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Homotopy Analysis Transform Method for Integro-Differential Equations

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Abstract

In this article, we propose a reliable combination between the homotopy analysis method (HAM) and Laplace transformation method (LTM) to find the analytic approximate solution for integro-differential equations. This study represents significant features of HATM and its capability of handling integro-differential equations. Some illustrative examples are also presented to demonstrate the validity and applicability of this technique. Comparison between the obtained results by HATM and exact solution are shown integro-differential equations to illustrate the effective of this method. This method is reliable and capable of providing analytic treatment for solving such equations.

Keywords: *Integro-differential equations, Homotopy analysis method, Laplace transform method.*

1 Introduction

This paper deals with one of the most applied problems in the engineering sciences. It is concerned with the integro-differential equations where both differential and integral operators will appear in the same equation. This type of equations was introduced by Volterra for the first time in the early 1900. Volterra investigated the population growth, focusing his study on the hereditary influences; where through his research work the topic of integro-differential equations was established [1]. Scientists and engineers come across the integro-differential equations through their research work in heat and mass diffusion processes, electric circuit problems, neutron diffusion, and biological species coexisting together with increasing and decreasing rates of generating. Applications of the integro-differential equations in electromagnetic theory and dispersive waves and ocean circulations are enormous. More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations literatures [2]. It's important to note that in the integro-differential equations, the unknown function $u(x)$ and one or more of its derivatives such as $u'(x), u''(x), \ldots$ appear out and under the integral sign as well. One quick source of integro-differential equations can be clearly seen when we convert the differential equation to an integral equation by using Leibnitz rule. The integro-differential equation can be viewed in this case as an intermediate stage when finding an equivalent Volterra integral equation to the given differential equation. The following are the examples of linear integro-differential equations [3-7]:

$$u'(x) = f(x) - \int_0^x (x-t)u(t)dt, \; u(0) = 0 \tag{1.1}$$

$$u''(x) = g(x) + \int_0^x (x-t)u(t)dt, \quad u(0) = 0, \; u'(0) = -1 \tag{1.2}$$

$$u'(x) = e^x - x + \int_0^1 xtu(t)dt, \quad u(0) = 0 \tag{1.3}$$

$$u''(x) = h(x) + \int_0^x tu'(t)dt, \quad u(0) = 0, \; u'(0) = 1 \tag{1.4}$$

It is clear from the above examples that the unknown function $u(x)$orone of its derivatives appear under the integral sign, and the other derivativesappear out the integral sign as well. These examples can be classified as the Volterra and Fredholm integro-differential equations. Equations (1.1) and (1.2) are the Volterra type whereas equations (1.3) and (1.4) are of Fredholm type integro-differential equations. It is to be noted that these equations are linear integro-differential equations. However, nonlinear integro-differential equations also arise in many scientific and engineering problems [8-10]. Our concern in this paper will be linear integro-differential equations and we will be concerned with the different solution techniques. To obtain a solution of the integro-differential equation, we need to specify the initial conditions to determine the unknown constants [11].

The homotopy analysis method (HAM) [12] has been proved to be one of the useful techniques to solve numerous linear and nonlinear functional equations. As mentioned in [13, 14], unlike all previous analytic techniques, the homotopy analysis method provides great freedom to express solutions of a given nonlinear problem by means of different base functions. Also this method provides a way to adjust and control the convergence region and the rate of convergence of solution series, by introducing the auxiliary parameter h [15-18]. By properly choosing the base functions, initial approximations, auxiliary linear operators, and auxiliary parameter h, HAM gives rapidly convergent successive approximations of the exact solution. A systematic description of this analytic technique, for nonlinear problems, can be found in [13].In recent years many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [19-23].

The main aim of this article is to present analytical and approximate solution of integro-differential equations by using new mathematical tool like homotopy analysis transform method. The proposed method is coupling of the homotopy analysis method HAM and Laplace transform method [24-27].We have studied some of linear and nonlinear integro-differential equations with the help of homotopy analysis transform method.

This paper is organized as follows. In Section 2, a short description of the basic ideas of the homotopy analysis method will be stated and homotopy analysis transform method is applied to construct approximate solution. In Section 3 is devoted to the convergence analysis of the method. In Section 4, our numerical findings are reported and demonstrate the accuracy of the proposed scheme, by considering three numerical examples. Finally, conclusions are stated in the last section.

