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Minimality and Maximality of Ordered Quasi-Ideals in Ordered Ternary Semigroups

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Abstract

The notion of ternary semigroups was introduced by Lehmer in 1932. Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup. Our aim in this paper is to develop a body of results on the minimality and maximality of ordered quasi-ideals in ordered ternary semigroups, that can be used like the more classical results on unordered structures which studied by Choosuwan and Chinram in 2012.

Keywords: *ordered ternary semigroup, ordered quasi-ideal, minimality and maximality.*

1 Introduction and Preliminaries

The literature of ternary algebraic system was introduced by Lehmer [18] in 1932. He investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroups was known to Banach (cf. [20]). He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. The study of ordered ternary semigroups began about 2000 by several authors, for example, Iampan [15], Chinram [8], Yaqoob, Abdullah, Rehman and Naeem [26], and Akram and Yaqoob [1]. The theory of different types of ideals in (ordered) semigroups and in (ordered) ternary semigroups was studied by several researches such as: In 1965, Sioson [23] studied ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In 1995, Dixit and Dewan [11] studied the properties of quasi-ideals and bi-ideals in ternary semigroups. In 1998, the concept and notion of ordered quasi-ideals in ordered semigroups was introduced by Kehayopulu [17] as follows: Let S be an ordered semigroup. A subsemigroup Q of S is called an *ordered quasi-ideal* of S if $(SQ] \cap (QS] \subseteq Q$, and $(Q] \subseteq Q$. In 2000, Cao and Xu [4] characterized minimal and maximal left ideals in ordered semigroups, and gave some characterizations of minimal and maximal left ideals in ordered semigroups. In 2002, Arslanov and Kehayopulu [2] gave some characterizations of minimal and maximal ideals in ordered semigroups. In 2004, Iampan and Siripitukdet [16] characterized (0-)minimal and maximal ordered left ideals in ordered Γ-semigroups, and gave some characterizations of (0-)minimal and maximal ordered left ideals in ordered Γ-semigroups. In 2007, Iampan [13] characterized (0-)minimal and maximal lateral ideals in ternary semigroups. In 2008, Iampan [14] characterized (0-)minimal and maximal ordered quasi-ideals in ordered semigroups, and gave some characterizations of (0-)minimal and maximal ordered quasi-ideals in ordered semigroups. Dutta, Kar and Maity [12] studied some interesting properties of regular ternary semigroups, completely regular ternary semigroups, intra-regular ternary semigroups and characterized them by using various ideals of ternary semigroups. In 2009, Bashir and Shabir [3] introduced the notions of pure ideals, weakly pure ideals in ternary semigroups. They also defined purely prime ideals of a ternary semigroup and studied some properties of these ideals. In 2010, Iampan [15] introduced the concept of ordered ideal extensions in ordered ternary semigroups. In 2011, Saelee and Chinram [21] studied rough, fuzzy and rough fuzzy bi-ideals in ternary semigroups. In 2012, Changphas [5] studied minimal quasi-ideals in ternary semigroups. Choosuwan and Chinram [9] gave some characterizations of minimal and maximal quasi-ideals in ternary semigroups. Chinram, Baupradist and Saelee [7] characterized minimal and maximal bi-ideals in ordered ternary semigroups. Daddi and Pawar introduced the concepts of ordered quasi-ideals, ordered bi-ideals in ordered ternary semigroups, and studied their properties. Lekkoksung and Lekkoksung [19] gave some characterizations of intra-regular ordered ternary semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, and bi-ideals and right ideals in ordered ternary semigroups. Changphas [6] studied the properties of quasi-ideals and bi-ideals in ordered ternary semigroups. In 2013, Sanborisoot and Changphas [22] introduced the concepts of pure ideals, weakly pure ideals and purely prime ideals in ordered ternary semigroups.

The notion of quasi-ideals in semigroups was first introduced by Steinfeld [24] in 1956, and it has been widely studied. In 1956, Steinfeld [25] gave some characterizations of 0-minimal quasi-ideals in semigroups. The concept of a

(0-)minimal and a maximal one-sided ideal or ideal is the really interested and important thing in the many algebraic structures. The main purpose of this paper is to develop a body of results on the minimality and maximality of ordered quasi-ideals in ordered ternary semigroups, that can be used like the more classical results on unordered structures which studied by Choosuwan and Chinram [9].

Before going to prove the main results we need the following definitions that we use later.

Definition 1.1. *A nonempty set T is called a ternary semigroup if there exists a ternary operation $[\,]: T\times T\times T \to T$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,*

$$[x_1x_2[x_3x_4x_5]] = [x_1[x_2x_3x_4]x_5] = [[x_1x_2x_3]x_4x_5].$$

For nonempty subsets A, B and C of a ternary semigroup T, let

$$[ABC] := \{[abc] \mid a \in A, b \in B, \text{ and } c \in C\}.$$

If $A = \{a\}$, then we write $[\{a\}BC]$ as $[aBC]$ and similarly if $B = \{b\}$ or $C = \{c\}$, we write $[AbC]$ and $[ABc]$, respectively. For the sake of simplicity, we write $[x_1x_2x_3]$ as $x_1x_2x_3$ and $[ABC]$ as ABC.

