_1\beta = T_6 \cup T_5 \cup T_4 \cup T_3 \cup T_1 = T_1 \Rightarrow T_1 \in V(D, \beta)$,
$T_0 \in Q \Rightarrow \overline{T}_0\beta = T_6 \cup T_5 \cup T_4 \cup T_3 \cup T_2 \cup T_1 = T_0 \Rightarrow T_0 \in V(D, \beta)$.

Hence, $Q \subseteq V(D, \beta)$. Also,

$$\begin{aligned} Z \in V(D, \beta) &\Rightarrow Z = Y\beta,\ \exists Y \in D \\ &\Rightarrow Z = Y\beta = \bigcup_{y\in Y} y\beta = \bigcup_{y\in Y} f(y) \in Q \end{aligned}$$

since $f(y) \in Q$ and Q is closed set-theoretic union. Therefore, $V(D, \beta) \subseteq Q$. Hence $V(D, \beta) = Q$.

Moreover, $\beta = \bigcup_{T\in V(X^*,\beta)} \left(Y_T^\beta \times T\right)$ is a quasinormal representation since $\emptyset \notin Q$. From the definition of β, $f(x) = x\beta$ for all $x \in X$. It is easily seen that $V(X^*,\beta) = V(D,\beta) = Q$. We get $\beta = \bigcup_{i=0}^{6} \left(Y_i^\beta \times T_i\right)$.

On the other hand

$$\begin{aligned}
&t \in \overline{T}_6 \Rightarrow t\beta = f(t) = T_6 \Rightarrow t \in Y_6^\beta \Rightarrow \overline{T}_6 \subseteq Y_6^\beta,\\
&t \in \overline{T}_5 \Rightarrow t\beta = f(t) = T_5 \Rightarrow t \in Y_5^\beta \Rightarrow \overline{T}_5 \subseteq Y_5^\beta,\\
&t \in \overline{T}_3 = \left(\overline{T}_3\backslash\overline{T}_4\right) \cup \overline{T}_5 \cup \overline{T}_6 \Rightarrow t\beta = f(t) \in \{T_6, T_5, T_4, T_3\}\\
&\qquad \Rightarrow t \in Y_6^\beta \cup Y_5^\beta \cup Y_4^\beta \cup Y_3^\beta\\
&\qquad \Rightarrow \overline{T}_3 \subseteq Y_6^\beta \cup Y_5^\beta \cup Y_4^\beta \cup Y_3^\beta,\\
&t \in \overline{T}_2 = \overline{T}_6 \cup \overline{T}_5 \cup \left(\overline{T}_2\backslash\overline{T}_1\right) \cup \left((\overline{T}_2 \cap \overline{T}_1)\backslash\overline{T}_4\right) \Rightarrow t\beta = f(t) \in \{T_6, T_5, T_4, T_3, T_2\}\\
&\qquad \Rightarrow t \in Y_6^\beta \cup Y_5^\beta \cup Y_4^\beta \cup Y_3^\beta \cup Y_2^\beta\\
&\qquad \Rightarrow \overline{T}_2 \subseteq Y_6^\beta \cup Y_5^\beta \cup Y_4^\beta \cup Y_3^\beta \cup Y_2^\beta,\\
&t \in \overline{T}_1 = \overline{T}_6 \cup \overline{T}_5 \cup \left(\overline{T}_1\backslash\overline{T}_2\right) \cup \left((\overline{T}_2 \cap \overline{T}_1)\backslash\overline{T}_4\right) \Rightarrow t\beta = f(t) \in \{T_6, T_5, T_4, T_3, T_1\}\\
&\qquad \Rightarrow t \in Y_6^\beta \cup Y_5^\beta \cup Y_4^\beta \cup Y_3^\beta \cup Y_1^\beta\\
&\qquad \Rightarrow \overline{T}_5 \subseteq Y_6^\beta \cup Y_5^\beta \cup Y_4^\beta \cup Y_3^\beta \cup Y_1^\beta.
\end{aligned}$$

Also, by using $f_2(a) = T_3$, $\exists\ a \in \overline{T}_3\backslash\overline{T}_4$, we obtain $Y_3^\beta \cap \overline{T}_3 \neq \emptyset$. Similarly, from properties of f_3, f_4, be seen $Y_2^\beta \cap \overline{T}_2 \neq \emptyset$ and $Y_1^\beta \cap \overline{T}_1 \neq \emptyset$. Therefore, the mapping $\varphi : Q \to D' = \left\{\overline{T}_0, \overline{T}_1, \ldots, \overline{T}_6\right\}$ to be defined $\varphi(T_i) = \overline{T}_i$ satisfy the conditions in (3.10) for β. Hence φ is complete $\beta-$isomorhism because of $\varphi(T)\beta = \overline{T}\beta = T$, for all $T \in V(D,\beta)$. By Theorem 3.4, $\beta \in R_\varphi(Q, D')$. □

Therefore, there is one to one correspondence between the elements of $R_\varphi(Q, D')$ and the set of ordered systems of disjoint mappings.