2 Preliminaries and Notations

In order to elucidate the solution procedure of the homotopy analysis transform method,We consider the following integro- differential equations of second kind:

$$y^{n}(x) = f(x) + \int_{0}^{x} K(x,t)y(t)dt, \ \ 0 \leq x \leq 1$$

with initial conditions

$y(a) = \alpha_0, y^{'}(a) = \alpha_1, \ldots \ldots \ldots y^{n-1}(a) = \alpha_{n-1}$.(2.1)To solve the general *nth*-order integro-differential equation (2.1) using, the homotopy analysis transform method, we recall that the Laplace transforms of the derivatives of are defined by

$$L[y^n(x)] = s^n L[y(x)] - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots \ldots\ldots - y^{n-1}(0).$$

Now applying the Laplce transform on both side in Eq. (2.1) we have

$$L[y^n(x)] = L[f(x)] + L[\int_0^x K(x,t)y(t)dt], \quad 0 \le x \le 1 \tag{2.2}$$

We define the nonlinear operator

$$N[\varphi(x;q)] = L[\varphi^{(n)}(x;q)] - L[f(x)] - L[\int_0^x K(x,t)\varphi(x;q)dt] \tag{2.3}$$

where $q \in (0,1)$ be an embedding parameter and $\varphi(x;q)$ is the real function of x and q . By means of generalizing the traditional homotopy methods, the great mathematician Liao [13-14] constract the zero order deformation equation

$$(1-q)L[\varphi(x;q) - y_0(x)] = \hbar q H(x) N[\varphi(x;q)], \tag{2.4}$$

where is a nonzero auxiliary parameter, $H(x) \neq 0$ an auxiliary function,$y_0(x)$is an initial guess of $y(x)$ and $\varphi(x;q)$ is an unknown function.It is important that one has great freedom to choose auxiliary thing in HATM. Obviously, when $q = 0$ and $q = 1$, it holds:

$$\varphi(x;0) = y_0(x), \; \varphi(x;1) = y(x), \tag{2.5}$$

respectively. Thus, as q increases from 0 to 1 , the solution varies from the initial guess to the solution .Expanding $\varphi(x;q)$ in Taylor's series with respect to q , we have:

$$\varphi(x;q) = y_0(x,t) + \sum_{m=1}^{\infty} q^m y_m(x) \tag{2.6}$$

Where

$$y_m(x) = \frac{1}{m!}\frac{\partial^m \varphi(x;q)}{\partial q^m}\Big|_{q=0} \tag{2.7}$$

If the auxiliary linear operator, the initial guess , the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen , the series (2.6) converges at $q = 1$, we have:

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) \tag{2.8}$$

which must be one of the solutions of the original integral equations. Define the vectors:

$$\vec{y}_n = \{y_0(x), y_1(x), \ldots\ldots\ldots, y_n(x)\} \tag{2.9}$$

Differentiating equation (2.4) m -times with respect to the embedding parameter q , then setting $q=0$ and finally dividing them by $m!$, we obtain the m^{th} -order deformation equation.

$$L[y_m(x) - \chi_m y_{m-1}(x)] = \hbar q H(x) R_m(\vec{y}_{m-1}, x) \tag{2.10}$$

Where

$$R_m(\vec{y}_{m-1}, x) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\varphi(x;q)}{\partial q^{m-1}}\Big|_{q=0} \tag{2.11}$$

And

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{2.12}$$

In this way, it is easily to obtain $y_m(x)$ for $m \ge 1$, at m^{th} $-$order , we have

$$y(x) = \sum_{m=0}^{M} y_m\,(x), \tag{2.13}$$

when M $\to \infty$ we get an accurate approximation of the original Eq (2.1).

3 Convergence Analyses

The convergence of the method is established by Theorem 3.1 in [28] and [29]. In fact, on each interval the inequality $\|y_{i+1}\|_2 < \alpha \|y_i\|_2$ is required to hold for $i = 0,1,2, \ldots, n$ where $0 < \alpha < 1$ is a constant and is the maximum order of the approximant used in the computation. Of course, this is only a necessary condition for convergence, because it would be necessary to compute $\|y_i\|_2$for every $i = 0,1,2, \ldots, n$ in order to conclude that the series is convergent.

Definition 3.1: *Let*$\varphi_n(x), n = 1,2, \ldots\ldots$ *be the successive approximations to the solution y(x) of a problem. If the positive constants L, P exist such that*

$$L = \lim_{n\to\infty} \frac{|\varphi_{n+1}(x_i) - y(x_i)|}{|\varphi_n(x_i) - y(x_i)|^p},$$

Then we call p the (estimated) Local order of convergence (EOC) at the point x_i.. The constant *L* is called convergence Factor atx_i.