Definition 1.2. *A nonempty subset S of a ternary semigroup T is called a ternary subsemigroup of T if $SSS \subseteq S$.*

For any positive integers m and n with $m \leq n$ and any elements $x_1, x_2, ..., x_{2n}$ and x_{2n+1} of a ternary semigroup [23], we can write

$$\begin{aligned}[x_1x_2\ldots x_{2n+1}] &= [x_1\ldots x_mx_{m+1}x_{m+2}\ldots x_{2n+1}]\\ &= [x_1\ldots[[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}]\ldots x_{2n+1}].\end{aligned}$$

Example 1.3. *[11] Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.*

Example 1.4. *[11] Let $O = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), A_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), A_3 = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, and $A_4 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. Then $T = \{O, I, A_1, A_2, A_3, A_4\}$ is a ternary semigroup under matrix multiplication.*

Definition 1.5. *A partially ordered ternary semigroup T is called an ordered ternary semigroup if for any $a, b, x, y \in T$,*

$$a \leq b \Rightarrow axy \leq bxy, xay \leq xby, \text{ and } xya \leq xyb.$$

For a subset H of an ordered ternary semigroup T, we denote

$$(H] := \{t \in T \mid t \leq h \text{ for some } h \in H\}.$$

If $H = \{a\}$, we also write $(\{a\}]$ as $(a]$.

Definition 1.6. *An element z of an ordered ternary semigroup T is called a zero element if*

(1) $zxy = xzy = xyz = z$ *for all* $x, y \in T$*, and*

(2) $z \leq x$ *for all* $x \in T$*.*

If $z \in T$ *is a zero element, it is denoted by* 0*.*

Definition 1.7. *A nonempty subset I of an ordered ternary semigroup T is called an ordered left (resp., ordered lateral, ordered right) ideal of T if*

(1) $TTI \subseteq I$ *(resp.,* $TIT \subseteq I, ITT \subseteq I$*), and*

(2) $(I] \subseteq I$*.*

A nonempty subset I of an ordered ternary semigroup T is called an *ordered ideal* of T if I is an ordered left, an ordered right and an ordered lateral ideal of T.

Definition 1.8. *A nonempty subset Q of an ordered ternary semigroup T is called an ordered quasi-ideal of T if*

(1) $(TTQ] \cap (TQT] \cap (QTT] \subseteq Q$*,*

(2) $(TTQ] \cap (TTQTT] \cap (QTT] \subseteq Q$*, and*

(3) $(Q] \subseteq Q$*.*

We can easily prove that $\{0\}$ is the smallest ordered quasi-ideal of an ordered ternary semigroup T with a zero element and it is called a *zero ordered quasi-ideal* of T. Moreover, $0 \in Q$ for all ordered quasi-ideal Q of T.

Definition 1.9. *A nonempty subset B of an ordered ternary semigroup T is called an ordered bi-ideal of T if*

(1) $BTBTB \subseteq B$*, and*

(2) $(B] \subseteq B$*.*

We have the following lemma.

Lemma 1.10. *[10] For subsets A, B and C of an ordered ternary semigroup T, the following statements hold.*

(1) $A \subseteq (A]$.

(2) If $A \subseteq B$, *then* $(A] \subseteq (B]$.

(3) $((A]] = (A]$.

(4) $(A](B](C] \subseteq (ABC]$ *and* $((A](B](C]] \subseteq (ABC]$.

(5) $(A \cup B] = (A] \cup (B]$.

(6) $(A \cap B] \subseteq (A] \cap (B]$.

Lemma 1.11. *Let T be an ordered ternary semigroup. Then the following statements hold.*

(1) Every ordered left, ordered lateral and ordered right ideal of T is an ordered quasi-ideal of T.

(2) The intersection of an ordered left, an ordered lateral and an ordered right ideal of T is an ordered quasi-ideal of T.

(3) Every ordered quasi-ideal of T is an ordered bi-ideal of T.

Proof. Let L, R and M be an ordered left, an ordered right and an ordered lateral ideal of T, respectively.
(1) We see that $(L] = L, (R] = R$ and $(M] = M$. Thus $(TTL] \cap (TLT \cup TTLTT] \cap (LTT] \subseteq (TTL] \subseteq (L] = L$, $(TTR] \cap (TRT \cup TTRTT] \cap (RTT] \subseteq (RTT] \subseteq (R] = R$, and $(TTM] \cap (TMT \cup TTMTT] \cap (TTM] \subseteq (TMT \cup T(TMT)T] \subseteq (M \cup TMT] \subseteq (M \cup M] = (M] = M$. Hence, L, R and M are ordered quasi-ideals of T.
(2) Suppose that $Q = L \cap M \cap R$ and let $l \in L, m \in M$ and $r \in R$. Then $rml \in RML \subseteq TTL \cap TMT \cap RTT \subseteq L \cap M \cap R = Q$, so $Q \neq \emptyset$. We see that $(Q] = (L \cap M \cap R] \subseteq (L] \cap (M] \cap (R] = L \cap M \cap R = Q$. Thus

$$\begin{aligned}(TTQ] \cap (TQT \cup TTQTT] \cap (TTQ] &\subseteq (TTL] \cap (TMT \cup TTMTT] \cap (RTT] \\ &\subseteq (L] \cap (M] \cap (R] \\ &= L \cap M \cap R \\ &= Q.\end{aligned}$$

Hence, Q is an ordered quasi-ideal of T.
(3) Let B be an ordered quasi-ideal of T. Then $BTBTB \subseteq (TTT)TB \subseteq TTB$, $BTBTB \subseteq TTBTT \subseteq TBT \cup TTBTT$ and $BTBTB \subseteq BT(TTT) \subseteq BTT$. Since B is an ordered quasi-ideal of T, we have

$$\begin{aligned}BTBTB &\subseteq TTB \cap (TBT \cup TTBTT) \cap BTT \\ &\subseteq (TTB] \cap (TBT \cup TTBTT] \cap (BTT] \\ &\subseteq B\end{aligned}$$

and $(B] = B$. Hence, B is an ordered bi-ideal of T. □

Theorem 1.12. *Let A be a nonempty subset of an ordered ternary semigroup T. Then the following statements hold.*

(1) $(TTA], (ATT]$ and $(TAT \cup TTATT]$ are an ordered left, an ordered right and an ordered lateral ideals of T, respectively.

