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Certain Classes of Solutions of the Quasilinear Systems of Differential Equations

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Abstract

The aim of this paper is to establish the existence, behavior and approximation of certain classes of solutions of the quasilinear systems of differential equations. Behavior of integral curves in neighborhoods of an arbitrary or certain curve is considered. The approximate solutions with precise error estimates are determined. The theory of qualitative analysis of differential equations and topological retraction method are used.

Keywords: *Quasilinear differential systems, existence and behavior of solutions, approximation of solutions.*

1 Introduction

Let us consider the systems of differential equations

$$\dot{x}_i = [p_i(t) + h_i(x,t)]\, x_i, \quad i = 1, \cdots, n, \tag{1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^\tau$, $h_i \in C(\Omega, \mathbb{R})$, $p_i \in C(I, \mathbb{R})$, $i = 1, \cdots, n$, $D \subset \mathbb{R}^n$ is open set, $\Omega = D \times I$, $I = \langle a, \infty\rangle$, $a \in \mathbb{R}$. The functions h_i satisfy the Lipschitz's condition with respect to the variable x on D.

Above conditions for the functions p_i and h_i $(i = 1, \cdots, n)$, grant the existence and unique solutions of every Cauchy's problem for system (1) in Ω.

Many authors considered the systems of differential equations in the form (1), for example, K. Sigmund, Y. Takeuchi, N. Adachi, A. Tineo ([4], [5], [6]).

The systems of type (1) are considered very often. This model is used in physics, biophysics, chemistry, biochemistry and economy.

Let

$$\Gamma = \{(x,t) \in \Omega : x = \varphi(t), \, t \in I\},$$

be a curve in Ω, for some $\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$, $\varphi_i \in C^1(I, \mathbb{R})$. We shall consider the behavior of integral curves $(x(t), t)$, $t \in I$, of the systems (1), with respect to the sets $\omega, \sigma \subset \Omega$, which are the appropriate neighborhoods of curve Γ, in the forms

$$\omega = \{(x,t) \in \Omega : \|x - \varphi(t)\| < r(t)\}, \tag{2}$$

$$\sigma = \{(x,t) \in \Omega : |x_i - \varphi_i(t)| < r_i(t), \, i = 1, \cdots n\} \tag{3}$$

($\|\cdot\|$ is Euklidian norm on $\mathbb{R}^n$), where $r, r_i \in C^1(I, \mathbb{R}^+)$, $i = 1, \cdots, n$, $\mathbb{R}^+ = \langle 0, \infty \rangle$. The qualitative analysis theory of differential equations and topological retraction method of T.Ważewski [9], are used.

2 Notation and Preliminaries

Let us denote the tangent vector field to an integral curve $(x(t), t)$, $t \in I$, of (1) by T. We have

$$T(x,t) = ((p_1(t) + h_1(x,t))\, x_1, \cdots, (p_n(t) + h_n(x,t))\, x_n, 1). \tag{4}$$

The boundary surfaces of ω and σ are, respectively,

$$W = \left\{(x,t) \in Cl\omega \cap \Omega : B(x,t) := \sum_{i=1}^{n} (x_i - \varphi_i(t))^2 - r^2(t) = 0\right\}, \tag{5}$$

$$\begin{aligned} W_i^k &= \left\{(x,t) \in Cl\sigma \cap \Omega : B_i^k(x,t) := (-1)^k (x_i - \varphi_i(t)) - r_i(t) = 0\right\}, \\ k &= 1, 2, \;\; i = 1, \cdots, n. \end{aligned} \tag{6}$$

The vectors ∇B and ∇B_i^k are the external normals on surfaces W and W_i^k, respectively,

$$\frac{1}{2} \nabla B(x,t) = \begin{pmatrix} x_1 - \varphi_1(t), \cdots, x_i - \varphi_i(t), \cdots x_n - \varphi_n(t), \\ -\sum_{i=1}^{n} (x_i - \varphi_i(t))\, \varphi_i'(t) - r(t)\, r'(t) \end{pmatrix}, \tag{7}$$

$$\nabla B_i^k(x,t) = (-1)^k \left(\delta_{1i}, \cdots, \delta_{ii}, \cdots, \delta_{ni}, -\varphi_i' - (-1)^k r_i'\right), \tag{8}$$

where δ_{mi} is the Kronecker delta symbol. Considering the sign of the scalar products

$$P(x,t) = \left(\frac{1}{2}\nabla B(x,t), T(x,t)\right) \quad \text{on } W$$

and

$$P_i^k(x,t) = \left(\nabla B_i^k(x,t), T(x,t)\right) \quad \text{on } W_i^k,\ k = 1,2,\ i = 1,\cdots,n,$$

we shall establish the behavior of integral curves of (1) with respect to the sets ω and σ, respectively.