Definition 3.2: *The relative errors* δ_n*of the* n*terms approximation of HATM, which is defined as:*

$$\delta(x_j) = \frac{|u_{exact}(x_j) - u_{app}(x_j)|}{|u_{exact}(x_j)|},$$

4 Applications

In order to elucidate the solution procedure of the homotopy analysis transform method for solving n[th]-order integro-differential equations is illustrated in the three examples in this sections which shows the effectiveness and generalizations of our proposed method given below. We show the high accuracy of the solution results from applying the present method to our problem (2.1) compared with the exact solution; the maximum error is defined as:

$$E_n = \|y_{Exact} - \varphi_n(x)\|_\infty ,$$

Where $n = 1,2...$ represents the number of iterations [29-31].

Example 1

We use the proposed method to find the approximate solutions of the following second-order integro-differential equation by using the HATM [29]

$$\begin{cases} y''(x) = e^x - x + \int_0^1 xty(t)dt, \\ y(0) = 1, \qquad y'(0) = 1. \end{cases} \tag{4.1}$$

which has the exact solution $y(x) = e^x$

As mentioned above, taking theLaplace transform of both sides of Eq. (4.1) gives:

$$\mathrm{L}[y''(x)] = \mathrm{L}[e^x - x] + \frac{1}{s^2}\int_0^1 ty(t)dt$$

$$s^2\mathrm{L}[y(x)] - sy(0) - y'(0) - \frac{1}{s-1} + \frac{1}{s^2} - \frac{1}{s^2}\int_0^1 ty(t)dt = 0. \tag{4.2}$$

Using given the initial condition Eq. (4.1) becomes

$$\mathrm{L}[y(x)] - \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^2(s-1)} + \frac{1}{s^4} - \frac{1}{s^4}\int_0^1 ty(t)dt = 0 \tag{4.3}$$

Form Eq. (4.3), we define a nonlinear operator as

$$N[\varphi(x,t;q)] = \mathrm{L}[\varphi(x,t;q)] - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^2(s-1)}\right) - \frac{1}{s^4}\int_0^1 t[\varphi(x,t;q)]dt = 0 \tag{4.4}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathrm{L}[\varphi(x,t;q) - y_0(x,t)] = q\hbar N[\varphi(x,t;q)] \tag{4.5}$$

With initial conditions, where $q \in [0,1]$ is an embedding parameter, $\hbar$is a non-zero auxiliary function, L is Laplace transformation operator, $y0(x,t)$ is an initial guess of $y(x,t)$ and $\varphi(x,t;q)$ is unknown function. When $q = 0$ and $q = 1$ we have:

$$\varphi(x,t;0) = y_0(x,t), \varphi(x,t;1) = y(x,t) \tag{4.6}$$

Expanding $\phi(x,t;q)$ in Taylor series with respect to q, we obtain

$$\varphi(x,t;q) = y_0(x,t) + \sum_{m=1}^{\infty} y_m\,(x,t)q^m \tag{4.7}$$

Where

$$y_m(x,t) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\varphi(x,t;q)}{\partial q^{m-1}}|q = 0 \tag{4.8}$$

The above series is convergent at $q = 1$, then

$$\varphi(x,t;q) = y_0(x,t) + \sum_{m=1}^{\infty} y_m\,(x,t) \tag{4.9}$$

We define the vector

$$\vec{y}_{m-1} = \{y_0(x,t), y_1(x,t) \dots, y_{m-1}(x,t)\} \tag{4.10}$$

The m_th order deformation equation is

$$y_m(x,t) = \chi_m y_{m-1}(x,t) + \hbar \mathrm{L}^{-1}\left(R_m\big(\tilde{y}_{m-1}(x,t)\big)\right) \tag{4.11}$$

Where

$$R_m(\vec{y}_{m-1}) = L[y_{m-1}] - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^2(s-1)}\right)(1-\chi_m) - \frac{1}{s^4}\int_0^1 t y_{m-1}\,dt \tag{4.12}$$

Using the Mathematical package, we obtain the solution as,

$$y_0(x) = 0, y_1(x) = -e^x h + \frac{hx^3}{6},$$

$$y_2(x) = -e^x h(1+h) + \frac{h(30+59h)x^3}{180},$$

$$y_3(x) = -e^x h(1+h)^2 + \frac{h(900+h(3540+2611h))x^3}{5400}, \tag{4.13}$$

At $h = -1$ the solution is given by

$$\varphi_n(x) = \sum_{\mathrm{i}=0}^{n-1} y_{\mathrm{i}}\,(x) = e^x - \frac{x^3}{3!\,30^{n-1}}, \quad \mathrm{n} = 1,2, \dots$$

$$y(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty}\left(e^x - \frac{x^3}{3!30^{n-1}}\right) = e^x, \quad (4.14)$$

that converges to the exact solution, we notice that the result obtained by the present method is very superior (lower error combined with less number of iterations) to that obtained by HPM and VIM. From **Table 1**, it can be deduced that, the error decreased monastically with the increment of the integer.

