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A Fixed Point Theorem of Strict Generalized Type Weakly Contractive Maps in Orbitally Complete Metric Spaces When the Control Function is not Necessarily Continuous

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Abstract

K.P.R. Sastry, Ch. Srinivasa Rao, N. Appa Rao [5] introduced the notation of a control function and proved a fixed point theorem for a strict generalized weakly contractive map of an orbitally complete metric space when the control *function is not assumed to be continuous. In this paper we introduce the notation of a generalized type weakly contractive map of an orbitally complete metric space and prove a fixed point theorem for such maps without assuming the continuity of the control function. Our result answers an open problem raised in Sastry et al. [5], in the affirmative.*

Keywords: *weakly contractive maps, generalized weakly contractive maps, fixed point, T-orbitally complete metric spaces, strict generalized weakly contractive map, control function, strict generalized type weakly contractive map.*

1 Introduction

In 1997, Alber and Cuerre-Delabriere [1] introduced the concept of weakly contractive maps in a Hilbert space and proved the existence of fixed points. In 2001, Rhoades [4] extended this concept to Banach spaces and established the existence of fixed points.

Throughout this paper, (X,d) is a metric space, and $T: X \to X$ a self map of X. Let $\mathbb{R}^+ = [0,\infty)$, $\mathbb{N}$, the set of all natural numbers and $\mathbb{R}$, the set of all real numbers. We write

$\Psi = \{ \psi: [0,\infty) \to [0,\infty) / \psi \text{ is strictly increasing and } \psi(0) = 0\}$
Members of Ψ are called control functions.

$\Phi = \{ \varphi: [0,\infty) \to [0,\infty) / \varphi \text{ is continuous , non decreasing and } \varphi(t) = 0 \Leftrightarrow t = 0\}$

Definition 1.1 (Rhoades, [4]): *A self map $T: X \to X$ is said to be a weakly contractive map if there exists a $\varphi \in \Phi$ with $lim_{t\to\infty} \varphi(t) = \infty$ such that*

$d(Tx,Ty) \le d(x,y) - \varphi\big(d(x,y)\big)$ for all $x,y \in X$ …. (1.1.1)

Here we observe that every contractive map T on X with contractive constant k is a weakly contractive map with $\varphi(t) = (1-k)t,\ t > 0$. But its converse is not true.

Rhoades [4] proved the following theorem.

Theorem 1.2 (Rhoades [4], Theorem 1.1): *Let (X,d) be a complete metric space and T a weakly contractive self map on X. Then T has a unique fixed point in X.*

Babu and Alemayehu [2] introduced the notion of a generalized weakly contractive map.

Definition 1.3 (Babu and Alemayehu, [2]): *A map $T: X \to X$ is said to be a generalized weakly contractive map if there exists a $\varphi \in \Phi$ such that*

$d(Tx,Ty) \leq M(x,y) - \varphi(M(x,y))$ for all $x,y \in X$ where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}((d(x,Ty) + d(y,Tx))\right\}$$

Remark 1.4 (Babu and Alemayehu, [2]): Every weakly contractive map defined on a bounded metric space with a positive diameter is a generalized weakly contractive map, but its converse is not true.

Theorem 1.5 (Babu and Alemayehu [2], Theorem 1.3): *Let (X,d) be a complete metric space and $T: X \to X$ be a self map. If T is a generalized weakly contractive map on X, then T has a unique fixed point in X.*

If X is a complete bounded metric space, Theorem 1.2 follows as a corollary to

Theorem 1.5: *In fact in this case, Theorem 1.5 is a generalization of Theorem 1.2 (Example 3.2 of Babu and Alemayehu [2]).*

Definition 1.6: *Let $T: X \to X$ .For $x \in X$, $O(x) = O_T(x) = \{T^n x \,/ n = 0,1,2, \dots.\}$ is called the orbit of x, where $T^0 = I$, the identity map of X.*

Let (X,d) be a complete metric space and $T: X \to X$. Then X is said to be T-orbitally complete, if, for $x \in X$, every Cauchy sequence which is contained in $O(x)$ converges to a point of X. In other words, $\overline{O(x)}$ is a complete metric space.

Babu and Sailaja [3] proved the existence of fixed points of a generalized weakly contractive map T in T-orbitally complete metric spaces.

Theorem 1.7 (Babu and Sailaja [3], Theorem 2.1): *Let (X,d) be a metric space and $T: X \to X$. Suppose X is a T-orbitally complete metric space. Assume that for some $x_0 \in X$, there exists a $\varphi \in \Phi$ such that $d(Tx,Ty) \leq M(x,y) - \varphi(M(x,y))$ for all $x,y \in \overline{O(x_0)}$ …* (1.7.1)

Where $M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}((d(x,Ty) + d(y,Tx))\right\}$

Then the sequence $\{T^n x_0\}$ is a Cauchy sequence in X. Let $\lim_{n\to\infty} T^n x_0 = z, z \in X$.

