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Some Double Sequence Spaces Defined by a Modulus Function

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Abstract

In the present paper we construct some new double sequence spaces defined by a modulus function. Further we give its some topological and algebraic properties.

Keywords: *Double sequence space, Modulus Function, vector valued double sequence space.*

1 Introduction

Some works on double sequences were studied by Hardy[5], Moricz[6], Moricz and Rhoades[7]. Hardy introduced the notion of regular convergence for double sequences. Hill [8] was the first who applied methods of functional analysis to doubles sequences. He described the topological dual of the space of all regularly convergent double sequences and perfectness of matrices with respect to the regular convergence. Türkmenoğlu [10] showed under which conditions that $C_{0p}(t)$, $\mathcal{L}_u$ and $C_{0bp}(t)$ are paranormed double sequence spaces, determined their duals and gave some inclusion relations between those spaces. Boos, Leiger and Zeller [11] defined the concept of ν-SM method by the application domain of a matrix sequence $\mathcal{A} = \left(\mathcal{A}^{(\nu)}\right)$ of infinite matrices, gave the consistency theory for such type methods and introduced the notion of

C_e-convergence for double sequences. By using gliding hump method, Zeltser [12] characterized the class of four-dimensional matrix mappings from λ into μ, where $\lambda, \mu \in \{C_e, C_{be}\}$. By C_e and C_{be}, we denote the spaces of all C_e-convergent and of all bounded C_e-convergent double sequences, respectively. Also employing the same arguments, Zeltser [13] gave the theorems determining the necessary and sufficient conditions for C_e-SM and C_{be}-SM methods to be concervative and coercive.

Modulus function introduced by Nakano [1] and used to solve some structural problems of the scalar FK-spaces theory. Ruckle [2] constructed a class of scalar FK-spaces $L(f)$, where f is a modulus function. $L(f)$ is a generalization of the spaces $l_p, (0 < p \leq 1)$.

Yılmaz [3] introduced and investigated the sequence space $\lambda(X_k, r, f, s)$ defined by a modulus function and constructed its FK-structure under some conditions.

In the present paper we introduce some new double sequence spaces by using a modulus function f and investigate some properties of these sequence spaces.

Let $\Omega(X)$ be the space of all X-valued double sequences and X be a Banach space. The topology of $\Omega(X)$ is a locally convex topology produced by the family of all seminorms defined by

$$p_{ij}(x) = \|x_{ij}\|.$$

Since the family of $\{p_{ij} : i, j \in \mathbb{N} \times \mathbb{N}\}$ are countable, this topology is metrizable, where $\|,\|$ is the norm on X.

Also, total paranorm generating this metric, which is constructed by the Frechet combination of seminorms $\{p_{ij} : i, j \in \mathbb{N} \times \mathbb{N}\}$, is given by

$$g(x) = \sum \frac{1}{2^{i+j}} \frac{\|x_{ij}\|}{1 + \|x_{ij}\|}.$$

A locally convex double sequence space (E, τ) is said to be a DK-space, if all the seminorms defined by

$$\begin{aligned} r_{kl} &: E \to \mathbb{R} \\ x &= (x_{ij}) \to |x_{kl}|, \end{aligned}$$

are continuous. A DK-space with a Frechet topology is called an FDK-space. A normed FDK-space is called BDK-space [4].

For $\lambda(X) \subset \Omega(X)$, a Frechet sequence space $\lambda(X)$ is called an FDK-space, if the coordinate maps

$$\begin{aligned} f_{kl} : \lambda(X) &\to X \\ x &\to f_{kl}(x) = x_{kl} \end{aligned}$$

are continuous.

2 The Double Sequence Spaces $\mathcal{L}_u(f)$

Let f be a modulus and let us define the double squence space $\mathcal{L}_u(f)$ by

$$\mathcal{L}_u(f) = \{x \in \Omega : \sum_{m,n\in\mathbb{N}} f(|x_{mn}|) < \infty\}.$$

It is easy to show that $\mathcal{L}_u(f)$ is a linear space, where Ω denotes the space of all real or complex valued double sequences.