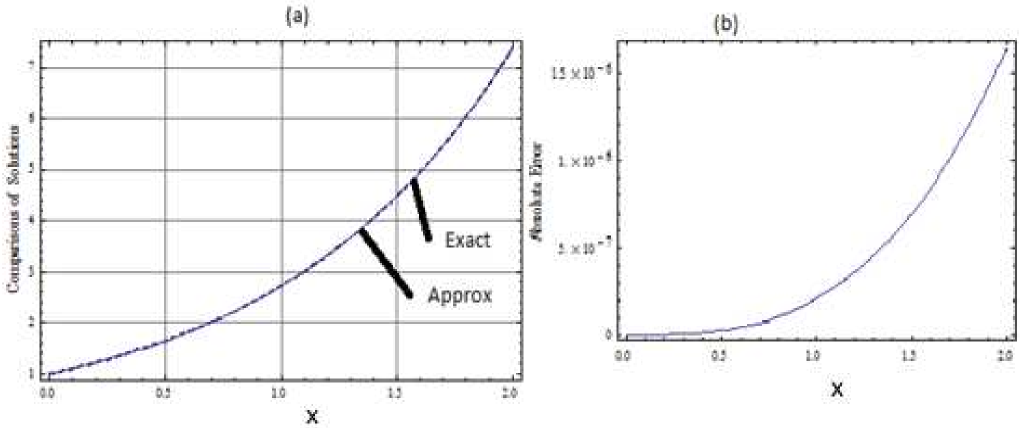

Figure 1: (a) The exact solution is compared with the approximate solution when $h = -1$

(b) AbsoluteError of 5th-order approximate solution with $h = -1$.

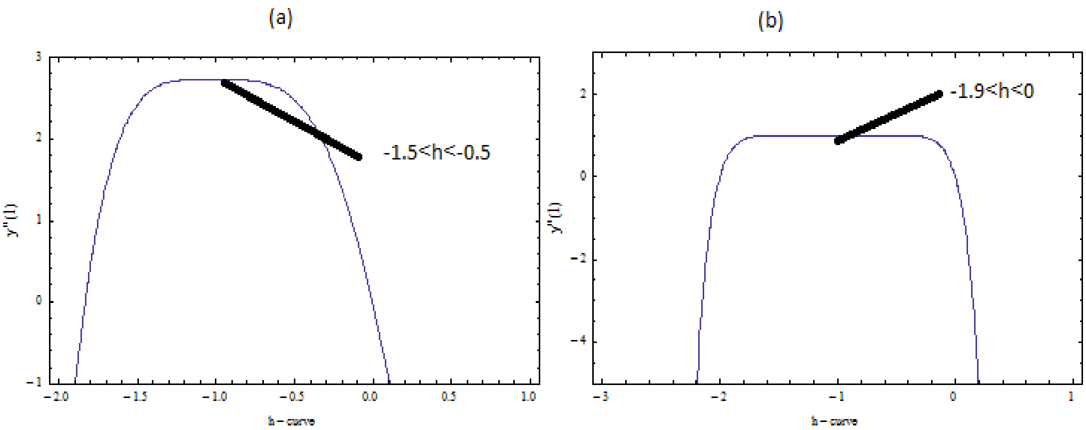

Figure 2: (a) and **(b)** the h-curve of the 5th and 10th order approximate solution (4.13) at x = 1.

Table 1: Comparison of relative errors δ(x) for Example 1

	$\hbar=-1.25$	$\hbar=-1.15$	$\hbar=-1$	$\hbar=-0.9$
$x_1=0.1$	$6.14E-6$	$1.93E-7$	$6.47E-8$	$2.01E-8$
$x_2=0.2$	$4.74E-6$	$1.35E-7$	$1.27E-9$	$1.27E-8$
$x_3=0.3$	$2.14E-5$	$1.57E-6$	$1.51E-7$	$1.36E-7$
$x_4=0.4$	$9.40E-5$	$3.60E-6$	$3.76E-7$	$5.25E-7$
$x_5=0.5$	$2.06E-4$	$3.444E-7$	$2.51E-7$	$2.29E-6$
$x_6=0.6$	$2.28E-4$	$3.07E-5$	$4.06E-7$	$3.98E-6$
$x_7=0.7$	$7.46E-4$	$1.73E-4$	$9.09E-6$	$1.59E-5$
$x_8=0.8$	$1.97E-2$	$9.06E-4$	$2.84E-6$	$7.76E-4$
$x_9=0.9$	$4.95E-3$	$3.26E-4$	$1.42E-5$	$3.67E-4$
$x_{10}=1.0$	$2.99E-3$	$9.76E-4$	$2.61E-5$	$5.65E-4$