(2) $(TTA \cup A], (ATT \cup A]$ and $(TAT \cup TTATT \cup A]$ are an ordered left, an ordered right and an ordered lateral ideals of T containing A, respectively.

Proof. (1) Since $A \neq \emptyset$, we have $(TTA] \neq \emptyset, (ATT] \neq \emptyset$ and $(TAT \cup TTATT] \neq \emptyset$. We see that $((TTA]] = (TTA], ((ATT]] = (ATT]$ and $((TAT \cup TTATT]] = (TAT \cup TTATT]$. Thus $TT(TTA] = (T](T](TTA] \subseteq ((TTT)TA] \subseteq (TTA]$, $(ATT]TT = (ATT](T](T] \subseteq (AT(TTT)] \subseteq (ATT]$ and

$$\begin{aligned} T(TAT \cup TTATT]T &= (T](TAT \cup TTATT](T] \\ &\subseteq (T(TAT \cup TTATT)T] \\ &\subseteq (T(TAT)T \cup T(TTATT)T] \\ &= ((TTT)A(TTT) \cup TTATT] \\ &\subseteq (TAT \cup TTATT]. \end{aligned}$$

Hence, $(TTA], (ATT]$ and $(TAT \cup TTATT]$ are an ordered left, an ordered right and an ordered lateral ideals of T, respectively.
(2) The proof is almost similar to the proof of (1). □

Theorem 1.13. *If Q is an ordered quasi-ideal of an ordered ternary semigroup T, then it is the intersection of an ordered left, an ordered right and an ordered lateral ideal of T.*

Proof. Assume that Q is an ordered quasi-ideal of T and let $L = (TTQ \cup Q], R = (QTT \cup Q]$ and $M = (TQT \cup TTQTT \cup Q]$. By Theorem 1.12 (2), we have L, R and M are an ordered left, an ordered right and an ordered lateral ideals of T containing Q, respectively. Thus $Q \subseteq L \cap M \cap R$. Since Q is an ordered quasi-ideal of T, we have

$$\begin{aligned} L \cap M \cap R &= (TTQ \cup Q] \cap (TQT \cup TTQTT \cup Q] \cap (QTT \cup Q] \\ &= ((TTQ] \cap (TQT \cup TTQTT] \cap (QTT]) \cup (Q] \\ &\subseteq Q \cup (Q] \\ &= Q. \end{aligned}$$

Hence, $Q = L \cap M \cap R$, so Q is the intersection of an ordered left, an ordered right and an ordered lateral ideal of T. □

Theorem 1.14. *Let T be an ordered ternary semigroup. Then the intersection of arbitrary nonempty family of ordered quasi-ideals of T is either empty or an ordered quasi-ideal of T.*

Proof. Let $\{Q_i \mid i \in I\}$ be a nonempty family of ordered quasi-ideals of T and let $Q = \bigcap_{i\in I} Q_i \neq \emptyset$. We claim that Q is an ordered quasi-ideal of T. Since Q_i is an ordered quasi-ideal of T for all $i \in I$, we have $(TTQ] \cap (TQT \cup TTQTT] \cap (QTT] \subseteq (TTQ_i] \cap (TQ_iT \cup TTQ_iTT] \cap (Q_iTT] \subseteq Q_i$ for all $i \in I$. Thus

$$(TTQ] \cap (TQT \cup TTQTT] \cap (QTT] \subseteq \bigcap_{i\in I} Q_i = Q$$

and $(Q] = (\bigcap_{i\in I} Q_i] \subseteq \bigcap_{i\in I}(Q_i] = \bigcap_{i\in I} Q_i = Q$. Hence, Q is an ordered quasi-ideal of T. □

Definition 1.15. *Let A be a nonempty subset of an ordered ternary semigroup T. The intersection of all ordered quasi-ideals of T containing A is called the ordered quasi-ideal of T generated by A and is denoted by $Q(A)$. Moreover, $Q(A)$ is the smallest ordered quasi-ideal of T containing A. If $A = \{a\}$, we also write $Q(\{a\})$ as $Q(a)$.*

Theorem 1.16. *Let A be a nonempty subset of an ordered ternary semigroup T. Then $Q(A) = (A] \cup ((TTA] \cap (TAT \cup TTATT] \cap (ATT])$. In particular, $Q(a) = (a] \cup ((TTa] \cap (TaT \cup TTaTT] \cap (aTT])$ for all $a \in T$.*

Proof. By Theorem 1.12 (2), we have $(A \cup TTA], (A \cup ATT]$ and $(A \cup TAT \cup TTATT]$ are an ordered left, an ordered right and an ordered lateral ideals of T containing A, respectively. By Lemma 1.11 (2), we have $(A \cup TTA] \cap (A \cup TAT \cup TTATT] \cap (A \cup ATT]$ is an ordered quasi-ideal of T containing A. Thus

$$\begin{aligned} Q(A) &\subseteq (A \cup TTA] \cap (A \cup TAT \cup TTATT] \cap (A \cup ATT] \\ &= (A] \cup ((TTA] \cap (TAT \cup TTATT] \cap (ATT]). \end{aligned}$$

By the proof of Theorem 1.13, we have

$$\begin{aligned} &(A] \cup ((TTA] \cap (TAT \cup TTATT] \cap (ATT]) \\ &= (A \cup TTA] \cap (A \cup TAT \cup TTATT] \cap (A \cup ATT] \\ &\subseteq (Q(A) \cup TT(Q(A))] \cap (Q(A) \cup T(Q(A))T \cup TT(Q(A))TT] \cap \\ &\quad (Q(A) \cup (Q(A))TT] \\ &\subseteq Q(A). \end{aligned}$$

Hence, $Q(A) = (A] \cup ((TTA] \cap (TAT \cup TTATT] \cap (ATT])$. □

2 Minimality of Ordered Quasi-Ideals in Ordered Ternary Semigroups

In this section, we characterize the relationship between the minimality of ordered quasi-ideals and a quasi-simple and a 0-quasi-simple ordered ternary semigroups.

