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On Quasi Statistical Convergence of Double Sequences

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Abstract

In this paper we recall the definition of statistical convergence for single and double sequences. Introducing the notion of the quasi statistical convergence on double sequence we compare it with statistical convergence and give a characterization of quasi statistical convergence on double sequence. Furthermore we establish a decomposition theorem for quasi statistical convergence and some results are obtained relating to strong quasi summability for double sequences.

Keywords: *Natural Density, Statistical and Quasi-statistical convergence of double sequence, Decomposition theorem, Strong Quasi Summability.*

1 Introduction

The concept of statistical convergence was introduced by Fast [3] and Steinhaus [10] and later on re-introduced by Schoenberg [9] independently based on the notion of asymptotic density of the subset of natural numbers. Properties of statistical convergence sequences were studied in Conor [2], Salat [8], Fridy ([4],[5]), Fridy and Miller [14] gave a characterization of statistical convergence of bounded sequence using a family of metric summability method. This idea was first introduced by Buck [1] and has been further studied by Freedman [6], Conor [2], Salat [8], Mursaleen [7], Tripathy [12] and many others ([13],[14],[15],[16]). Over the years and under different aspects statistical

convergence has been discussed in the theory of Fourier analysis, Egrodic theory and Number theory.
Sakaoglu and Yurdakadim [11] introduce the concept of quasi-statistical convergence for single sequences. Motivated by their definitions, we introduce the concept of quasi statistical convergence for double sequences.
Our aim in this paper is to establish some results relating to statistical convergence and quasi statistical convergence for double sequences. By introducing the concept of strong quasi summability, we study the inclusion theorem between quasi-statistical convergence and strong quasi summability for double sequences.

2 Definitions and Notations

Firstly we recall the following definitions from [4],[7].
Note that a subset E of the set $\mathbf{N}$ of natural numbers is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_n \frac{1}{n} \mid \{k \leq n : k \in E\} \mid \text{exist},$$

where the vertical bars denotes the cardinality of the enclosed set.
A sequence $x = \{x_k\}$ is said to be statistically convergent to the number L if for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} \mid \{k \leq n :\mid x_k - L \mid \geq \epsilon\} \mid = 0$$

and is denoted by $st - \lim x = L$ or $x_k \to L(st)$.
Let $K \subseteq \mathbf{N}\times\mathbf{N}$ be a two-dimensional set of positive integers and let $K(m,n)$ be the numbers of $(i,j) \in K$ such that $i \leq m$; $j \leq n$. Then the two-dimensional analogue of natural density can be defined as follows:
The lower asymptotic density of a set $K \subseteq \mathbf{N} \times \mathbf{N}$ is defined as
$\underline{\delta_2}(K) = \liminf_{m,n} \frac{K(m,n)}{mn}$.
In this case if the sequence $\{\frac{K(m,n)}{mn}\}$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as
$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K)$.
A double sequence $x = \{x_{ij}\}$ is said to be convergent in Pringshem's sense if for every $\epsilon > 0$ there exist $N \in \mathbf{N}$ such that $|x_{ij} - L| < \epsilon$ for $i, j \geq N$. The number L is called the Pringshem's limit of x.
Here by the convergence of double sequence, we mean the convergence in Pringsheim's sense.
A real double sequence $x = \{x_{ij}\}$ is said to be statistically convergent to the number L if for each $\epsilon > 0$ the set $K_\epsilon = \{(i,j), i \leq m \; and \; j \leq n :\mid x_{ij} - L \mid \geq \epsilon\}$ has a double natural density zero. We denote this as $st_2 - \lim_{ij} x_{ij} = L$.
A double sequence x is said to be bounded if there exists a positive number

M such that $|x_{ij}| < M$ for all i,j.
If a real double sequence $x = \{x_{ij}\}$ be such that $\{x_{ij}\}$ satisfies the property T for all i, j except a set of natural density zero, then we say that $x = \{x_{i,j}\}$ satisfies T for a.a. i, j.
a.a. i, j is the abbreviated form of "almost all i, j".
Note that if x is statistically convergent to the number L , then L is determined uniquely.
Also if x is bounded convergent double sequence then it is also statistical convergent to the same number. If x is unbounded convergent double sequence, then x is convergent statistically.