The results of this paper are based on the following Lemmas 1 and 2 (see [1], [7]) and Lemma 3 (see [2], [8]). In the following $(n_1,\cdots,n_n)$, denote a permutation of indices $(1,\cdots,n)$.

Lemma 1. *If, for the system* (1), *the scalar product*

$$P(x,t) = \left(\frac{1}{2}\nabla B(x,t), T(x,t)\right) < 0 \quad \text{on } W,$$

then the system (1) *has the* $n-$parameter *class of solutions belonging to the set* ω (graphs of solutions belong to ω) *for all* $t \in I$.

According to this Lemma, the set W is a set of point of strict entrance of integral curves of the system (1) with respect to the sets ω and Ω. Hence, all solutions of system (1) which satisfy condition

$$\|x(t_0) - \varphi(t_0)\| < r(t_0), \quad t_0 \in I, \tag{9}$$

also satisfy condition

$$\|x(t) - \varphi(t)\| < r(t), \quad \text{for every } t > t_0, \tag{10}$$

Hence, for any point $P_0 = (x^0, t_0) \in \omega$, the integral curve passing trough P_0 belongs to ω *for all* $t \geq t_0$.

Lemma 2. *If, for the system* (1), *the scalar product*

$$P(x,t) = \left(\frac{1}{2}\nabla B(x,t), T(x,t)\right) > 0 \quad \text{on } W,$$

then the system (1) *has at least one solution on* I *whose graph belongs to the set* ω *for all* $t \in I$.

According to Lemma 2, the set W is a set of points of strict exit of integral curves of the system (1) with respect to the sets ω and Ω. Hence, according to T. Wazewski's retraction method [9], system (1) has at least one solutions belonging to the set ω for every $t \in I$.

Lemma 3. *If, for the system* (1) , *the scalar products*

$$P_i^k = \left(\nabla B_i^k, T\right) < 0 \quad on\ W_i^k,\ k = 1,2,\ i = n_1, \cdots, n_p, \tag{11}$$

and

$$P_i^k = \left(\nabla B_i^k, T\right) > 0 \quad on\ W_i^k,\ k = 1,2,\ i = n_{p+1}, \cdots, n_n, \tag{12}$$

where $p \in \{0, 1, \cdots, n\}$, *then the system* (1) *has a* $p-$parameter *class of solutions which belongs to the set* σ *(graphs of solutions belong to* σ*) for all* $t \in I$.

The case $p = 0$ means that the system (1) has at least one solution belonging to the set σ for all $t \in I$. The conditions (11) and (12) imply that the set $U = \bigcup_{i=n_1}^{n_p} (W_i^1 \cup W_i^2)$ has no point of exit and $V = \bigcup_{i=n_{p+1}}^{n_n} (W_i^1 \cup W_i^2)$ is the set of points of strict exit from set σ with respect to the set Ω, for integral curves of system (1), which according to the retraction method [9], makes the statement of Lemma valid (see [1], [2], [8]). In the case $p = n$ this Lemma gives the statement of Lemma 1, and for $p = 0$ the statement of Lemma 2 in [7].

3 The $n-$Parameter Classes of Solutions

First, let us consider the behavior of integral curves of the system (1) with respect to the set ω.