Example 2

We use the proposed method to find the approximate solutions of the following integro-differential equation by using the HATM [29]

$$\begin{cases} y^{(8)}(x) = -8e^x + x^2 + y(x) + \int_0^1 x^2 y'(t)dt, \\ y(0)=1,\quad y'(0)=0,\quad y''(0)=-1,\quad y'''(0)=-2 \\ y^{(4)}(0)=-3,\quad y^{(5)}(0)=-4,\quad y^{(6)}(0)=-5,\quad y^{(7)}(0)=-6 \end{cases} \tag{4.15}$$

Which has the exact solution $y(x) = e^x - xe^x$Applying Laplace transform, we have

$$\mathrm{L}[y^{(8)}(x)] = \mathrm{L}[-8e^x + x^2] + \mathrm{L}[y(x)] + \frac{2}{s^3}\int_0^1 y'(t)dt$$

Which satisfies

$$s^8\mathrm{L}[y(x)] - s^7 y(0) - s^6 y'(0) - s^5 y''^{(0)} - s^4 y'''^{(0)} - s^3 y^{(4)}(0) - s^2 y^{(5)}(0) - s y^{(6)}(0) - y^{(7)}(0) + \frac{8}{(s-1)} - \frac{2}{s^3} - \mathrm{L}[y(x)] - \frac{2}{s^3}\int_0^1 y^{'(t)}dt = 0 \tag{4.16}$$

Using given the initial condition Eq. (4.15) becomes

$$L[y(x)] + \left(-\frac{s^7}{(s^8-1)} + \frac{s^5}{(s^8-1)} + \frac{2s^4}{(s^8-1)} + \frac{3s^3}{(s^8-1)} + \frac{4s^2}{(s^8-1)} + \frac{5s}{(s^8-1)} + \frac{6}{(s^8-1)} + \frac{8}{(s^8-1)(s-1)} - \frac{2}{s^3(s^8-1)}\right) - \frac{2}{s^3(s^8-1)}\int_0^1 y^{'(t)}dt = 0 \tag{4.17}$$

We define a nonlinear operator as:

$$N[\varphi(x,t;q)] = L[\varphi(x,t;q)] + \left(-\frac{s^7}{(s^8-1)} + \frac{s^5}{(s^8-1)} + \frac{2s^4}{(s^8-1)} + \frac{3s^3}{(s^8-1)} + \frac{4s^2}{(s^8-1)} + \frac{5s}{(s^8-1)} + \frac{6}{(s^8-1)} + \frac{8}{(s^8-1)(s-1)} - \frac{2}{s^3(s^8-1)}\right) - \frac{2}{s^3(s^8-1)}\int_0^1 y^{'(t)}dt = 0 \tag{4.18}$$

The *m*-th order deformation equation is:

$$y_m(x,t) = \chi_m y_{m-1}(x,t) + \hbar L^{-1}\left(R_m\left(\tilde{y}_{m-1}(x,t)\right)\right) \tag{4.19}$$

Where

$$R_m(\vec{y}_{m-1}) = L[y_{m-1}] + \left(-\frac{s^7}{(s^8-1)} + \frac{s^5}{(s^8-1)} + \frac{2s^4}{(s^8-1)} + \frac{3s^3}{(s^8-1)} + \frac{4s^2}{(s^8-1)} + \frac{5s}{(s^8-1)} + \frac{6}{(s^8-1)} + \frac{8}{(s^8-1)(s-1)} - \frac{2}{s^3(s^8-1)}\right) - \frac{2}{s^3(s^8-1)}\int_0^1 y^{'}(t)dt = 0 \tag{4.20}$$