Theorem 2.1 $\mathcal{L}_u(f)$ *is a paranormed space with the function*

$$g(x) = \sum_{m,n\in\mathbb{N}} f(|x_{mn}|).$$

Proof: It is obvious that g is well-defined by the definition of $\mathcal{L}_u(f)$ .Let us verify that g provides the paranorm conditions:

i) $g(\theta) = \sum_{m,n\in\mathbb{N}} f(|\theta|) = 0.$

ii) For each $x \in \mathcal{L}_u(f)$, it is clear that

$$g(x) = g(-x).$$

iii) For all $x, y \in \mathcal{L}_u(f)$, we have

$$\begin{aligned} g(x+y) &= \sum_{m,n\in\mathbb{N}} f(|x_{mn} + y_{mn}|) \leq \sum_{m,n\in\mathbb{N}} f(|x_{mn}| + |y_{mn}|) \\ &\leq \sum_{m,n\in\mathbb{N}} f(|x_{mn}|) + \sum_{m,n\in\mathbb{N}} f(|y_{mn}|) = g(x) + g(y). \end{aligned}$$

iv) (a) Suppose that λ is a scalar and $g(x) \to 0$. Then we get

$$g(\lambda x) = \sum_{m,n\in\mathbb{N}} f(|\lambda| |x_{mn}|) \leq K. \sum_{m,n\in\mathbb{N}} f(|x_{mn}|),$$

where K is a positive integer such that $|\lambda| \leq K$. So $g(\lambda x) \to 0$.

(b) Suppose that $\lambda^r \to 0$ and $x \in \mathcal{L}_u(f)$. Then there exist positive numbers ε and k such that

$$\sum_{m,n=k+1}^{\infty} f(|x_{mn}|) < \frac{\varepsilon}{2},$$

by virtue of the the fact $f(|x_{mn}|) < \infty$. Now let us write

$$h(t) = \sum_{m,n=1}^{k} f(t|x_{mn}|).$$

Then h is continuous at 0. Therefore there exists a number δ such that $0<\delta<1$, for each $0<t<\delta$

$$|h(t)|<\frac{\varepsilon}{2}.$$

Then there is a number N such that for each $r>N$, we get

$$|\lambda^r|<\delta.$$

Since $\lambda^r\to 0$, for each $r>N$, then we have

$$\begin{aligned} g(\lambda^r x) &= \sum_{m,n=1}^{k} f(|\lambda^r x_{mn}|)+\sum_{m,n=k+1}^{\infty} f(|\lambda^r x_{mn}|) \\ &\leq \frac{\varepsilon}{2}+\sum_{m,n=k+1}^{\infty} f(|x_{mn}|) \\ &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \end{aligned}$$

which completes the proof.

It can be verified easily that g is total, i.e., for each $x\in\mathcal{L}_u(f)$, then

$$g(x)=0\Rightarrow x=\theta.$$

Theorem 2.2 *The double sequence space $\mathcal{L}_u(f)$ is a DK-space.*

Proof: For each $k,l\in\mathbb{N}$, the functions

$$\begin{aligned} P_{kl}:\mathcal{L}_u(f) &\to \mathbb{C} \\ x &\to P_{kl}(x)=x_{kl} \end{aligned}$$

are continuous. For all $\varepsilon>0$, there exists a $\delta>0$ such that $g(x)<\delta$. Choosing $\delta=f(\varepsilon)$, we have

$$\sum_{k,l}^{\infty} f(|x_{kl}|)<f(\varepsilon)\Rightarrow f(|x_{kl}|)<f(\varepsilon).$$