Theorem 1 *Let* $m, M \in C(\Omega, R)$ *and*

$$\varphi_i(t) = C_i \exp\left[\int p_i(t)\, dt\right], \quad i = 1, \cdots, n, \quad C_i \in \mathbb{R}. \tag{13}$$

(i) *If the conditions*

$$\begin{aligned} p_i(t) + h_i(x,t) &\leqslant M(x,t), \quad i = 1, \cdots, n\ , \\ \sum_{i=1}^{n} |h_i(x,t)\, \varphi_i(t)| &\leqslant -M(x,t)\, r(t) + r'(t)\ , \end{aligned}$$

are satisfied on W, *then the system* (1) *has the* $n-$*parameter class of solutions* $x(t)$ *belonging to the set* ω *for all* $t \in I$, *i.e. every solutions* $x(t)$ *of system* (1) *which satisfies the condition* (9) *also satisfies* (10) *for every* $t \geq t_0$.

(*ii*) *If the conditions*

$$m(x,t) \leqslant p_i(t) + h_i(x,t), \quad i = 1, 2, \cdots, n \ ,$$

$$\sum_{i=1}^{n} |h_i(x,t)\,\varphi_i(t)| \leqslant m(x,t)\,r(t) + r'(t) \ ,$$

are satisfied on W, *then the system* (1) *has at least one solution* $x(t)$ *which satisfies the condition* (10).

Proof. Firstly, we can note that $p_i(t)\,\varphi_i(t) - \varphi_i'(t) = 0,\ i = 1, \cdots, n,$ $t \in I$. For the scalar product $P(x,t)$ we have

$$\begin{aligned} P &= \sum_{i=1}^{n} (p_i + h_i)(x_i - \varphi_i)\,x_i - \sum_{i=1}^{n} (x_i - \varphi_i)\,\varphi_i' - rr' \\ &= \sum_{i=1}^{n} (p_i + h_i)(x_i - \varphi_i)^2 + \sum_{i=1}^{n} [(p_i + h_i)\,\varphi_i - \varphi_i']\,(x_i - \varphi_i) - rr' \\ &= \sum_{i=1}^{n} (p_i + h_i)(x_i - \varphi_i)^2 + \sum_{i=1}^{n} h_i\varphi_i\,(x_i - \varphi_i) - rr'. \end{aligned}$$

Now, according to the conditions of this Theorem, the following estimates for P on W are valid in cases (i) and(ii), respectively:

(i)

$$\begin{aligned} P &\leq r^2 \sum_{i=1}^{n} (p_i + h_i) + r \sum_{i=1}^{n} |h_i\varphi_i| - rr' \\ &\leq Mr^2 + r \sum_{i=1}^{n} |h_i\varphi_i| - rr' < 0, \end{aligned} \tag{14}$$

(ii)

$$P \geq mr^2 - r \sum_{i=1}^{n} |h_i\varphi_i| - rr' > 0. \tag{15}$$

According to the Lemma 1, the estimate (14) implies that the system (1) has the $n-$parameter class of solutions $x(t)$ belonging to the corresponding set ω for all $t \in I$, and the estimate (15) implies that the system (1) has at least one solution $x(t)$ which satisfies that condition. This confirms the statements of the Theorem. ∎

4 The $p-$Parameter Classes of Solutions

Let us now consider the behavior of integral curves of the system (1) with respect to the set σ, where φ is defined by (13).

Theorem 2 *If, on* $W_i^k,\ k=1,2,$

$$|h_i(x,t)\varphi_i(t)| < -(p_i(t)+h_i(x,t))r_i(t)+r_i'(t) \tag{16}$$

for $i=n_1,\cdots,n_p,$ *and*

$$|h_i(x,t)\varphi_i(t)| < (p_i(t)+h_i(x,t))r_i(t)-r_i'(t) \tag{17}$$

for $i=n_{p+1},\cdots,n_n$, *then the system* (1) *has a* $p-$*parameter class of solutions* $x(t)$ *satisfying the condition*

$$|x_i(t)-\varphi_i(t)| < r_i(t), \qquad i=1,\ldots,n, \quad t\in I,$$

where φ *is defined by* (13).