Definition 2.1. *Let T be an ordered ternary semigroup without a zero element. Then T is called quasi-simple if T has no proper ordered quasi-ideals.*

Theorem 2.2. *Let T be an ordered ternary semigroup without a zero element. Then the following statements are equivalent.*

(1) T *is quasi-simple.*

(2) $(TTa] \cap (TaT \cup TTaTT] \cap (aTT] = T$ *for all* $a \in T$.

(3) $Q(a) = T$ *for all* $a \in T$.

Proof. (1)⇒(2) Assume that T is quasi-simple and let $a \in T$. By Theorem 1.12 (1), we have $(TTa], (aTT]$ and $(TaT \cup TTaTT]$ are an ordered left, an ordered right and an ordered lateral ideals of T, respectively. By Lemma 1.11 (2), we have $(TTa] \cap (TaT \cup TTaTT] \cap (aTT]$ is an ordered quasi-ideal of T. Since T is quasi-simple, we have

$$(TTa] \cap (TaT \cup TTaTT] \cap (aTT] = T.$$

(2)⇒(3) Assume that $(TTa] \cap (TaT \cup TTaTT] \cap (aTT] = T$ for all $a \in T$. Let $a \in T$. Then $(TTa] \cap (TaT \cup TTaTT] \cap (aTT] = T$. By Theorem 1.16, we get

$$\begin{aligned} T &= (TTa] \cap (TaT \cup TTaTT] \cap (aTT] \\ &\subseteq (a] \cup ((TTa] \cap (TaT \cup TTaTT] \cap (aTT]) \\ &= Q(a). \end{aligned}$$

Hence, $T = Q(a)$.
(3)⇒(1) Assume that $Q(a) = T$ for all $a \in T$. Let Q be an ordered quasi-ideal of T and let $a \in Q$. Then $Q(a) = T$, and so $Q(a) \subseteq Q \subseteq T$. Hence, $T = Q$. Therefore, T is quasi-simple. □

Definition 2.3. *Let T be an ordered ternary semigroup with a zero element, $T^3 \neq \{0\}$ and $|T| > 1$. Then T is called 0-quasi-simple if T has no nonzero proper ordered quasi-ideals.*

Theorem 2.4. *Let T be an ordered ternary semigroup with a zero element, $T^3 \neq \{0\}$ and $|T| > 1$. Then T is 0-quasi-simple if and only if $Q(a) = T$ for all $a \in T \setminus \{0\}$.*

Proof. Assume that T is 0-quasi-simple and let $a \in T \setminus \{0\}$. Then $Q(a) \neq \{0\}$. Since T is 0-quasi-simple, we have $Q(a) = T$.

Conversely, assume that $Q(a) = T$ for all $a \in T \setminus \{0\}$. Let Q be a nonzero ordered quasi-ideal of T and $a \in Q \setminus \{0\}$. Then $Q(a) = T$ and $Q(a) \subseteq Q \subseteq T$. This implies that $T = Q$. Hence, T is 0-quasi-simple. □

Definition 2.5. *An ordered quasi-ideal Q of an ordered ternary semigroup T without a zero element is called a minimal ordered quasi-ideal of T if there is no an ordered quasi-ideal A of T such that $A \subset Q$. Equivalently, if for any ordered quasi-ideal A of T such that $A \subseteq Q$, we have $A = Q$.*

We also define a minimal ordered left, a minimal ordered lateral and a minimal ordered right ideal of an ordered ternary semigroup without a zero element in the same way of a minimal ordered quasi-ideal.

Theorem 2.6. *Let Q be an ordered quasi-ideal of an ordered ternary semigroup T without a zero element. Then Q is a minimal ordered quasi-ideal of T if and only if it is the intersection of a minimal ordered left, a minimal ordered right and a minimal ordered lateral ideal of T.*

Proof. Assume that Q is a minimal ordered quasi-ideal of T. Then

$$(TTQ] \cap (TQT \cup TTQTT] \cap (QTT] \subseteq Q.$$

By Theorem 1.12 (1), $(TTQ], (QTT]$ and $(TQT \cup TTQTT]$ are an ordered left, an ordered right and an ordered lateral ideal of T, respectively, By Lemma 1.11 (2), $(TTQ] \cap (TQT \cup TTQTT] \cap (QTT]$ is an ordered quasi-ideal of T. Since Q is a minimal ordered quasi-ideal of T, we have

$$(TTQ] \cap (TQT \cup TTQTT] \cap (QTT] = Q.$$

We claim that $(TTQ]$ is a minimal ordered left ideal of T. Let L be an ordered left ideal of T such that $L \subseteq (TTQ]$. Then $(TTL] \subseteq (L] = L \subseteq (TTQ]$. Thus

$$(TTL] \cap (TQT \cup TTQTT] \cap (QTT] \subseteq (TTQ] \cap (TQT \cup TTQTT] \cap (QTT] = Q.$$

Since $(TTL] \cap (TQT \cup TTQTT] \cap (QTT]$ is an ordered quasi-ideal of T and Q is a minimal ordered quasi-ideal of T, we have $(TTL] \cap (TQT \cup TTQTT] \cap (QTT] = Q$. Thus $Q \subseteq (TTL]$ and so $(TTQ] \subseteq (TT(TTL]] \subseteq (TT(L]] = (TTL] \subseteq L$. Hence, $L = (TTQ]$. Therefore, $(TTQ]$ is a minimal ordered left ideal of T. A similar proof holds for the other two case, $(QTT]$ and $(TQT \cup TTQTT]$ are minimal ordered right and minimal ordered lateral ideal of T, respectively.