Since f is increasing, we obtain

$$|x_{kl}|=|P_{kl}(x)|<\varepsilon.$$

Theorem 2.3 *$\mathcal{L}_u(f)$ is complete.*

Proof: Let $\left(x^{l}\right)$ be a Cauchy sequence in $\mathcal{L}_{u}(f)$. For each $\varepsilon>0, \exists n_{0}\in\mathbb{N}$ $\ni$ for each $l,r>n_{0}$, $g\left(x^{l}-x^{r}\right)<\varepsilon$. Since, for each i,j, coordinate functions $f_{ij}(x)=x_{ij}$ are continuous on $\mathcal{L}_{u}(f)$, then $\left(x_{ij}^{l}\right)$ is a Cauchy sequence in $\mathbb{C}$, $(l>n_{0})$. Because of the completeness of $\mathbb{C}$, the Cauchy sequence converges to a point. Let x_{ij} be such a point. Construct the double sequence $x=(x_{ij})$ with these limit points. Then we have

$$\begin{aligned} g\left(x^{l}-x^{r}\right) &< \varepsilon \Rightarrow \sum_{i,j} f\left(\left|x_{ij}^{l}-x_{ij}^{r}\right|\right)<\varepsilon \Rightarrow \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left|x_{ij}^{l}-x_{ij}^{r}\right|\right)<\varepsilon \\ &\Rightarrow \lim_{r}\sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left|x_{ij}^{l}-x_{ij}^{r}\right|\right)=\sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left|x_{ij}^{l}-x_{ij}\right|\right)<\varepsilon \\ &\Rightarrow g\left(x^{l}-x\right)<\varepsilon\,(\forall\ l>n_{0}) \Rightarrow x^{l}\to x. \end{aligned}$$

Also, we get

$$\begin{aligned} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f\left(\left|x_{ij}\right|\right) &= \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left|x_{ij}-x_{ij}^{l}-x_{ij}^{l}\right|\right) \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f\left(\left|x_{ij}-x_{ij}^{l}\right|\right)+\sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left|x_{ij}^{l}\right|\right)<\varepsilon, \end{aligned}$$

which implies $x\in\mathcal{L}_{u}(f)$.

Corollary 2.4 $\mathcal{L}_{u}(f)$ *is an FDK-space.*

Theorem 2.5 $\mathcal{L}_{u}(f)\subseteq\mathcal{L}_{u}$, *where* $\mathcal{L}_{u}=\{x\in\Omega:\sum_{i,j=1}^{\infty}\mid x_{ij}\mid<\infty\}$.

Proof: Suppose that $x\in\mathcal{L}_{u}(f)$ but $x\notin\mathcal{L}_{u}$. Then we have

$$\sum_{k,l=1}^{\infty} f\left(\left|x_{kl}\right|\right)<\infty$$

and

$$\sum_{k,l=1}^{\infty}\left(\left|x_{kl}\right|\right)=\infty.$$

From the last equation above, one can see that there is a double sequence of natural numbers (n_{i},m_{j}) such that,

$$\sum_{k=n_{i}}^{n_{i+1}}\sum_{l=m_{j}}^{m_{j+1}}\left(\left|x_{kl}\right|\right)>1\Rightarrow f(1)<f\left(\sum_{k=n_{i}}^{n_{i+1}}\sum_{l=m_{j}}^{m_{j+1}}\left(\left|x_{kl}\right|\right)\right)=\sum_{k=n_{i}}^{n_{i+1}}\sum_{l=m_{j}}^{m_{j+1}} f\left(\left|x_{kl}\right|\right).$$

Then we obtain

$$\sum_{k,l=1}^{\infty} f(|x_{kl}|) < \infty$$

and

$$\sum_{k=n_i}^{n_{i+1}} \sum_{l=m_j}^{m_{j+1}} f(|x_{kl}|) \to 0, (i, j \to \infty),$$

which implies a contradiction such that $f(1) = 0$. Hence, we get $x \in \mathcal{L}_u$. This completes the proof.

We note that

$$f(x) = x \Rightarrow \mathcal{L}_u(f) = \mathcal{L}_u.$$

If f is unbounded, $\mathcal{L}_u(f) \subset \mathcal{L}_u$.