Proof. For the scalar product $P_i^k = \left(\nabla B_i^k, T\right)$ on $W_i^k,\ k=1,2,$ we have

$$\begin{aligned} P_i^k &= (-1)^k(p_i+h_i)x_i-(-1)^k\varphi_i'-r' \\ &= (-1)^k[(p_i+h_i)x_i-\varphi_i']-r' \\ &= (-1)^k(p_i+h_i)(x_i-\varphi_i)+(-1)^k[(p_i+h_i)\varphi_i-\varphi_i']-r_i' \\ &= (p_i+h_i)r_i+(-1)^k[(p_i+h_i)\varphi_i-\varphi_i']-r_i' \\ &= (p_i+h_i)r_i+(-1)^k h_i\varphi_i-r_i'. \end{aligned}$$

According to conditions (16) and (17) on $W_i^k,\ k=1,2,$ we have

$$P_i^k \leqslant (p_i+h_i)r_i+|h_i\varphi_i|-r_i' < 0$$

for $i=n_1,\cdots,n_p$, and

$$P_i^k \geqslant (p_i+h_i)r_i-|h_i\varphi_i|-r_i' > 0$$

for $i=n_{p+1},\cdots,n_n$. Hence, in direction of p axis we have $P_i^k<0$ on W_i^k, and in the direction of other $n-p$ axis $P_i^k>0$ on $W_i^k,\ k=1,2.$ These estimates, according to the Lemma 3, confirm that the system (1) has the $p-$parameter class of solutions $x(t)$ belonging to the corresponding set σ for all $t\in I$. ■

5 Applications

The predator-prey ecosystem model, applying appropriate substitution ([3]), can be written in the form

$$\begin{aligned} \dot{x}_1 &= (1 - x_1 - a x_2)\ x_1, \\ \dot{x}_2 &= (-b + a x_1)\ x_2, \quad a, b \in \mathbb{R}^+. \end{aligned} \tag{18}$$

Let us consider the integral curves of system (18) in neighborhoods of his curves of stationary point: $(0, 0, t)$, $(1, 0, t)$ and $\left(\frac{b}{a}, \frac{a-b}{a^2}, t\right)$, $t \in I$.

Theorem 3 *Let* $\alpha, \beta \in \mathbb{R}^+$.

(i) System (18) *has a one-parameter class of solutions* $x(t)$ *which satisfy condition*

$$|x_1(t)| < \alpha e^{-\beta t}, \quad |x_2(t)| < \alpha e^{-\beta t} \quad \text{for } t > 0,$$

where

$$1 + \beta > \alpha(1 + a), \ b > \alpha a + \beta.$$

(ii) System (18) *has a one-parameter class of solutions* $x(t)$ *which satisfy condition*

$$|x_1(t) - 1| < \alpha e^{-\beta t}, \quad |x_2(t)| < \alpha e^{-\beta t} \quad \text{for } t > 0,$$

where

$$\beta > (1 + a)(1 + \alpha), \quad b > a(1 + \alpha) + \beta.$$

(iii) System (18) *has at least one solution* $x(t)$ *which satisfy condition*

$$\left| x_1(t) - \frac{b}{a} \right| < \alpha e^{-\beta t}, \quad \left| x_2(t) - \frac{a - b}{a^2} \right| < \alpha e^{-\beta t} \quad \text{for } t > 0, \tag{19}$$

where

$$\beta > \max\left\{ (1 + a)\left(\frac{b}{a} + \alpha\right), \ a\alpha + \frac{|a - b|}{a} \right\}.$$

Proof. Let (x_1^0, x_2^0, t), $t \in I$ be a curve of stationary point of (18). For the scalar products $P_i^k(x_1, x_2, t) = \left(\nabla B_i^k, T\right)$ on W_i^k, $k = 1, 2$, $i = 1, 2$, we have

$$\begin{aligned} P_1^k(x_1, x_2, t) &= (1 - x_1 - a x_2)\, r + (-1)^k (1 - x_1 - a x_2)\, x_1^0 - r', \\ P_2^k(x_1, x_2, t) &= (-b + a x_1)\, r + (-1)^k (-b + a x_1)\, x_2^0 - r', \end{aligned}$$

where $r = r_1 = r_2 = \alpha e^{-\beta t}$ for all cases (i), (ii) and (iii).