Conversely, let $Q = L \cap M \cap R$ where L, R and M are a minimal ordered left, a minimal ordered right and a minimal ordered lateral ideal of T, respectively. By Lemma 1.11 (2), we have Q is an ordered quasi-ideal of T. Let A be an ordered quasi-ideal of T such that $A \subseteq Q$. By Theorem 1.12 (1), we have $(TTA], (ATT]$ and $(TAT \cup TTATT]$ are an ordered left, an ordered right and an ordered lateral ideal of T, respectively. Now,

$$(TTA] \subseteq (TTQ] \subseteq (TTL) \subseteq (L] = L.$$

Since L is a minimal ordered left ideal of T, we have $(TTA] = L$. Similarly, $(ATT] = R$ and $(TAT \cup TTATT] = M$. Since A is an ordered quasi-ideal of T, we have

$$Q = L \cap M \cap R = (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A.$$

This implies that $A = Q$. Hence, Q is a minimal ordered quasi-ideal of T. □

Definition 2.7. *A nonzero ordered quasi-ideal Q of an ordered ternary semigroup T with a zero element is called a 0-minimal ordered quasi-ideal of T if there is no a nonzero ordered quasi-ideal A of T such that $A \subset Q$. Equivalently, if for any nonzero ordered quasi-ideal A of T such that $A \subseteq Q$, we have $A = Q$.*

We also define a 0-minimal ordered left, a 0-minimal ordered lateral and a 0-minimal ordered right ideal of an ordered ternary semigroup with a zero element in the same way of a 0-minimal ordered quasi-ideal.

Theorem 2.8. *Let T be an ordered ternary semigroup with a zero element. Then the intersection of a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral ideal of T is either $\{0\}$ or a 0-minimal ordered quasi-ideal of T.*

Proof. Let $Q = L \cap M \cap R \neq \{0\}$ where L, R and M are a 0-minimal ordered left, a 0-minimal ordered right and a 0-minimal ordered lateral ideal of T, respectively. By Lemma 1.11 (2), we have Q is an ordered quasi-ideal of T. Let A be a nonzero ordered quasi-ideal of T such that $A \subseteq Q$. By Theorem 1.12 (1), we have $(TTA], (ATT]$ and $(TAT \cup TTATT]$ are an ordered left, an ordered right and an ordered lateral ideal of T, respectively. Thus we have the following two cases:

Case 1: $(TTA] = \{0\}, (ATT] = \{0\}$, or $(TAT \cup TTATT] = \{0\}$.

If $(TTA] = \{0\}$, then $(TTA] = \{0\} \subseteq A$. Thus A is a nonzero ordered left ideal of T. Since $A \subseteq Q \subseteq L$ and L is a 0-minimal ordered left ideal of T, we have $A = L$. This implies that $A = Q$. Similarly, if $(ATT] = \{0\}$ or $(TAT \cup TTATT] = \{0\}$, then $A = Q$.

Case 2: $(TTA] \neq \{0\}, (ATT] \neq \{0\}$, and $(TAT \cup TTATT] \neq \{0\}$.

Now,

$$(TTA] \subseteq (TTQ] \subseteq (TTL) \subseteq (L] = L.$$

Since L is a 0-minimal ordered left ideal of T, we have $(TTA] = L$. Similarly, $(ATT] = R$ and $(TAT \cup TTATT] = M$. Since A is an ordered quasi-ideal of T, we have

$$Q = L \cap M \cap R = (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A.$$

This implies that $A = Q$. Hence, Q is a 0-minimal ordered quasi-ideal of T. □

Theorem 2.9. *Let Q be an ordered quasi-ideal of an ordered ternary semigroup T without a zero element. If Q is quasi-simple, then Q is a minimal ordered quasi-ideal of T.*

Proof. Assume that Q is quasi-simple and let A be an ordered quasi-ideal of T such that $A \subseteq Q$. Now,

$$(QQA] \cap (QAQ \cup QQAQQ] \cap (AQQ] \subseteq (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A$$

and $(A] \cap Q \subseteq (A] = A$. Thus A is an ordered quasi-ideal of Q. Since Q is quasi-simple, we have $A = Q$. Hence, Q is a minimal ordered quasi-ideal of T. □

Theorem 2.10. *Let Q be a nonzero ordered quasi-ideal of an ordered ternary semigroup T with a zero element. If Q is 0-quasi-simple, then Q is a 0-minimal ordered quasi-ideal of T.*

Proof. Assume that Q is 0-quasi-simple and let A be a nonzero ordered quasi-ideal of T such that $A \subseteq Q$. Now,

$$(QQA] \cap (QAQ \cup QQAQQ] \cap (AQQ] \subseteq (TTA] \cap (TAT \cup TTATT] \cap (ATT] \subseteq A$$

and $(A] \cap Q \subseteq (A] = A$. Thus A is a nonzero ordered quasi-ideal of Q. Since Q is 0-quasi-simple, we have $A = Q$. Hence, Q is a 0-minimal ordered quasi-ideal of T. □

Theorem 2.11. *Let T be an ordered ternary semigroup without a zero element having proper ordered quasi-ideals. Then every proper ordered quasi-ideal of T is minimal if and only if the intersection of any two distinct proper ordered quasi-ideals is empty.*

Proof. Let Q_1 and Q_2 be two distinct proper ordered quasi-ideals of T. By assumption, we have that Q_1 and Q_2 are minimal. If $Q_1 \cap Q_2 \neq \emptyset$, then by Theorem 1.14, $Q_1 \cap Q_2$ is an ordered quasi-ideal of T. Since $Q_1 \cap Q_2 \subseteq Q_1$ and Q_1 is minimal, we have $Q_1 \cap Q_2 = Q_1$. Since $Q_1 \cap Q_2 \subseteq Q_2$ and Q_2 is minimal, we have $Q_1 = Q_1 \cap Q_2 = Q_2$. That is a contradiction. Hence, $Q_1 \cap Q_2 = \emptyset$.