Using the Mathematica package, we obtain the solution as:

$$y_0(x) = 0$$

$$y_1(x) = e^x hx + hx^2 + \frac{1}{2}h(\mathrm{Cos}[x] - 3\mathrm{Cosh}[x] - 2\mathrm{Sinh}[x] - 2\mathrm{Sin}[\frac{x}{\sqrt{2}}]\mathrm{Sinh}[\frac{x}{\sqrt{2}}])$$

$$y_2(x) = e^x h(1+h)x - \frac{ht^2\left(h+e\left(-4+h\left(e-2\left(6+\text{Cos}[1]-2\text{Sin}\left[\frac{1}{\sqrt{2}}\right]\text{Sinh}\left[\frac{1}{\sqrt{2}}\right]\right)\right)\right)\right)}{4e}$$
$$+ \frac{h(\text{Cosh}[x](h+e(-12+h(-20+e-2\text{Cos}[1]+4\text{Sin}[\frac{1}{\sqrt{2}}]\text{Sinh}[\frac{1}{\sqrt{2}}]))) - 8e(1+h)\text{Sinh}[x]}{-(h+e(-4+h(-12+e-2\text{Cos}[1]+4\text{Sin}[\frac{1}{\sqrt{2}}]\text{Sinh}[\frac{1}{\sqrt{2}}])))(\text{Cos}[x]-2\text{Sin}[\frac{x}{\sqrt{2}}]\text{Sinh}[\frac{x}{\sqrt{2}}]))} \Big/ 8e$$

(4.21)

At $h = -1$ the solution is given by

$$\varphi_n(x) = \sum_{\text{i}=0}^{n-1} y_\text{i}\,(x) = , \quad \text{n} = 1,2,\ldots$$

$$y(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \sum_{\text{i}=0}^{n-1} y_\text{i}\,(x) \;=\; e^x - xe^x, \qquad (4.22)$$

that converges to the exact solution, we notice that the result obtained by the present method is very superior (lower error combined with less number of iterations) to that obtained by HPM and VIM. From **table 2**, it can be deduced that, the error decreased monastically with the increment of the integer.

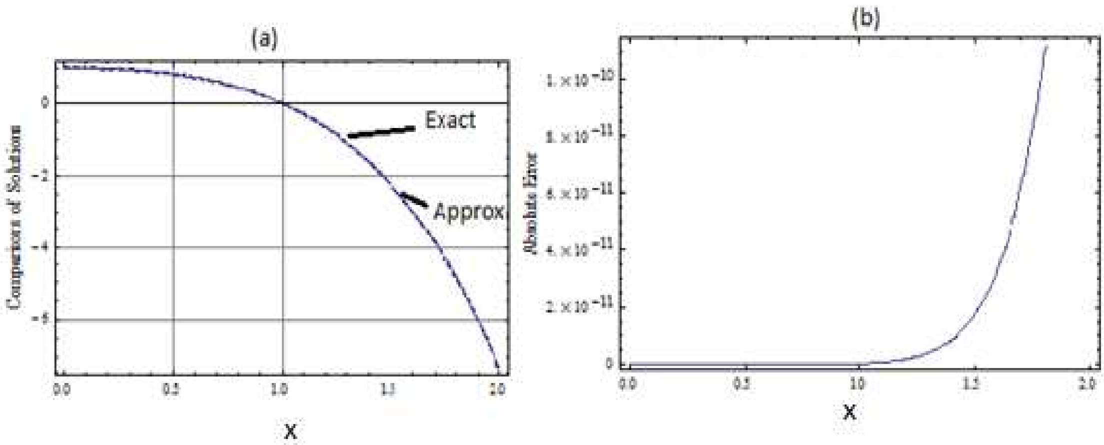

Figure 3: (a) The exact solution is compared with the approximate solutionwhen$h = -1$

(b) Absolute Error of 5th-order approximate solution with $h = -1$.

Table 2: Maximum error and EOC at $h = -1$

x	E_1	E_2	E_3
0.2	5.6437E−14	0.3111E−19	0.99999
0.4	5.7792E−11	0.3185E−16	0.99999
0.6	3.3326E−09	0.1837E−14	0.99999
0.8	5.9179E−08	0.3262E−13	0.99999
1.0	5.5115E−07	0.3038E−12	1.00000

According to the requirements of our test, $\frac{\|y_{i+1}\|_2}{\|y_i\|_2} < 1$ for all $i = 0,1,2,\cdots,n$.

From Table **2**, it can be deduced that, the error decreased monotically with the increment of the integer n.