(i) For $(x_1^0, x_2^0, t) = (0, 0, t)$ we have

$$\begin{aligned} P_1^k(x_1, x_2, t) &= (1 - x_1 - ax_2)\ r - r' \geq (1 - r - ar)\ r - r' \\ &= \alpha e^{-\beta t}\left[1 - (1 + a)\,\alpha e^{-\beta t} + \beta\right] \\ &\geq \alpha e^{-\beta t}\left[1 - (1 + a)\,\alpha + \beta\right] > 0, \end{aligned}$$

$$\begin{aligned} P_2^k(x_1, x_2, t) &= (-b + ax_1)\ r - r' \leq (-b + ar)\ r - r' \\ &= \alpha e^{-\beta t}\left(-b + a\alpha e^{-\beta t} + \beta\right) \\ &< \alpha e^{-\beta t}\left(-b + a\alpha + \beta\right) < 0. \end{aligned}$$

(ii) For $(x_1^0, x_2^0, t) = (1, 0, t)$ we have

$$\begin{aligned} P_1^k(x_1, x_2, t) &= (1 - x_1 - ax_2)\ r + (-1)^k(1 - x_1 - ax_2) - r' \\ &\geq (-r - ar)\ r - (r + ar) - r' \\ &= \alpha e^{-\beta t}\left[-(1 + a)\left(1 + \alpha e^{-\beta t}\right) + \beta\right] \\ &\geq \alpha e^{-\beta t}\left[-(1 + a)(1 + \alpha) + \beta\right] > 0, \end{aligned}$$

$$\begin{aligned} P_2^k(x_1, x_2, t) &= (-b + ax_1)\ r - r' = [a - b + a(x_1 - 1)]\ r - r' \\ &\leq (a - b + ar)\ r - r' = \alpha e^{-\beta t}\left(a - b + a\alpha e^{-\beta t} + \beta\right) \\ &< \alpha e^{-\beta t}\left(a - b + a + \beta\right) < 0. \end{aligned}$$

(iii) For $(x_1^0, x_2^0, t) = \left(\frac{b}{a}, \frac{a-b}{a^2}, t\right)$ we have

$$\begin{aligned}
P_1^k(x_1, x_2, t) &= (1 - x_1 - ax_2)\, r + (-1)^k (1 - x_1 - ax_2) \frac{b}{a} - r' \\
&= \left[-\left(x_1 - \frac{b}{a}\right) - a\left(x_2 - \frac{a-b}{a^2}\right)\right] r - \\
&\quad - (-1)^k \left[\left(x_1 - \frac{b}{a}\right) + a\left(x_2 - \frac{a-b}{a^2}\right)\right] \frac{b}{a} - r' \\
&\geq (-r - ar)\ r - (r + ar) \frac{b}{a} - r' \\
&= \alpha e^{-\beta t}\left[-(1+a)\left(\frac{b}{a} + \alpha e^{-\beta t}\right) + \beta\right] \\
&\geq \alpha e^{-\beta t}\left[-(1+a)\left(\frac{b}{a} + \alpha\right) + \beta\right] > 0,
\end{aligned}$$

$$\begin{aligned}
P_2^k(x_1, x_2, t) &= (-b + ax_1)\, r + (-1)^k (-b + ax_1) \frac{a-b}{a^2} - r' \\
&= a\left(x_1 - \frac{b}{a}\right) r + (-1)^k \left(x_1 - \frac{b}{a}\right) \frac{a-b}{a} - r' \\
&\geq -ar^2 - r\frac{|a-b|}{a} - r' = \alpha e^{-\beta t}\left[-a\alpha e^{-\beta t} - \frac{|a-b|}{a} + \beta\right] \\
&\geq \alpha e^{-\beta t}\left[-a\alpha - \frac{|a-b|}{a} + \beta\right] > 0.
\end{aligned}$$

According to Lemma 3, the above estimates for P_i^k $(k = 1, 2,\ i = 1, 2)$ imply the statement of the Theorem.

Notice that the statement of the theorem does not guarantee the existence of a new solution that satisfies the condition (19) . It guarantees the existence at least one solution that satisfies the (19) , and that is exactly stationary solution $\left(\frac{b}{a}, \frac{a-b}{a^2}, t\right)$, $t \in I$. ■

Remark: We can note that the obtained results also contain an answers to the questions on approximation and stability or instability of solutions $x(t)$ whose existence is established. The errors of the approximation and the functions of stability or instability (including autostability and stability along the coordinates) are defined by the functions $r(t)$ and $r_i(t)$, $i = 1, \cdots, n$ (see [2], [7]).

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