Conversely, let Q be a proper ordered quasi-ideal of T and let A be an ordered quasi-ideal of T such that $A \subseteq Q$. Then A is a proper ordered quasi-ideal of T. If $A \neq Q$, then by assumption, $A = A \cap Q = \emptyset$. That is a contradiction. Hence, $A = Q$. Therefore, Q is a minimal ordered quasi-ideal of T. □

Theorem 2.12. *Let T be an ordered ternary semigroup with a zero element having nonzero proper ordered quasi-ideals. Then every nonzero proper ordered quasi-ideal of T is 0-minimal if and only if the intersection of any two distinct nonzero proper ordered quasi-ideals is $\{0\}$.*

Proof. Let Q_1 and Q_2 be two distinct nonzero proper ordered quasi-ideals of T. By assumption, we have that Q_1 and Q_2 are 0-minimal. If $Q_1 \cap Q_2 \neq \{0\}$, then by Theorem 1.14, $Q_1 \cap Q_2$ is a nonzero ordered quasi-ideal of T. Since $Q_1 \cap Q_2 \subseteq Q_1$ and Q_1 is 0-minimal, we have $Q_1 \cap Q_2 = Q_1$. Since $Q_1 \cap Q_2 \subseteq Q_2$ and Q_2 is 0-minimal, we have $Q_1 = Q_1 \cap Q_2 = Q_2$. That is a contradiction. Hence, $Q_1 \cap Q_2 = \{0\}$.

Conversely, let Q be a nonzero proper ordered quasi-ideal of T and let A be a nonzero ordered quasi-ideal of T such that $A \subseteq Q$. Then A is a nonzero proper ordered quasi-ideal of T. If $A \neq Q$, then by assumption, $A = A \cap Q = \{0\}$. That is a contradiction. Hence, $A = Q$. Therefore, Q is a 0-minimal ordered quasi-ideal of T. □

3 Maximality of Ordered Quasi-Ideals in Ordered Ternary Semigroups

In this section, we characterize the relationship between the maximality of ordered quasi-ideals and the union $\mathcal{U}$ of all proper ordered quasi-ideals in ordered ternary semigroups without a zero element and the union $\mathcal{U}_0$ of all nonzero proper ordered quasi-ideals in ordered ternary semigroups with a zero element.

Definition 3.1. *A proper ordered quasi-ideal Q of an ordered ternary semigroup T is called a maximal ordered quasi-ideal of T if there is no a proper ordered quasi-ideal A of T such that $Q \subset A$. Equivalently, if for any proper ordered quasi-ideal A of T such that $Q \subseteq A$, we have $A = Q$. Equivalently, if for any ordered quasi-ideal A of T such that $Q \subset A$, we have $A = T$.*

Theorem 3.2. *Let Q be a proper ordered quasi-ideal of an ordered ternary semigroup T. If either*

(1) $T \setminus Q = \{a\}$ for some $a \in T$ or

(2) $T \setminus Q \subseteq (TTb] \cap (TbT \cup TTbTT] \cap (bTT]$ for all $b \in T \setminus Q$,

then Q is a maximal ordered quasi-ideal of T.

Proof. Let A be an ordered quasi-ideal of T such that $Q \subset A$.
Case 1: $T \setminus Q = \{a\}$ for some $a \in T$.

Since $Q \subset A$, we have $\emptyset \neq A \setminus Q \subseteq T \setminus Q = \{a\}$. Thus $A \setminus Q = \{a\}$. Hence, $A = Q \cup (A \setminus Q) = Q \cup \{a\} = Q \cup (T \setminus Q) = T$.
Case 2: $T \setminus Q \subseteq (TTb] \cap (TbT \cup TTbTT] \cap (bTT]$ for all $b \in T \setminus Q$.
Let $b \in A \setminus Q \subseteq T \setminus Q$ because $A \setminus Q \neq \emptyset$. Thus

$$\begin{aligned} T \setminus Q &\subseteq (TTb] \cap (TbT \cup TTbTT] \cap (bTT] \\ &\subseteq (TTA] \cap (TAT \cup TTATT] \cap (ATT] \\ &\subseteq A. \end{aligned}$$

Hence, $T = Q \cup (T \setminus Q) \subseteq Q \cup A = A$. This implies that $A = T$.

Therefore, Q is a maximal ordered quasi-ideal of T. □

Theorem 3.3. *If Q is a maximal ordered quasi-ideal of an ordered ternary semigroup T and $Q \cup Q(a)$ is an ordered quasi-ideal of T for all $a \in T \setminus Q$, then either*

(1) $T \setminus Q \subseteq (a]$ and $a^3 \in Q$ for some $a \in T \setminus Q$, and $(TTb] \cap (TbT \cup TTbTT] \cap (bTT] \subseteq Q$ for all $b \in T \setminus Q$ or

(2) $T \setminus Q \subseteq Q(a)$ for all $a \in T \setminus Q$.