Theorem 2.6 *The double sequence of unit vectors is bounded in* $\mathcal{L}_u(f)$.

Proof: Let δ be the double sequence of unit vectors, i.e.,

$$\delta = \begin{bmatrix} e_{11} & e_{12} & ... \\ e_{21} & e_{22} & ... \\ . & . & ... \end{bmatrix}, \qquad e_{ij} = \begin{bmatrix} 0 & 0 & ... & 0 & ... \\ 0 & 0 & ... & 0 & ... \\ . & . & . & 0 & ... \\ 0 & 0 & ... & 1 & ... \\ . & . & ... & 0 & ... \end{bmatrix},$$

where 1 is term of (i, j).

If $a \in \mathbb{C}$, then $g(ae_{11}) = ... = g(ae_{ij}) = f(|a|)$. Let us choose a $\lambda \in \mathbb{C}$ such that $f(|\lambda|) < \epsilon$. Then, for $0 \leq a \leq \lambda$, we write

$$g(ae_{11}) = f(|a|) \leq f(|\lambda|) < \epsilon.$$

This implies that $ae_{11} \in \{x : g(x) < \varepsilon\}$. Since we have

$$g(ae_{ij}) = g(ae_{kl}) = f(|a|),$$

for $(i, j) \neq (k, l)$, then we get $a\delta \in \{x : g(x) < \varepsilon\}$, which shows that each sphere centered at origin contains the sequence δ. Thus δ is a bounded double sequnce in $\mathcal{L}_u(f)$.

3 The Double Sequence Space $\mathcal{L}_u(X, f)$

Let X be a Banach Space. We define

$$\mathcal{L}_u(X, f) = \{x \in \Omega(X) : \sum_{m,n} f(\|x_{mn}\|) < \infty\},$$

where $\|,\|$ is the norm of X. It is obvious that $\mathcal{L}_u(X, f)$ is a linear space.

Theorem 3.1 $\mathcal{L}_u(X,f)$ *is paranormed space with*

$$p(x)=\sum_{i,j} f(\|x_{mn}\|).$$

Proof: i) If $x=\theta$, for each i,j, $x_{ij}=0$ and $p(\theta)=0$.
ii) It is obvious that $p(-x)=p(x)$.
iii) For $x,y\in \mathcal{L}_u(X,f)$, we write

$$\begin{aligned} p(x+y) &= \sum_{i,j} f(\|x_{ij}+y_{ij}\|)\leq \sum_{i,j} f(\|x_{ij}\|)+f(\|y_{ij}\|) \\ &= \sum_{i,j} f(\|x_{ij}\|)+\sum_{i,j} f\|y_{ij}\|). \end{aligned}$$

iv) Let $x=\left(x^l\right)\in \mathcal{L}_u(X,f)$ be a sequence and $\lambda=\left(\lambda^l\right)$ be a sequence of scalars. Assume that $\lambda\to\lambda^0$ and $p\left(x^l-x^0\right)\to 0$, $(l\to\infty)$. So scalar sequence $\lambda=\left(\lambda^l\right)$ is convergent, for each $l\in\mathbb{N}$. Then there exists a positive number K such that $\mid \lambda^l \mid\leq K$. Therefore we have

$$\begin{aligned} p\left(\lambda^l x^l-\lambda^0 x^0\right) &= \sum_{i,j} f\left(\left\|\lambda^l x^l_{ij}-\lambda^0 x^0_{ij}\right\|\right) \\ &= \sum_{i,j} f\left(\left\|\lambda^l x^l_{ij}-\lambda^l x^0_{ij}+\lambda^l x^0_{ij}-\lambda^0 x^0_{ij}\right\|\right) \\ &\leq \sum_{i,j} f(\left\|\lambda^l x^l_{ij}-\lambda^l x^0_{ij}\right\|)+f\left(\left\|\lambda^l x^0_{ij}-\lambda^0 x^0_{ij}\right\|\right) \\ &\leq \sum_{i,j} f(\left|\lambda^l\right|\left\|x^l_{ij}-x^0_{ij}\right\|)+\sum_{i,j} f(\left|\lambda^l-\lambda^0\right|\left\|x^0_{ij}\right\|) \\ &= K.p\left(x^l-x^0\right)+\sum_{i,j} f(\left|\lambda^l-\lambda^0\right|\left\|x^0_{ij}\right\|). \end{aligned} \quad (1)$$