Then z is a fixed point of T.

Further, z is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of T.

Corollary 1.8 (Babu and Sailaja [3], Corollary2.2): *Let (X,d) be a T-orbitally complete bounded metric space. Assume that for some $x_0 \in X$, there exists $\varphi \in \Phi$ such that*

$d(Tx,Ty) \leq d(x,y) - \varphi\big(d(x,y)\big)$ for all $x,y \in \overline{O(x_0)}$ …. (1.8.1)

Then the sequence $\{T^n x_0\}$ is Cauchy in$X$. Let $\lim_{n\to\infty} T^n x_0 = z, z \in X$.

Then z is a fixed point of T.

Further, z is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of T.

Definition 1.9: *Let (X,d) be a metric space and $T: X \to X$. We say that T is a strict generalized weakly contractive map if there exists a control function $\psi \in \Psi$ such that*

$d(Tx,Ty) \leq M(x,y) - \psi\big(M(x,y)\big)$ for all $x,y \in X$ ….. (1.9.1)

Where $M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}((d(x,Ty) + d(y,Tx))\right\}$

Using the above notion, Sastry et. al. [5] proved the following theorem.

Theorem 1.10: *Let (X,d) be a metric space and$T: X \to X$. Let (X,d) be T-orbitally complete. Assume that for some $x_0 \in X$, there exists a control function $\psi \in \Psi$ such that*

$d(Tx,Ty) \leq M(x,y) - \psi\big(M(x,y)\big)$ for all $x,y \in \overline{O(x_0)}$ ….. (2.2.1)

Where $M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}((d(x,Ty) + d(y,Tx))\right\}$

Then the sequence $\{T^n x_0\}$ is Cauchy in X. Let $\lim_{n\to\infty} T^n x_0 = z, z \in X$, then z is a fixed point of T.

Further, z is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of T.

Further Sastry et. al. [5] raised the following open problem: Is Theorem 1.10 true if $M\ (x,y)$ is replaced by $\alpha(x,y) = \frac{1}{2}\ (d(x,Ty) + d(y,Tx))$?

In this paper we prove a fixed point theorem which answers the above open problem in the affirmative.

In proving our main result, we make use of the following well known result; a proof can be found in Babu and Saliaja [3].

Lemma 1.11: *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n-1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist*

an $\varepsilon > 0$ *and sequences of positive integers* $\{m(k)\}$ *and* $\{n(k)\}$ *with* $m(k) > n(k) > k$ *such that* $d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon, d\left(x_{m(k)-1}, x_{n(k)}\right) < \varepsilon$ *and*
(i) $lim_{k\to\infty} d\left(x_{m(k)-1}, x_{n(k)+1}\right) = \varepsilon$
(ii) $lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon$ *and*
(iii) $lim_{k\to\infty} d\left(x_{m(k)-1}, x_{n(k)}\right) = \varepsilon$.

2 Main Results

Before we prove our main result, we first prove a lemma.

Lemma 2.1: *Suppose* $\psi: [0,\infty) \to [0,\infty)$ *is strictly increasing and* $\psi(0) = 0$. *If* $\{y_n\}$ *is a sequence in* $[0,\infty)$, *then* $\psi(y_n) \to 0 \Rightarrow y_n \to 0$.

Proof: Suppose $\psi(y_n) \to 0$ and y_n does not tend to zero. Then $\exists\, \gamma > 0$ and an infinite sequence n_k such that $\{y_{n_k}\} \geq \gamma$. Then $\psi(y_{n_k}) \geq \psi(\gamma)$.

Letting $k \to \infty$, we get $0 \geq \psi(\gamma)$ $(\because\ \psi(y_{n_k}) \to 0\ as\ k \to \infty)$
$\therefore\ \gamma = 0$, a contradiction.
$\therefore\ y_n \to 0$.

Now we state and prove our main result which answers the open problem of Sastry et.al [5] in the affirmative.

Theorem 2.2: *Let* (X,d) *be a complete metric space* $T: X \to X$ *and* T *is orbitally complete. Assume that for some* $x_0 \in X$, *there exists a* $\psi \in \Psi$ *such that*
$d(Tx,Ty) \leq \alpha(x,y) - \psi\big(\alpha(x,y)\big)\ \ \forall\ x,y \in \overline{O(x_0)}$ ….. (2.2.1)

Where $\alpha(x,y) = \frac{1}{2}[d(x,Ty) + d(y,Tx)]$

Then the sequence $\{T^n x_0\}$ is a Cauchy sequence in X. If $\lim_{n\to\infty} T^n x_0 = z$, $z \in X$, then z is a fixed point of T.