Remark 1: *If x is statistically convergent, then x need not be convergent and it is not necessarily bounded.*

Example 1: Let $x = \{x_{ij}\}$ be defined as

$$x_{ij} = \begin{cases} j, i = 1, \text{ for all } j \in \mathbf{N} \\ 0, \text{otherwise;} \end{cases}$$

Then x is statistically convergent to 0 because

$$\lim_{m,n} \frac{1}{mn}|\{(i, j), i \leq m, j \leq n :| x_{ij} - 0 |\geq \epsilon\}| \leq \lim_{m,n} \frac{n}{mn} = 0$$

Example 2: Let $x = \{x_{ij}\}$ be defined as

$$x_{ij} = \begin{cases} i, \text{ if } i \text{ is square, for all } j \in \mathbf{N} \\ 3, \text{otherwise;} \end{cases}$$

Then x is neither convergent nor bounded but statistically convergent to 3.

3 Main Results

We assume that $c = (c_{mn})$ is a double sequence of positive real numbers such that

$$\lim_{m,n} c_{mn} = +\infty \text{ and } \limsup_{m,n} \frac{c_{mn}}{mn} < \infty$$

We define the quasi-density of $E \subset \mathbf{N} \times \mathbf{N}$ corresponding to the sequence $c = (c_{mn})$ by

$$\delta_c(E) = \lim_{m,n} \frac{1}{c_{mn}} \mid \{(i, j), i \leq m, j \leq n : i, j \in E\} \mid \text{, if it exist.}$$

The sequence $x = (x_{ij})$ is said to be quasi-statistically convergent to L if for each $\epsilon > 0$ the set $E_\epsilon = \{(i,j), i \leq m, j \leq n :\mid x_{ij} - L \mid \geq \epsilon\}$ has quasi-density zero. In this case we write

$$st_2(q) - \lim x = L \text{ or } x_{ij} \to L(st_2(q)).$$

The next result establishes the relationship between quasi-statistical convergence and statistical convergence of double sequence.

Lemma 3.1 *Let $c = \{c_{mn}\}$ be the double sequence of positive real numbers such that $\lim_{mn} c_{mn} = \infty$ and $\limsup_{mn} \frac{c_{mn}}{mn} < \infty$. If $x = \{x_{ij}\}$ is quasi-statistically convergent to L then it is statistically convergent to L also.*

Proof: Let

$$st_2 - \lim x = L \text{ and } H = \sup_{mn} \tfrac{c_{mn}}{mn}.$$

Then

$$\frac{1}{mn} \mid \{(i,j) \in N \times N, i \leq m, j \leq n :\mid x_{ij} - L \mid \geq \epsilon\} \mid$$

$$\leq \frac{H}{c_{mn}} \mid \{(i,j) \in N \times N, i \leq m, j \leq n :\mid x_{ij} - L \mid \geq \epsilon\} \mid .$$

Hence the lemma follows.

We give an example in order to show that the converse of the above lemma does not hold good.

Example 3: Let $c = \{c_{mn}\}$ be a double sequence of real numbers such that $\lim_{mn} c_{mn} = \infty$ and $\lim_{mn} \frac{\sqrt{mn}}{c_{mn}} = \infty$.

We can select a subsequence $\{c_{m_p n_q}\}$ of c such that $c_{m_p n_q} > 1$ for each $p \in \mathbf{N}, q \in \mathbf{N}$.
Let us consider the double sequence $x = \{x_{ij}\}$ defined by

$$x_{ij} = \begin{cases} c_{ij}, \text{ if } i,j \text{ are square and } c_{ij} \in \{c_{m_p n_q} : p,q \in \mathbf{N}\} \\ 3, \text{ if } i,j \text{ are square and } c_{ij} \text{ are not in } \{c_{m_p n_q} : p,q \in \mathbf{N}\} \\ 0, \text{otherwise.} \end{cases}$$

Clearly x is statistically convergent to 0.
We now show that x is not quasi-statistically convergent to 0.
Choose $\epsilon = 1$.