Since K is a constant and $p\left(x^l-x^0\right)\to 0$ $(l\to\infty)$, the first term in the right hand side of inequality in (1) tends to zero. Also, for each $l\in\mathbb{N}$, there exists a $T\geq 0$ such that

$$Tx^0=\left(Tx^0_{ij}\right)\in \mathcal{L}_u(X,f).$$

For $\varepsilon>0$, there exist numbers i_0,j_0 such that for each $l\in\mathbb{N}$, we get

$$\begin{pmatrix} \sum\limits_{i>i_0}\sum_{j>j_0} f(\left|\lambda^l-\lambda^0\right|\left\|x^0_{ij}\right\|) \\ +\sum\limits_{i=0}^{i_0}\sum_{j=j_0+1}^{\infty} f(\left|\lambda^l-\lambda^0\right|\left\|x^0_{ij}\right\|) \\ +\sum\limits_{i=i_0+1}^{\infty}\sum_{j=0}^{j_0} f(\left|\lambda^l-\lambda^0\right|\left\|x_{ij}\right\|) \end{pmatrix} \leq \begin{pmatrix} \sum\limits_{i>i_0}\sum_{j>j_0} f(T\left\|x^0_{ij}\right\|) \\ +\sum\limits_{i=0}^{i_0}\sum_{j=j_0+1}^{\infty} f(T\left\|x^0_{ij}\right\|) \\ +\sum\limits_{i=i_0+1}^{\infty}\sum_{j=0}^{j_0} f(T\left\|x^0_{ij}\right\|) \end{pmatrix}$$

$$< \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{2}$$

On the other hand, since

$$\lim_{l\to\infty}\sum_{i=0}^{i_0}\sum_{j=0}^{j_0} f(\left|\lambda^l-\lambda^0\right|\left\|x_{ij}^0\right\|)=\sum_{i=0}^{i_0}\sum_{j=0}^{j_0} f(\lim_{l\to\infty}\left|\lambda^l-\lambda^0\right|\left\|x_{ij}^0\right\|)=0.$$

For the same ε, there exists an l_0 such that for each $l>l_0$

$$\sum_{i=0}^{i_0}\sum_{j=0}^{j_0} f(\left|\lambda^l-\lambda^0\right|\left\|x_{ij}^0\right\|)<\frac{\varepsilon}{2}.$$

Then we get

$$\sum_{i,j} f(\left|\lambda^l-\lambda^0\right|\left\|x_{ij}^0\right\|)\leq\left(\begin{array}{c}\sum\limits_{i>i_0}\sum_{j>j_0} f(T\left\|x_{ij}^0\right\|)\\+\sum\limits_{i=0}^{i_0}\sum_{j=j_0+1}^{\infty} f(T\left\|x_{ij}^0\right\|)\\+\sum\limits_{i=i_0+1}^{\infty}\sum_{j=0}^{j_0} f(T\left\|x_{ij}^0\right\|)\\+\sum\limits_{i=0}^{i_0}\sum_{j=0}^{j_0} f(\left|\lambda^l-\lambda^0\right|\left\|x_{ij}^0\right\|)\end{array}\right)<\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{2}=\varepsilon,$$

for each $\varepsilon>0$ and $l>l_0$. Thus the second term in inequality (1) tends to zero, i.e.,

$$p\left(\lambda^l x^l-\lambda^0 x^0\right)\to 0 \qquad (l\to\infty).$$

So the proof is completed.