Further, z is unique in the sense that $\overline{O(x_0)}$ contains one and only one fixed point of T.

Proof: Let $y = Tx$ in (2.2.1). Then

$$d(Tx,TTx) \leq \frac{1}{2}[d(x,TTx) + d(Tx,Tx)] - \psi\left(\frac{1}{2}[d(x,TTx) + d(Tx,Tx)]\right)$$

$$= \frac{1}{2}\{d(x,TTx)\} - \psi\left(\frac{1}{2}\{d(x,TTx)\}\right) \quad \ldots\ldots \quad (2.2.2)$$

If R.H.S of (2.2.2) is 0, then $d(Tx,TTx) = 0 \Rightarrow TTx = Tx$

$\therefore Tx$ is a fixed point of T.

Suppose $d(Tx,TTx) \neq 0$.

Then (2.2.2) $\Rightarrow \psi\left(\frac{1}{2}(d(x,TTx)\right) \leq \frac{1}{2}\ d(x,TTx) - d(Tx,TTx)$ ….. (2.2.3)

$$\leq \frac{1}{2}(d(x,Tx) + d(Tx,TTx)) - d(Tx,TTx)$$

$$= \frac{1}{2}\left(d(x,Tx) - d(Tx,TTx)\right) \quad …. \quad (2.2.4)$$

Now $\psi\left(\frac{1}{2}(d(x,TTx)\right) = 0 \;\Rightarrow\; d(x,TTx) = 0$

$\Rightarrow d(Tx,TTx) = 0$ (from (2.2.3)), contradicting our supposition.

$\therefore\;\; 0 < \psi\left(\frac{1}{2}\ (dx,TTx)\right) \leq \frac{1}{2}\{d(x,Tx) - d(Tx,TTx)\}$ (from (2.2.4))

$\Rightarrow d(Tx,TTx) < d(x,Tx)$

$\therefore\;\; d(Tx,TTx) \leq d(x,Tx)$ ……. (2.2.5)

with equality $\Leftrightarrow x$ is a fixed point of T.

Let $x_0 \in X$, write $T^n x_0 = x_n,\; n = 0,1,2,...$

Write $\alpha_n = d(x_n, x_{n+1})$. Then from (2.2.5),

$$\alpha_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, TTx_n) \leq d(x_n, Tx_n) = \alpha_n.$$

$\therefore\;\; \alpha_n$ is a decreasing sequence and hence tends to a limit, say, a.
$\therefore\;\; \psi(\alpha_n)$ is a decreasing sequence and hence tends to a limit ,say, b.
$\therefore\;\; \alpha_n > a \;\Rightarrow\; \psi(\alpha_n) \geq \psi(\alpha) \Rightarrow b \geq \psi(a)$

Now
$\alpha_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, TTx_n) \leq \frac{1}{2}\ d(x_n, TTx_n) - \psi\left(\frac{1}{2}d(x_n, TTx_n)\right)$

from (2.2.2) …… (2.2.6)

$$\leq \frac{1}{2}\left(d(x_n, Tx_n) + d(Tx_n, TTx_n)\right) - \psi\left(\frac{1}{2}d(x_n, TTx_n)\right)$$
$$= \frac{1}{2}(\alpha_n + \alpha_{n+1}) - \psi\left(\frac{1}{2}d(Tx_n, TTx_n)\right)$$

$\Rightarrow \psi\left(\frac{1}{2}d(Tx_n, TTx_n)\right) \leq \frac{1}{2}(\alpha_n + \alpha_{n+1}) - \alpha_{n+1}$
$= \frac{1}{2}(\alpha_n - \alpha_{n+1}) \rightarrow 0 \;\; as \;\; n \rightarrow \infty$
$\therefore \;\; \psi\left(\frac{1}{2}d(Tx_n, TTx_n)\right) \rightarrow 0 \;\; as \;\; n \rightarrow \infty$ …… (2.2.7)

$\therefore \;\; d(Tx_n, TTx_n) \rightarrow 0 \;\; as \;\; n \rightarrow \infty$ ……. (2.2.8)

($\because$ ψ is strictly increasing and $\psi(0) = 0$, bt Lemma 2.1)

Now from (2.2.6), (2.2.7) and (2.2.8), we get
$a \leq \alpha_{n+1} \leq \frac{1}{2} d(x_n, TTx_n) - \psi\left(\frac{1}{2}d(x_n, TTx_n)\right) \rightarrow 0 \;\; as \;\; n \rightarrow \infty$
$\therefore \;\; a = 0$