Proof. Assume that Q is a maximal ordered quasi-ideal of an ordered ternary semigroup T and $Q \cup Q(a)$ is an ordered quasi-ideal of T for all $a \in T \setminus Q$. Then we consider the following two cases:
Case 1: $(TTa] \cap (TaT \cup TTaTT] \cap (aTT] \subseteq Q$ for some $a \in T \setminus Q$.
Then $a^3 \in (TTa] \cap (TaT \cup TTaTT] \cap (aTT] \subseteq Q$, so $a^3 \in Q$. By Theorem 1.16, we have

$$\begin{aligned} Q \cup (a] &= (Q \cup ((TTa] \cap (TaT \cup TTaTT] \cap (aTT])) \cup (a] \\ &= Q \cup (((TTa] \cap (TaT \cup TTaTT] \cap (aTT]) \cup (a]) \\ &= Q \cup Q(a). \end{aligned}$$

Thus $Q \cup (a]$ is an ordered quasi-ideal of T. Since $a \in T \setminus Q$, we have $Q \subset Q \cup (a]$. Since Q is a maximal ordered quasi-ideal of T, we have $Q \cup (a] = T$. Thus $T \setminus Q \subseteq (a]$. Next, we let $b \in T \setminus Q$. Then $b \leq a$. Thus

$$(TTb] \cap (TbT \cup TTbTT] \cap (bTT] \subseteq (TTa] \cap (TaT \cup TTaTT] \cap (aTT] \subseteq Q.$$

Case 2: $(TTa] \cap (TaT \cup TTaTT] \cap (aTT] \nsubseteq Q$ for all $a \in T \setminus Q$.
Let $a \in T \setminus Q$. Then $Q \subset Q \cup Q(a)$. Since $Q \cup Q(a)$ is an ordered quasi-ideal of T and Q is maximal, we have $Q \cup Q(a) = T$. Hence, $T \setminus Q \subseteq Q(a)$. □

For an ordered ternary semigroup T without a zero element, the union of all proper ordered quasi-ideals of T is denoted by $\mathcal{U}$.

Lemma 3.4. *Let T be an ordered ternary semigroup without a zero element. Then $T = \mathcal{U}$ if and only if $Q(a) \neq T$ for all $a \in T$.*

Proof. Assume that $T = \mathcal{U}$ and let $a \in T$. Then $a \in \mathcal{U}$, so $a \in Q$ for some proper ordered quasi-ideal Q of T. Hence, $Q(a) \subseteq Q \neq T$, that is $Q(a) \neq T$.

Conversely, assume that $Q(a) \neq T$ for all $a \in T$. Then $Q(a) \subseteq \mathcal{U}$ for all $a \in T$, so $a \in \mathcal{U}$ for all $a \in T$. Hence, $T = \mathcal{U}$. □

Theorem 3.5. *Let T be an ordered ternary semigroup without a zero element. Then one and only one of the following four conditions is satisfied:*

(1) $\mathcal{U}$ is not an ordered quasi-ideal of T.

(2) $Q(a) \neq T$ for all $a \in T$.

(3) There exists $a \in T$ such that $Q(a) = T, (a] \nsubseteq (TTa] \cap (TaT \cup TTaTT] \cap (aTT]$, and $a^3 \in \mathcal{U}$, T is not quasi-simple, $T \setminus \mathcal{U} = \{x \in T \mid Q(x) = T\}$, and $\mathcal{U}$ is the unique maximal ordered quasi-ideal of T.

(4) $T \setminus \mathcal{U} \subseteq Q(a)$ for all $a \in T \setminus \mathcal{U}$, T is not quasi-simple, $T \setminus \mathcal{U} = \{x \in T \mid Q(x) = T\}$, and $\mathcal{U}$ is the unique maximal ordered quasi-ideal of T.

Proof. Assume that $\mathcal{U}$ is an ordered quasi-ideal of T. We consider the following two cases:

Case 1: $\mathcal{U} = T$.

By Lemma 3.4, the condition (2) holds.

Case 2: $\mathcal{U} \neq T$.

Then T is not quasi-simple. We claim that $\mathcal{U}$ is the unique maximal ordered quasi-ideal of T. Let Q be an ordered quasi-ideal of T such that $\mathcal{U} \subset Q$. If $Q \neq T$, then $Q \subseteq \mathcal{U}$. That is a contradiction. Thus $Q = T$, so $\mathcal{U}$ is a maximal ordered quasi-ideal of T. Next, assume that A is a maximal ordered quasi-ideal of T. Then $A \neq T$, so $A \subseteq \mathcal{U} \subset T$. Since A is maximal, we have $A = \mathcal{U}$. Therefore, $\mathcal{U}$ is the unique maximal ordered quasi-ideal of T. Since $\mathcal{U} \neq T$, we have $Q(a) = T$ for all $a \in T \setminus \mathcal{U}$ and $Q(a) \neq T$ for all $a \in \mathcal{U}$. Thus $T \setminus \mathcal{U} = \{x \in T \mid Q(x) = T\}$ and so $\mathcal{U} \cup Q(x) = T$ is an ordered quasi-ideal of T for all $x \in T \setminus \mathcal{U}$. By Theorem 3.3, we have the following two cases:

(i) $T \setminus \mathcal{U} \subseteq (a]$ and $a^3 \in \mathcal{U}$ for some $a \in T \setminus \mathcal{U}$, and $(TTb] \cap (TbT \cup TTbTT] \cap (bTT] \subseteq \mathcal{U}$ for all $b \in T \setminus \mathcal{U}$ or

(ii) $T \setminus \mathcal{U} \subseteq Q(a)$ for all $a \in T \setminus \mathcal{U}$.

Assume (i) holds. Then $T = Q(a)$. If $(a] \subseteq (TTa] \cap (TaT \cup TTaTT] \cap (aTT]$, then by Theorem 1.16, we have

$$\begin{aligned} T &= Q(a) \\ &= (a] \cup ((TTa] \cap (TaT \cup TTaTT] \cap (aTT]) \\ &= (TTa] \cap (TaT \cup TTaTT] \cap (aTT] \\ &\subseteq \mathcal{U}. \end{aligned}$$

Thus $\mathcal{U} = T$. That is a contradiction. Hence, $(a] \nsubseteq (TTa] \cap (TaT \cup TTaTT] \cap (aTT]$, so the condition (3) holds.