Example 3

Let us test the homotopy analysis transform method on the following linear system of two Volterra's integro-differential equations [30-33]:

$$\begin{cases} u'(x) = 1 + x + x^2 - v(x) - \int_0^x \big(u(t) + v(t)\big)dt, \\ v'(x) = -1 - x + u(x) - \int_0^x \big(u(t) - v(t)\big)dt \end{cases} \tag{4.23}$$

with the initial conditions

$$u(0) = 1, \qquad v(0) = -1 \tag{4.24}$$

and with the exact solutions

$$u(x) = x + e^x, \qquad v(x) = x - e^x (4.25)$$

Applying the Laplace transform, of equation (4.23), we have

$$\begin{cases} \mathrm{L}[u'(x)] = \mathrm{L}[1 + x + x^2] - \mathrm{L}[v(x)] - \frac{1}{s}\int_0^x \big(u(t) + v(t)\big)dt, \\ \mathrm{L}[v'(x)] = \mathrm{L}[-1 - x] + \mathrm{L}[u(x)] - \frac{1}{s}\int_0^x \big(u(t) - v(t)\big)dt \end{cases} \tag{4.26}$$

$$\begin{cases} s\mathrm{L}[u(x)] - u(0)\left(-\frac{1}{s} - \frac{1}{s^2} - \frac{2}{s^3}\right) + \mathrm{L}[v(x)] + \frac{1}{s}\int_0^x \big(u(t) + v(t)\big)dt = 0, \\ s\mathrm{L}[v(x)] - v(0) - \left(-\frac{1}{s} - \frac{1}{s^2}\right) - \mathrm{L}[u(x)] + \frac{1}{s}\int_0^x \big(u(t) - v(t)\big)dt = 0 \end{cases} \tag{4.27}$$

Using given the initial condition Eq. (4.24) becomes

$$\begin{cases} \mathrm{L}[u(x)] + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{s^4}\right) + \frac{1}{s}\mathrm{L}[v(x)] + \frac{1}{s^2}\int_0^x \big(u(t) + v(t)\big)dt = 0, \\ \mathrm{L}[v(x)] + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right) - \frac{1}{s}\mathrm{L}[u(x)] + \frac{1}{s^2}\int_0^x \big(u(t) - v(t)\big)dt = 0 \end{cases} \tag{4.28}$$

We define a nonlinear operator as:

$$\begin{cases} N[\varphi_1(x,t;q), \varphi_2(x,t;q)] = \mathrm{L}[\varphi_1(x,t;q)] + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{s^4}\right) + \frac{1}{s}\mathrm{L}[\varphi_2(x,t;q)] \\ \quad + \frac{1}{s^2}\int_0^x (\varphi_1(x,t;q) + \varphi_2(x,t;q))\, dt = 0, \\ \quad N[\varphi_1(x,t;q), \varphi_2(x,t;q)] = \mathrm{L}[\varphi_2(x,t;q)] + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right) - \frac{1}{s}\mathrm{L}[\varphi_1(x,t;q)] \\ \quad + \frac{1}{s^2}\int_0^x (\varphi_1(x,t;q) - \varphi_2(x,t;q))\, dt = 0, \end{cases} \tag{4.29}$$

The m-th order deformation equation is:

$$\begin{cases} u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar \mathrm{L}^{-1}\left(R_{1m}\big(\tilde{u}_{m-1}(x,t)\big)\right) \\ v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar \mathrm{L}^{-1}\left(R_{2m}\big(\tilde{v}_{m-1}(x,t)\big)\right) \end{cases} \tag{4.30}$$

Where

$$\begin{cases} R_{1m}(\vec{u}_{m-1}) = \mathrm{L}[u_{m-1}] + \left(-\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{s^4}\right) + \frac{1}{s}\mathrm{L}[v_{m-1}] + \frac{1}{s^2}\int_0^x \big(u_{m-1}(t) + v_{m-1}(t)\big)\, dt, \\ R_{2m}(\vec{v}_{m-1}) = \mathrm{L}[v_{m-1}] + \left(\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\right) - \frac{1}{s}\mathrm{L}[u_{m-1}] + \frac{1}{s^2}\int_0^x \big(u_{m-1}(t) - v_{m-1}(t)\big), \end{cases} \tag{4.31}$$

Using the Mathematica package, we obtain the solution as

$$u_0(x) = e^x,$$

$$v_0(x) = -e^x,$$

$$u_1(x) = -2hx - \frac{hx^2}{2} - \frac{hx^3}{3},$$

$$v_1(t) = \frac{3hx^2}{2},$$

$$u_2(x) = -2h(1+h)x - \frac{1}{2}h(1+h)x^2 - \frac{1}{6}h(2+h)t^3 + \frac{h^2x^4}{12} - \frac{h^2x^5}{60},$$

$$v_2(x) = \frac{1}{2}h(3+5h)x^2 - \frac{h^2x^3}{6} - \frac{h^2x^4}{12} - \frac{h^2x^5}{60}, \tag{4.32}$$