Then we have

$\frac{1}{c_{mn}} \mid \{(i,j) \in \mathbf{N} \times \mathbf{N}, i \leq m, j \leq n : \mid x_{ij} - 0 \mid \geq 1\} \mid$

$= \frac{1}{c_{mn}} \mid \sqrt{mn} \mid$

$= \frac{1}{c_{mn}} (\sqrt{mn} - \lambda_{mn}), ...(1),$

where $0 \leq \lambda_{mn} \leq 1$ for each $m, n \in \mathbf{N}$.
Taking limit as $m, n \to \infty$ in both sides of (1) we see that x is not quasi-statistically convergent to 0.

The next result establishes the relationship between statistical convergence and quasi-statistical convergence of double sequence.

Lemma 3.2 *Let $c = \{c_{mn}\}$ be the double sequence of positive real numbers such that $\lim_{mn} c_{mn} = \infty$, $\limsup_{mn} \frac{c_{mn}}{mn} < \infty$ and $\mu = \inf \frac{c_{mn}}{mn} > 0$. If $x = \{x_{ij}\}$ is statistically convergent to L then it is quasi-statistically convergent to L.*

Proof: Since $x = \{x_{ij}\}$ is statistically convergent to L, therefore we can write for every $\epsilon > 0$, the set $\{(i,j) : i \leq m, j \leq n : \mid x_{ij} - L \mid \geq \epsilon\}$ has natural density 0,
i.e,

$$\lim_{mn} \frac{1}{mn} \{(i,j) : i \leq m, j \leq n : \mid x_{ij} - L \mid \geq \epsilon\} = 0.$$

Now

$\frac{1}{mn} \mid \{(i,j) \in \mathbf{N} \times \mathbf{N} : i \leq m, j \leq n ; \mid x_{ij} - L \mid \geq \epsilon\} \mid$

$\geq \frac{\mu}{c_{mn}} \mid \{(i,j) \in \mathbf{N} \times \mathbf{N} : i \leq m, j \leq n ; \mid x_{ij} - L \mid \geq \epsilon\} \mid$

Taking limit on both sides we see that $x = \{x_{mn}\}$ is quasi-statistically convergent to L.

Hence the lemma follows.

We now present our results as follows:

Theorem 3.3 *Let $c = \{c_{mn}\}$ be a sequence of positive real numbers such that* $\lim_{mn} c_{mn} = \infty$, $\limsup_{mn} \frac{c_{mn}}{mn} < \infty$ *and* $\mu = \inf \frac{c_{mn}}{mn} > 0$. *Then $x = \{x_{ij}\}$ is statistically convergent to L if and only if x is quasi-statistically convergent to L.*

Proof: The proof of the theorem follows from the lemma 3.1 and lemma 3.2.

Theorem 3.4 *Let $c = \{c_{mn}\}$ be a sequence of positive real numbers such that* $\lim_{mn} c_{mn} = \infty$ *and* $\limsup_{mn} \frac{c_{mn}}{mn} < \infty$. *A real double sequence $x = \{x_{ij}\}$ is quasi-statistically convergent to a number L if and only if there exist a subset $A = \{(i,j) \subseteq \boldsymbol{N} \times \boldsymbol{N}$ and $i,j = 1,2,...\}$ such that $\delta_c(A) = 1$ and $\{x_{ij}\}$ is convergent to L.*

Proof: We first suppose that x is quasi statistically convergent to L.
Set $A_p = \{(i,j) \in \mathbf{N} \times \mathbf{N} :\mid x_{ij} - L \mid \geq \frac{1}{p}\}$ and

$B_p = \{(i,j) \in \mathbf{N} \times \mathbf{N} :\mid x_{ij} - L \mid < \frac{1}{p}\}$, where $p = 1,2,3...$
Then $\delta_c(A_p) = 0$ and also we see that
(1) $B_1 \supset B_2 \supset ...... \supset B_r \supset B_{r+1} \supset ....$
(2) $\delta_c(B_p) = 1$, where $p = 1,2,3......$
We now show that for $(i,j) \in B_p$, $\{x_{ij}\}$ converges to L.
Let $\{x_{ij}\}$ be not convergent to L. Then there exist an $\epsilon > 0$ such that $\mid x_{ij} - L \mid \geq \epsilon$ for infinitely many terms.
Let $B_\epsilon = \{(i,j) \in \mathbf{N} \times \mathbf{N} :\mid x_{ij} - L \mid < \epsilon\}$ and $\epsilon > \frac{1}{p}$ for $p = 1,2,3.....$
Then we have
(3) $\delta_c(B_\epsilon) = 0$.
By (1) we can write $B_p \subset B_\epsilon$.
Hence $\delta_c(B_p) = 0$, which contradict (2).
Therefore $\{x_{ij}\}$ is convergent to L.
Conversely, let us assume that there exist a subset
$A = \{(i,j) \subset \mathbf{N} \times \mathbf{N} : i,j = 1,2.3,....\}$ such that $\delta_c(A) = 1$ and $\lim_{ij} x_{ij} = L$ i.e.,
there exist $N \in \mathbf{N}$ such that for $\epsilon > 0$, $\mid x_{ij} - L \mid < \epsilon$ for all $i,j \geq N$.