Theorem 3.2 *$\mathcal{L}_u(X,f)$ is an FDK-space.*

Proof: i) We know that $\mathcal{L}_u(X,f)$ is a linear space and paranormed space.
ii) Let us proof continuity of coordinate maps p_{kl} which are defined by

$$\begin{aligned} p_{kl} &: \mathcal{L}_u(X,f)\to X \\ p_{kl}(x) &= x_{kl}. \end{aligned}$$

For each $\varepsilon>0$, let us choose $\delta=f(\varepsilon)$ such that

$$p(x)=\sum_{k,l} f(\|x_{kl}\|)<\delta=f(\varepsilon).$$

We get

$$f(\|x_{kl}\|)<f(\varepsilon)$$

and

$$\|x_{kl}\|=\|p_{kl}(x)\|<\varepsilon.$$

iii) We will show that $\mathcal{L}_u(X,f)$ is complete. Let (x^l) be a Cauchy sequence in $\mathcal{L}_u(X,f)$. For each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that, for $l, r > n_0$

$$p\left(x^l - x^r\right) < \varepsilon.$$

Since functions

$$p_{ij}(x) = x_{ij}$$

are continuous for each $(i,j) \in \mathbb{N}\times\mathbb{N}$, $\left(x_{ij}^l\right)$ is a Cauchy sequence in X, (for each $(i,j) \in \mathbb{N}\times\mathbb{N}$ and $l > n_0$). Since X is a Banach space, the Cauchy sequence converges to a point x_{ij}. Let's set the double sequence $x = (x_{ij})$ by this limit points. Now we have

$$\begin{aligned} x^l &\to x \Rightarrow p\left(x^l - x^r\right) < \varepsilon \\ &\Rightarrow \sum_{i,j} f\left(\left\|x_{ij}^l - x_{ij}^r\right\|\right) < \varepsilon \\ &\Rightarrow \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left\|x_{ij}^l - x_{ij}^r\right\|\right) < \varepsilon \\ \lim_r \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\left\|x_{ij}^l - x_{ij}^r\right\|\right) &= \sum_{i,j} f\left(\left\|x_{ij}^l - x_{ij}^r\right\|\right) < \varepsilon \\ &\Rightarrow p\left(x^l - x\right) < \varepsilon, \text{ for each } l > n_0. \end{aligned}$$

On the other hand we obtain

$$\begin{aligned} \sum_{i,j} f\left(\| x_{ij}^l \|\right) &= \sum_i \sum_j f\left(\| x_{ij} - x_{ij}^l + x_{ij}^l \|\right) \\ &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f\left(\| x_{ij} - x_{ij}^l \|\right) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f\left(\| x_{ij}^l \|\right) \\ &< \varepsilon \end{aligned}$$

which shows that $x \in \mathcal{L}_u(X,f)$.

4 Double Sequence Spaces $\mathcal{L}_u(X)$ and $\mathcal{M}_u(X)$

Let's define double sequence spaces $\mathcal{L}_u(X)$ and $\mathcal{M}_u(X)$ as follow:

$$\begin{aligned} \mathcal{M}_u(X) &= \{x = (x_{mn}) \in \Omega(X) : \sup_{m,n\in\mathbb{N}} \|x_{mn}\| < \infty\}, \\ \mathcal{L}_u(X) &= \{x = (x_{mn}) \in \Omega(X) : \sum_{m,n} \|x_{mn}\| < \infty\}. \end{aligned}$$

Theorem 4.1 *The space* $s\mathcal{L}_u(X)$ *and* $\mathcal{M}_u(X)$ *are BDK-spaces.*

Proposition 4.2 *For each modulus function* f*,* $\mathcal{L}_u(X,f) \subset \mathcal{L}_u(X)$ *.*