Theorem 3.7. *Let X be a finite set and Q be $XI-$ semilattice. If*

$$D' = \left\{\overline{T}_6, \overline{T}_5, \overline{T}_4, \overline{T}_3, \overline{T}_2, \overline{T}_1, \overline{T}_0\right\}$$

is $\alpha-$ isomorphic to Q and $\Omega(Q) = m_0$, then

$$\begin{aligned}
R(D') = \ & m_0 \cdot 4 \cdot \left(4^{|\overline{T}_3\backslash\overline{T}_4|} - 3^{|\overline{T}_3\backslash\overline{T}_4|}\right) \cdot 4^{|(\overline{T}_2\cap\overline{T}_1)\backslash\overline{T}_3|}\cdot\\
& \left(5^{|\overline{T}_2\backslash\overline{T}_1|} - 4^{|\overline{T}_2\backslash\overline{T}_1|}\right) \cdot \left(5^{|\overline{T}_1\backslash\overline{T}_2|} - 4^{|\overline{T}_1\backslash\overline{T}_2|}\right) \cdot 7^{|X\backslash\overline{T}_0|}
\end{aligned}$$

Proof. Lemma 3.5 and Lemma 3.6 show us that the number of the ordered system of disjoint mappings $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha})$ is equal to $|R_\varphi(Q, D')|$, which $\alpha \in B_X(D)$ regular element, $V(D,\alpha) = Q$ and $\varphi : Q \to D'$ is a complete $\alpha-$isomorphism.

From the Theorem 2.2, the number of the mappings $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$and $f_{5\alpha}$ are respectively as

$$1,1, \left(4^{\left|\overline{T}_3\setminus\overline{T}_4\right|}-3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}\right)\cdot 4^{\left|(\overline{T}_2\cap\overline{T}_1)\setminus\overline{T}_3\right|},$$
$$\left(5^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-4^{\left|\overline{T}_2\setminus\overline{T}_1\right|}\right), \left(5^{\left|\overline{T}_1\setminus\overline{T}_2\right|}-4^{\left|\overline{T}_1\setminus\overline{T}_2\right|}\right), 7^{\left|X\setminus\overline{T}_0\right|}.$$

Now, we determine the number of regular elements

$$|R_\varphi(Q,D')|=\left(4^{\left|\overline{T}_3\setminus\overline{T}_4\right|}-3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}\right)\cdot 4^{\left|(\overline{T}_2\cap\overline{T}_1)\setminus\overline{T}_3\right|}\cdot$$
$$\left(5^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-4^{\left|\overline{T}_2\setminus\overline{T}_1\right|}\right)\cdot\left(5^{\left|\overline{T}_1\setminus\overline{T}_2\right|}-4^{\left|\overline{T}_1\setminus\overline{T}_2\right|}\right)\cdot 7^{\left|X\setminus\overline{T}_0\right|}.$$

The number of all automorphisms of the semilattice Q is $q=4$. These are

$$id_Q=\begin{pmatrix} T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\\ T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\end{pmatrix},$$
$$\tau_1=\begin{pmatrix} T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\\ T_5 & T_6 & T_4 & T_3 & T_2 & T_1 & T_0\end{pmatrix},$$
$$\tau_2=\begin{pmatrix} T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\\ T_6 & T_5 & T_4 & T_3 & T_1 & T_2 & T_0\end{pmatrix},$$
$$\tau_3=\begin{pmatrix} T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\\ T_5 & T_6 & T_4 & T_3 & T_1 & T_2 & T_0\end{pmatrix}.$$

Therefore by using Theorem 2.3,

$$\begin{aligned} R(D') = \ & m_0\cdot 4\cdot\left(4^{\left|\overline{T}_3\setminus\overline{T}_4\right|}-3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}\right)\cdot 4^{\left|(\overline{T}_2\cap\overline{T}_1)\setminus\overline{T}_3\right|}\cdot\\ & \left(5^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-4^{\left|\overline{T}_2\setminus\overline{T}_1\right|}\right)\cdot\left(5^{\left|\overline{T}_1\setminus\overline{T}_2\right|}-4^{\left|\overline{T}_1\setminus\overline{T}_2\right|}\right)\cdot 7^{\left|X\setminus\overline{T}_0\right|}\end{aligned}$$

is obtained. □

Example 1. *Let $X=\{1,2,3,4,5\}$ and*

$$D=\{\{1\},\ \{2\},\ \{1,2\},\ \{1,2,3\},\ \{1,2,3,4\},\ \{1,2,3,5\},\ \{1,2,3,4,5\}\}.$$

D is an $X-$semilattice of unions since D is closed the union of sets. Moreover D satisfies the conditions in (3.1) and $\{1\}\cap\{2\}=\emptyset$. Then, D is an $XI-$semilattice. Let $D=Q$. Therefore $|\Omega(Q)|=1$. Besides, the number of all automorphisms of Q is $q=4$. By using Theorem 3.7

$$\begin{aligned} |R(D)| = \ & 4\cdot\left(4^{\left|\overline{T}_3\setminus\overline{T}_4\right|}-3^{\left|\overline{T}_3\setminus\overline{T}_4\right|}\right)4^{\left|(\overline{T}_2\cap\overline{T}_1)\setminus\overline{T}_3\right|}\\ & \left(5^{\left|\overline{T}_2\setminus\overline{T}_1\right|}-4^{\left|\overline{T}_2\setminus\overline{T}_1\right|}\right)\cdot\left(5^{\left|\overline{T}_1\setminus\overline{T}_2\right|}-4^{\left|\overline{T}_1\setminus\overline{T}_2\right|}\right)\cdot 7^{\left|X\setminus\overline{T}_0\right|}\\ & = 4\cdot(4^1-3^1)\cdot 4^0\cdot(5^1-4^1)\cdot(5^1-1^1)\cdot 7^0\\ & = 4\end{aligned}$$

is obtained.

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