Now

$$A_\epsilon = \{(i,j) :\mid x_{ij} - L \mid \geq \epsilon\}$$

$$\subseteq \mathbf{N} \times \mathbf{N} - \{(i_{N+1}, j_{N+1}), (i_{N+2}, j_{N+2}), ...\}$$

Hence $\delta_c(A_\epsilon) \leq 1 - 1 = 0$.

Therefore x is quasi-statistically convergent to L.

Remark 2: *If* $st_2(q) - \lim x_{ij} = L$ *then there exist a sequence* $\{y_{ij}\}$ *such that* $\lim_{i,j\to\infty} y_{ij} = L$ *and* $\delta_c\{(i,j) : x_{ij} = y_{ij}\} = 1$ *i,e.* $x_{ij} = y_{ij}$ *for almost all* $i, j \in \boldsymbol{N}$.

3.1 Decomposition Theorem for Quasi-Statistical Convergence

Theorem 3.5 *Let* $c = \{c_{mn}\}$ *be a sequence of positive real numbers such that* $\lim_{mn} c_{mn} = \infty$ *and* $\limsup_{mn} \frac{c_{mn}}{mn} < \infty$. *If* $x = \{x_{ij}\}$ *is a quasi-statistically convergent to L, then there is a sequence y which converges to L and quasi-statistically null sequence z such that* $x = y + z$.

Proof: Since x is quasi-statistically convergent sequence, therefore there exists an increasing sequence of positive integers (K_p) such that $K_0 = 0$ and
$\frac{1}{c_{mn}}|\{(i,j) : i \le m, j \le n, |x_{ij} - L| \ge \frac{1}{p}\}| < \frac{1}{p}$, $m, n > K_p$, $p = 1, 2, 3, ...$
We define $y = \{y_{ij}\}$ and $z = \{z_{ij}\}$ as follows:
$z_{ij} = 0$ and $y_{ij} = x_{ij}$ if $K_0 < i, j < K_1$,
$z_{ij} = 0$ and $y_{ij} = x_{ij}$ if $|x_{ij} - L| < \frac{1}{p}$, $K_p < i, j \le K_{p+1}$, $p \ge 1$
$z_{ij} = x_{ij} - L$ and $y_{ij} = L$ if $|x_{ij} - L| \ge \frac{1}{p}$, $K_p < i, j < K_{p+1}$, $p \ge 1$.
Then we see that the representation $x = y + z$ is valid.
Firstly we show that y converges to L.
Let us choose $\epsilon > 0$ and p be such that $\epsilon > \frac{1}{p}$.
If $|x_{ij} - L| \ge \frac{1}{p}$ when $i \ge K_p$, $j \ge K_p$, then $|y_{ij} - L| = |L - L| = 0$.
If $|x_{ij} - L| < \frac{1}{p}$, $i, j \ge K_p$, then $|y_{ij} - L| = |x_{ij} - L| < \frac{1}{p} < \epsilon$.
Hence $lim_{i,j\to\infty} y_{ij} = L$.
Secondly to show z is quasi-statistically null double sequence, it is enough to show that $\lim_{m,n\to\infty} \frac{1}{c_{mn}}|\{(i,j) : i \le m, j \le n, z_{ij} \ne 0\}| = 0$.
Clearly for $\epsilon > 0$,
$\{(i,j) : i \le m, j \le n, |z_{ij} - 0| \ge \epsilon\} \subseteq \{(i,j) : i \le m, j \le n, z_{ij} \ne 0\}$.
Therefore
$|\{(i,j) : i \le m, j \le n, |z_{ij} - 0| \ge \epsilon\}| \le |\{(i,j) : i \le m, j \le n, z_{ij} \ne 0\}|$.
Let $\delta > 0$ and $p \in \mathbf{N}$ such that $\frac{1}{p} < \delta$.
We want to show that $\frac{1}{mn}|\{(i,j) : i \le m, j \le n, z_{ij} \ne 0\}| < \delta$ for all $m > K_p$, $n > K_p$.
Here we see that
$z_{ij} \ne 0$ if and only if $|x_{ij} - L| \ge \frac{1}{p}$, $K_p < i, j \le K_{p+1}$,
If $K_p < i, j \le K_{p+1}$, then
$\{(i,j) : i \le m, j \le n, z_{ij} \ne 0\} = \{(i,j) : i \le m, j \le n, |x_{ij} - L| \ge \frac{1}{p}\}$.
If $K_q < i \le K_{q+1}$, $K_q < j \le K_{q+1}$ and $q > p$, then