Now $\psi(\alpha_n) \geq b \Rightarrow \;\; \alpha_n \geq \psi^{-1}(b)$

Letting $n \rightarrow \infty$, we get $0 \geq \psi^{-1}(b)$

$\therefore \;\; \psi^{-1}(b) = 0 \;\; i.e \;\; b = 0$
$\therefore d(x_n, x_{n+1}) \rightarrow 0 \;\; as \;\; n \rightarrow \infty$ and $\psi(d(x_n, x_{n+1})) \rightarrow 0 \;\; as \;\; n \rightarrow \infty$

We now show that the sequence $\{x_n\} \subset O(x_0)$ is Cauchy.

Otherwise, by Lemma 1.11, there exists an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and

$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$, $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$ and

$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$ ….. (2.2.9)

Hence $\varepsilon < d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$

$$= d(Tx_{m(k)-1}, Tx_{n(k)}) + d(x_{n(k)+1}, x_{n(k)})$$
$$\leq \alpha(x_{n(k)-1}, x_{n(k)}) - \psi\left(\alpha(x_{n(k)-1}, x_{n(k)})\right) + d(x_{n(k)+1}, x_{n(k)})$$
$$= \frac{1}{2}\left[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})\right]$$
$$-\psi\left(\frac{1}{2}\left[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})\right]\right)$$
$$+d(x_{n(k)+1}, x_{n(k)})$$

… (2.2.10)

$$= M(k) - \psi(M(k)) + d(x_{n(k)+1}, x_{n(k)})$$

Where $M(k) = \frac{1}{2}\left[d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})\right]$

From (2.2.9), $M(k) \to \varepsilon \ as \ k \to \infty$

Consequently, $M(k) \leq \varepsilon + \psi\left(\frac{\varepsilon}{2}\right) \ and \ M(k) \geq \frac{3\varepsilon}{4}, for\ large\ k.$

$$\therefore (2.2.8) \leq \varepsilon + \psi\left(\frac{\varepsilon}{2}\right) - \psi\left(\frac{3\varepsilon}{4}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \ for\ large\ k$$
$$= \varepsilon - \left(\psi\left(\frac{3\varepsilon}{4}\right) - \psi\left(\frac{\varepsilon}{2}\right)\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \ for\ large\ k$$

$< \varepsilon \ \ since\ d\left(x_{n(k)+1}, x_{n(k)}\right) \to 0\ as\ k \to \infty$ and ψ is strictly increasing, which is a contradiction

Therefore $\{x_n\}$ is a Cauchy sequence.

Suppose $x_n \to z \in \overline{O(x_0)} \ \ and \ Tz \neq z$. Then

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq \alpha(x_n, z) - \psi\left(\alpha(x_n, z)\right)$$
$$= \frac{1}{2}\left(d(x_{n+1}, Tz) + d(z, Tx_n)\right) - \psi\left(\frac{1}{2}d(x_{n+1}, Tz) + d(z, Tx_n)\right)$$
$$\leq \frac{1}{2}\left(d(x_n, Tz) + d(z, Tx_n)\right) = \left(\frac{1}{2}d(x_n, Tz) + d(z, Tx_{n+1})\right)$$

On letting $n \to \infty$, we get $z(z, Tz) \leq \frac{1}{2}(d(z, Tz) + d(z, z)) = \frac{1}{2}\ d(z, Tz)$
$\therefore \ d(z, Tz) = 0$ and hence $Tz = x$.

Therefore z is a fixed point of T.
Uniqueness: Let x, y be fixed points of T in $\overline{O(x_0)}$.

Then from (2.2 .1), we have
$$d(x, y) = d(Tx, Ty) \leq \alpha(x, y) - \psi(\alpha(x, y))$$
$$= \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \psi(\frac{1}{2}(d(x, Ty) + d(y, Tx)))$$
$$= d(x, y) - \psi\left(d(x, y)\right) < d(x, y),\ if \ x \neq y,$$ a contradiction
$\therefore x = y$

Note: On similar lines, the following theorem, which is parallel to Theorem1.2 (Rhodes [4], Theorem1.1) can also proved.

Theorem 2.3: *Let (X, d) be a complete metric space $T: X \to X$ and T is orbitally complete. Assume that for some $x_0 \in X$, there exists a $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq d(x, y) - \psi\left(d(x, y)\right) \ \ \forall \ x, y \in \overline{O(x_0)}$$

Then the sequence $\{T^n x_0\}$ is a Cauchy sequence in X.

If $lim_{n\to\infty} T^n x_0 = z,\ z \in X$, *then* z *is a fixed point of* T.

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