Assume (ii) holds. Then the condition (4) holds. □

For an ordered ternary semigroup T with a zero element, the union of all nonzero proper ordered quasi-ideals of T is denoted by $\mathcal{U}_0$.

Lemma 3.6. *Let T be an ordered ternary semigroup with a zero element. Then $T = \mathcal{U}_0$ if and only if $Q(a) \neq T$ for all $a \in T$.*

Proof. The proof is almost similar to the proof of Lemma 3.4. □

Theorem 3.7. *Let T be an ordered ternary semigroup with a zero element. Then one and only one of the following four conditions is satisfied:*

(1) $\mathcal{U}_0$ is not an ordered quasi-ideal of T.

(2) $Q(a) \neq T$ for all $a \in T$.

(3) There exists $a \in T$ such that $Q(a) = T, (a] \nsubseteq (TTa] \cap (TaT \cup TTaTT] \cap (aTT]$, and $a^3 \in \mathcal{U}_0$, T is not 0-quasi-simple, $T \setminus \mathcal{U}_0 = \{x \in T \mid Q(x) = T\}$, and $\mathcal{U}_0$ is the unique maximal ordered quasi-ideal of T.

(4) $T \setminus \mathcal{U}_0 \subseteq Q(a)$ for all $a \in T \setminus \mathcal{U}_0$, T is not 0-quasi-simple, $T \setminus \mathcal{U}_0 = \{x \in T \mid Q(x) = T\}$, and $\mathcal{U}_0$ is the unique maximal ordered quasi-ideal of T.

Proof. The proof is almost similar to the proof of Theorem 3.5. □

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References

[1] M. Akram and N. Yaqoob, Intuitionistic fuzzy soft ordered ternary semigroups, *Int. J. Pure Appl. Math.*, 84(2013), 93-107.

[2] M. Arslanov and N. Kehayopulu, A note on minimal and maximal ideals of ordered semigroups, *Lobachevskii J. Math.*, 11(2002), 3-6.

[3] S. Bashir and M. Shabir, Pure ideals in ternary semigroups, *Quasigroups Relat. Syst.*, 17(2009), 149-160.

[4] Y. Cao and X. Xu, On minimal and maximal left ideals in ordered semigroups, *Semigroup Forum*, 60(2000), 202-207.

[5] T. Changphas, A note on minimal quasi-ideals in ternary semigroups, *Int. Math. Forum*, 7(2012), 539-544.

[6] T. Changphas, A note on quasi and bi-ideals in ordered ternary semigroups, *Int. J. Math. Anal.*, 6(2012), 527-532.

[7] R. Chinram, S. Baupradist and S. Saelee, Minimal and maximal bi-ideals in ordered ternary semigroups, *Int. J. Phys. Sci.*, 7(2012), 2674-2681.

[8] R. Chinram and S. Saelee, Fuzzy ideals and fuzzy filters of ordered ternary semigroups, *J. Math. Res.*, 2(2010), 93-97.

[9] P. Choosuwan and R. Chinram, A study on quasi-ideals in ternary semigroups, *Int. J. Pure Appl. Math.*, 77(2012), 639-647.

[10] V.R. Daddi and Y.S. Pawar, On ordered ternary semigroups, *Kyungpook Math. J.*, 52(2012), 375-381.

[11] V.N. Dixit and S. Dewan, A note on quasi and bi-ideals in ternary semigroups, *Int. J. Math. Math. Sci.*, 18(1995), 501-508.

[12] T.K. Dutta, S. Kar and B.K. Maity, On ideals in regular ternary semigroups, *Discuss. Math., Gen. Algebra Appl.*, 28(2008), 147-159.

[13] A. Iampan, Lateral ideals of ternary semigroups, *Ukr. Math. Bull.*, 4(2007), 525-534.

[14] A. Iampan, Minimality and maximality of ordered quasi-ideals in ordered semigroups, *Asian-Eur. J. Math.*, 1(2008), 85-92.

[15] A. Iampan, On ordered ideal extensions of ordered ternary semigroups, *Lobachevskii J. Math.*, 31(2010), 13-17.

[16] A. Iampan and M. Siripitukdet, On minimal and maximal ordered left ideals in po-Γ-semigroups, *Thai J. Math.*, 2(2004), 275-282.

[17] N. Kehayopulu, On completely regular ordered semigroups, *Sci. Math.*, 1(1998), 27-32.

[18] D.H. Lehmer, A ternary analoue of abelian groups, *Am. J. Math.*, 59(1932), 329-338.

[19] S. Lekkoksung and N. Lekkoksung, On intra-regular ordered ternary semigroups, *Int. J. Math. Anal.*, 6(2012), 69-73.

[20] J. Los, On the extending of model I, *Fundam. Math.*, 42(1955), 38-54.

[21] S. Saelee and R. Chinram, A study on rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups, *IAENG, Int. J. Appl. Math.*, 41(2011).

[22] J. Sanborisoot and T. Changphas, On pure ideals in ordered ternary semigroups, *Thai J. Math.*, (In Press).

[23] F.M. Sioson, Ideal theory in ternary semigroups, *Math. Jap.*, 10(1965), 63-84.

[24] O. Steinfeld, Über die quasiideale von halbgruppen, *Publ. Math.*, 4(1956), 262-275.

[25] O. Steinfeld, Über die quasiideale von ringen, *Acta Sci. Math.*, 17(1956), 170-180.

[26] N. Yaqoob, S. Abdullah, N. Rehman and M. Naeem, Roughness and fuzziness in ordered ternary semigroups, *World Appl. Sci. J.*, 17(2012), 1683-1693.