$\frac{1}{c_{mn}}|\{(i,j) : i \leq m, j \leq n, z_{ij} \neq 0\}$
$\leq \frac{1}{c_{mn}}|\{(i,j), i \leq m, j \leq n, |x_{ij} - L| \geq \frac{1}{q}\}|$
$< \frac{1}{q}$
$< \frac{1}{p}$
$< \delta$
Taking limit we see that
$\lim_{m,n\to\infty} \frac{1}{c_{mn}}|\{(i,j) : i \leq m, j \leq n, z_{ij} \neq 0\}| = 0.$
Thus $z = \{z_{ij}\}$ is quasi-statistically null sequence.
This completes the proof of the theorem.

Remark 3: *From the above decomposition theorem we can conclude that if the double sequence x is quasi statistically convergent to L, then there is a subsequence $y = \{y_{ij}\}$ of x such that y converges to L.*

4 Strong Quasi-Summability

Introducing the notion of the quasi-statistical convergence our object is to establish the relationship between quasi-statistical convergence and strong quasi summability of double sequence.

The following definitions come from [2] and [6].

Definition 4.1 *A double sequence $x = \{x_{ij}\}$ is said to be Cesaro summable to L iff $\lim_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = L$.*

Definition 4.2 *A double sequence $x = \{x_{ij}\}$ is said to be strong Cesaro summable to L iff $\lim_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} |x_{ij} - L| = 0$.*

Similarly we can define strong quasi summability for double sequences as follows:

Definition 4.3 *A double sequence $x = \{x_{ij}\}$ is said to be strongly quasi-summable to L iff $\lim_{m,n} \frac{1}{c_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} |x_{ij} - L| = 0$.*

Theorem 4.4 *Let $c = \{c_{mn}\}$ be the sequence of positive real number such that $\lim_{m,n} c_{mn} = +\infty$ and $\limsup_{mn} \frac{c_{mn}}{mn} < \infty$. If $x = \{x_{ij}\}$ is strongly quasi summable to L then it is quasi-statistically convergent to L.*

Proof: Since $x = \{x_{ij}\}$ is strongly quasi summable to L, therefore

$$\lim_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} |x_{ij} - L| = 0............(1).$$

Now

$\frac{1}{c_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}-L|$

$\geq \frac{1}{c_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}-L|$, when $|x_{ij}-L|\geq\epsilon$

$\geq \frac{\epsilon}{c_{mn}}\{i\leq m, j\leq n : |x_{ij}-L|\geq\epsilon\}$.

Using (1) we see that

$$\delta_c\{(i,j)\in\mathbf{N}\times\mathbf{N} : i\leq m, j\leq n, |x_{ij}-L|\geq\epsilon\}=0.$$

Therefore x is quasi statistically convergent to L.