Proof: We assume that $x \in \mathcal{L}_u(X,f)$ and $x \notin \mathcal{L}_u(X)$. Then increasing sequences (k_n) and (l_n) can be found such that

$$\begin{aligned} \sum_{i=k_{n-1}}^{k_n-1} \sum_{j=l_{n-1}}^{l_n-1} \|x_{ij}\| &\geq 1 \\ &\Rightarrow f(1) \leq f\left(\sum_{i=k_{n-1}}^{k_n-1} \sum_{j=l_{n-1}}^{l_n-1} \|x_{ij}\|\right) \\ &\leq \sum_{i=k_{n-1}}^{k_n-1} \sum_{j=l_{n-1}}^{l_n-1} f(\|x_{ij}\|) \\ &\Rightarrow \sum_{i=k_{n-1}}^{k_n-1} \sum_{j=l_{n-1}}^{l_n-1} f(\|x_{ij}\|) < \infty \\ \lim_n \sum_{i=k_{n-1}}^{k_n-1} \sum_{j=l_{n-1}}^{l_n-1} f(\|x_{ij}\|) &= 0 \end{aligned}$$

It means $f(1) = 0$. This contradicts with f being a modulus function. Then $x \in \mathcal{L}_u(X)$.

Proposition 4.3 *For each* $x \in \mathcal{L}_u(X)$*,* x *can be written as follows:*

$$x = \sum_{i,j \in \mathbb{N}\times\mathbb{N}} I_{ij}(x_{ij}),$$

where

$$\begin{aligned} I_{ij} &: X \to \mathcal{L}_u(X) \\ I_{ij}(t) &= y \ni y_{ij} = t \text{ and } y_{kl} = 0, \text{if } (k,l) \neq (i,j). \end{aligned}$$

Note that the meaning of the above expression is that the net $\{S_F(x) : F \in \mathcal{F}\}$ converges to a point x according to norm topology of $\mathcal{L}_u(X)$, where $\mathcal{F}$ is a family of all finite subsets of $\mathbb{N}\times\mathbb{N}$. This family is directed with relation $\subseteq$.

Theorem 4.4 *For each* $x \in \mathcal{L}_u(X,f)$*,* x *can be represented as follows:*

$$x = \sum_{(i,j)\in\mathcal{F}} I_{ij}(x_{ij}),$$

where

$$\begin{aligned} I_{ij} &: X \to \mathcal{L}_u(X,f) \\ I_{ij}(t) &= y \ni y_{ij} = t \text{ and } y_{kl} = 0, \text{if } (k,l) \neq (i,j). \end{aligned}$$

Proof: We will show that for all given $\varepsilon > 0$, there exists an $F_0 = F_0(\varepsilon) \in \mathcal{F}$ such that when $F_0 \subseteq F$, $p(x - S_F(x)) < \varepsilon$. Due to the definition of I_{ij}, we can define a function as follow:

$$\begin{array}{rcl} S_F & : & \mathcal{L}_u(X,f) \to \mathcal{L}_u(X,f) \\ S_F(x_{ij}) & = & x_{ij}, \quad ((i,j) \in F) \\ S_F(x_{ij}) & = & 0. \quad\quad ((i,j) \notin F) \end{array}$$

Then we write

$$\begin{aligned} p(x - S_F(x)) &= \sum_{i,j \in \mathbb{N}\times\mathbb{N}} f\left(\|x_{ij} - \{S_F(x)\}_{i,j}\|\right) \\ &= \sum_{i,j \in \mathbb{N}\times\mathbb{N}\setminus F} f\left(\|x_{ij}\|\right), \end{aligned}$$

which implies $x \in \mathcal{L}_u(X,f)$. So $\sum_{i,j} f(\|x_{ij}\|) < \infty$.

Thus for $\varepsilon > 0$, there exists an $F_0(\varepsilon) \ni \sum_{i,j} f(\|x_{ij}\|) < \varepsilon$. Therefore for all $F \supseteq F_0$, we have

$$\sum_{i,j \in \mathbb{N}\times\mathbb{N}\setminus F} f(\|x_{ij}\|) = p(S_F(x) - x) < \varepsilon.$$

It means that the net $(S_F(x) : F \in \mathcal{F})$ converges to x. This completes the proof.

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