At $h = -1$ the solution is given by:

$$\varphi_n(x) = \sum_{i=0}^{n-1} u_i(x) = , \quad n = 1,2,\ldots$$

$$u(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} u_i(x) = x + e^x,$$

$$\varphi_n(x) = \sum_{i=0}^{n-1} v_i(x) = , \quad n = 1,2,\ldots$$

$$v(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} v_i(x) = x - e^x, \tag{4.33}$$

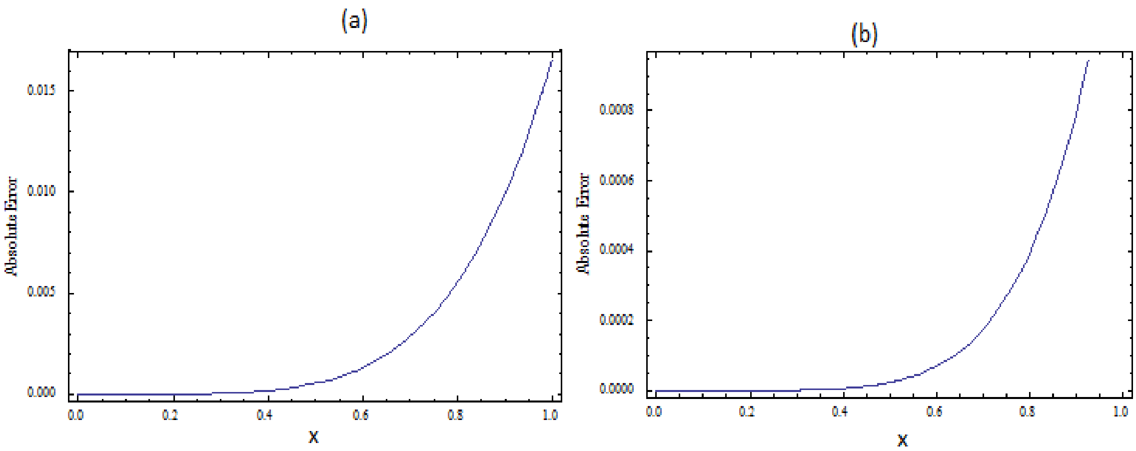

Figure 4: (a) and **(b)** Absolute Error of 5th-order approximate solution $u(x)$and $v(x)$ with $h = -1$.

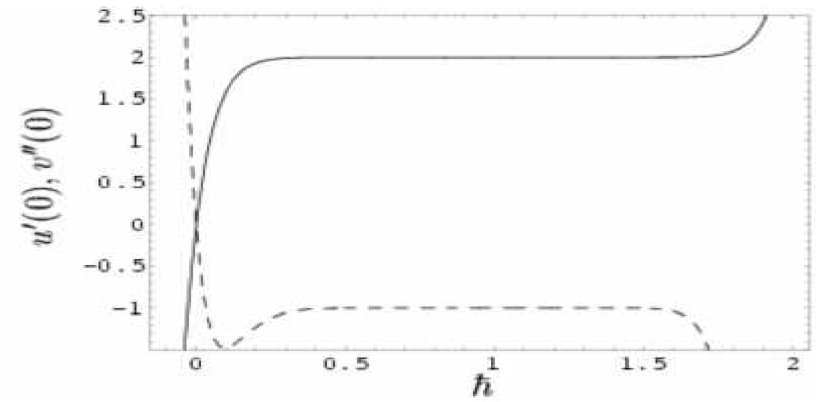

Fig. 5 $\hbar$-curves; solid line: 15th-order approximation of $u'(0)$; dashed line: 15th-order approximation of $v''(0)$.

that converges to the exact solution, we notice that the result obtained by the present method is very superior (lower error combined with less number of iterations) to that obtained by HPM and VIM.

5 Conclusions

In this paper, we presented the application of the homotopy analysis transform method (HATM) for solving a special form of nonlinear integro-differential equation. The sufficient condition for the convergence of the method is illustrated and then verified for three examples. As we can see in figures (1-5), HATM solutions have a good agreement with the numerical results provided that appropriate values for the convergence control parameter h are chosen. The ability of the HATM is mainly due to the fact that the method provides a way to ensure the convergence of series solution. The solution obtained with the help of HATM is more general as compared to HPTM,ADM and VIM solution. We can easily recover all results of HPTM, ADM and VIM by assuming$h = -1$. It is also demonstrated that the Adomian decomposition method and the homotopy perturbation method are special cases of the HATM. The HATM is clearly a very efficient and powerful technique for finding the numerical solutions of the proposed equation.

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