Theorem 4.5 *Let $c=\{c_{mn}\}$ be the double sequence of positive real numbers such that $\lim_{mn} c_{mn}=\infty$ and $\limsup_{mn}\frac{c_{mn}}{mn}<\infty$ and $\mu=\inf\frac{c_{mn}}{mn}>0$. Let the double sequence $x=\{x_{mn}\}$ be strongly cesero summable to L. Then x is quasi statistically convergent to L.*

Proof: Let $K_\epsilon=\{(i,j) : i\leq m, j\leq n, |x_{ij}-L|\geq\epsilon\}$.
Since x is strongly Cesaro summable to L, we have
$\lim\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}-L|=0$.......(1)

Now

$$\begin{aligned}\lim\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}-L| &= \frac{1}{mn}\{\sum_{(i,j)\in K_\epsilon}|x_{ij}-L|+\sum_{(i,j)\notin K_\epsilon}|x_{ij}-L|\}\\ &\geq \frac{1}{mn}|\{(i,j) : i\leq m, j\leq n, |x_{ij}-L|\geq\epsilon\}|\\ &\geq \frac{\mu}{c_{mn}}\{(i,j) : i\leq m, j\leq n; |x_{ij}-L|\geq\epsilon\}\end{aligned}$$

Using (1), we see that

$$\delta_c\{(i,j)\in\mathbf{N}\times\mathbf{N} : i\leq m, j\leq n, |x_{ij}-L|\geq\epsilon\}=0 \text{ is zero.}$$

Therefore x is quasi statistically convergent to L.

Remark 4: *From above theorems we see that if x is either strong quasi summable or strong Cesaro summable sequence then it is quasi statistically convergent to L.*
But if x is quasi statistically convergent sequence then it is neither strongly quasi summable nor strongly Cesaro summable sequence. For example

Example 4: Let $c=\{c_{mn}\}$ be the sequence of positive real numbers such that $\lim c_{mn}=+\infty$ and $\limsup\frac{c_{mn}}{mn}<\infty$, $d=\inf\frac{c_{mn}}{mn}$.

Let $x = (x_{ij})$ be defined by

$$x_{ij} = \begin{cases} j, \text{ if } i = 1 \text{ , for all } j \\ i, \text{ if } j = 1 \text{ , for all } i \\ 0, \text{otherwise;} \end{cases}$$

Then $\lim_{i,j} x_{i,j} = 0$ but

$$\lim_{m,n} \frac{1}{c_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \lim_{m,n} \frac{1}{mnc_{mn}} (\frac{m^2+n^2+m+n-2}{2})$$

which fails to tend to a finite limit as m, n tends to ∞.
Therefore x is not strongly quasi summable and also x is not strongly Cesaro summable sequence.
But x is quasi statistically convergent to 0 because

$\lim_{m,n} \frac{1}{c_{mn}} |\{(i,j) : |x_{ij} - 0| \geq \epsilon\}|$

$\leq \lim_{m,n} \frac{1}{d(mn)} (m + n - 1)$

$= 0$
Schoenberg [9] established the following theorem for single sequence.

Theorem 4.6 *[9] If $x = \{x_{ij}\}$ is a bounded statistically convergent sequence then it is Cesaro summable sequence.*

Combining this result with the lemma 3.1 we have the following corollary:

Corollary 4.7 *[11] If $x = \{x_{ij}\}$ is a bounded sequence and quasi statistically convergent to L then it is strongly Cesaro summable to L.*

Keeping this result in mind we now establish the following result for double sequence:

Theorem 4.8 *Let $x = \{x_{ij}\}$ be a bounded sequence and quasi-statistically convergent to L. Let $c = \{c_{mn}\}$ be a sequence of positive real numbers such that* $\lim c_{mn} = +\infty$ *and* $\limsup \frac{c_{mn}}{mn} < \infty$ *and* $\liminf \frac{c_{mn}}{mn} > 0$. *Then x is strongly quasi-summable to L.*

Proof: Since x is bounded, there exists a positive real numbers M such that $|x_{ij} - L| \leq M$ for all $i, j \in \mathbf{N}$.
Since x is quasi statistically convergent to L , therefore the double natural density $\delta_c\{(i,j) : i \leq m, j \leq n, |x_{ij} - L| > \epsilon\} = 0$.......(1).

Now

$\frac{1}{c_{mn}}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_{ij}-L|$

$< \epsilon\frac{mn}{c_{mn}} + M\frac{1}{c_{mn}}|\{(i,j) : i \le m, j \le n, |x_{ij}-L| > \epsilon\}|.$

Using (1) and taking limit on both sides, we see that x is strongly quasi-summable to L.

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