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Application of Mixed Quadrature Rules in the Adaptive Quadrature Routine

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Abstract

A model is set up which embodies the basic features of Adaptive quadrature routines involving mixed rules. Not before mixed quadrature rules have been used for fixing termination criterion in adaptive quadrature routines. Three mixed quadrature rules are constructed for this purpose. The first is the convex combination of Lobatto 4-point rule and the Clenshaw-Curtis 5-point rule, the second is the linear combination of the first mixed quadrature rule and the Lobatto 5-point rule and the third is the linear combination of the second mixed quadrature rule and the Kronrod extension of Lobatto 4-point rule. Adaptive quadrature routines being recursive by nature, a termination criterion is formed taking into account the three mixed quadrature rules. The algorithm presented in this paper has been 'C' programmed and successfully tested on different integrals. The efficiency of the process is reflected in the table at the end.

Keywords: *Lobatto 4-point rule, Lobatto 5-point rule, Kronrod extension of Lobatto 4-point rule, Clenshaw-Curtis 5-point rule, mixed quadrature rule, Adaptive quadrature.*

1 Introduction

Given a real integrable function f , an interval [a, b] and a prescribed tolerance $\in$, it is desired to compute an approximation P to the integral $I = \int_a^b f(x)\,dx$ so that $|P - I| \leq \in$. This can be done following adaptive integration schemes [2, 3, 6, 7]. In adaptive integration, the points at which the integrand is evaluated are so chosen in a way that depends on the nature of the integrand. The basic principle of adaptive quadrature routines is discussed in the following manner.

A fundamental additive property of a definite integral is the basic for adaptive quadrature. If c is any point between a and b, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The idea is that if we can approximate each of the two integrals on the right to within a specified tolerance, then the sum gives us the desired result. If not, we can recursively apply the additive property to each of the intervals $[a,c]$ and $[c,b]$ Adaptive subdivision of course has geometrical appeal. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

In this paper we design an algorithm for numerical computation of integrals in the adaptive quadrature routines involving mixed rules. The literature of the mixed quadrature rule [4,5] involves construction of a symmetric quadrature rule of higher precision as a linear / convex combination of two other rules of equal lower precision.

As the Lobatto 4-point rule $\left(R_{L_4}(f)\right)$ [8] and Clenshaw-Curtis 5-point $\left(R_{CC_5}(f)\right)$ [5] rule are of same precision (i.e. precision 5) one can form a mixed quadrature rule $\left(R_{L_4CC_5}(f)\right)$ of precision 7 by taking the convex combination of these two rules. Similarly one can form a mixed quadrature rule $\left(R_{L_4CC_5L_5}(f)\right)$ of precision 9 by taking the linear combination of the Lobatto 5-point rule $\left(R_{L_5}(f)\right)$ and the mixed quadrature rule $\left(R_{L_4CC_5}(f)\right)$ each of which is of precision 7.. As the mixed quadrature rule $\left(R_{L_4CC_5L_5}(f)\right)$ and the kronrod extension of the Lobatto 4-point rule $\left(R_{KEL_4}(f)\right)$ [7] are of same precision (i.e. precision 9) one can form a mixed quadrature rule ($R_{L_4CC_5L_5KEL_4}(f)$) of precision 11 by taking the linear combination of these two rules.

To prepare an algorithm for adaptive quadrature routines in evaluating an integral $I=\int_a^b f(x)\,dx$, we use the following three mixed quadrature rules.

(i) $R_{L_4CC_5}(f)$ as (I_1)

(ii) $R_{L_4CC_5L_5}(f)$ as (I_2)

(iii) $R_{L_4CC_5L_5KEL_4}(f)$ as (I_3).

1.1 A Simple Adaptive Strategy

The input to this schemes is $a,b,\in,n,f$. The output is $P\approx\int_a^b f(x)\,dx$ with the error hopefully less than $\in;n$ is the number of intervals initially chosen. A simple adaptive strategy is outlined in the following four-step algorithm.

Step1: An approximation I_1 to $I=\int_a^b f(x)\,dx$ is computed.

Step 2: The interval is divided into pieces, $[a,c]$ and $[c,b]$ where $c=(a+b)/2$, and then $I_2\approx\int_a^c f(x)\,dx$ and $I_3\approx\int_c^b f(x)\,dx$ are computed.

Step 3: I_2+I_3 is compared with I_1, to estimate the error in I_2+I_3.

Step 4: If $|\text{estimated error}|\le\in/2$ (termination criterion), then I_2+I_3 is accepted as an approximation to $\int_a^b f(x)\,dx$. Otherwise the same procedure is applied to $[a,c]$ and $[c,b]$, allowing each piece a tolerance of $\in/2$.

Adaptive quadrature routines essentially consist of applying the rules $R_{L_4CC_5}(f)$, $R_{L_4CC_5L_5}(f)$ and $R_{L_4CC_5L_5KEL_4}(f)$ to each of the subintervals covering $[a,b]$ until the termination criterion is satisfied. If the termination criterion is not satisfied on one or more of the subintervals, then those subintervals must be further subdivided and the entire process repeated.

2 Construction of the Mixed Quadrature Rule of Precision Seven

We choose the Lobatto 4-point rule $\left(R_{L_4}(f)\right)$:

$$I(f)=\int_{\alpha-h}^{\alpha+h} f(x)\,dx \approx R_{L_4}(f)$$

$$=\frac{h}{6}\left[f(\alpha-h)+5f\left(\alpha-\frac{1}{\sqrt{5}}h\right)+5f\left(\alpha+\frac{1}{\sqrt{5}}h\right)+f(\alpha+h)\right] \tag{2.1}$$

and the Clenshaw-Curtis 5-point rule $\left(R_{CC_5}(f)\right)$:

$$I(f)=\int_{\alpha-h}^{\alpha+h} f(x)\,dx \approx R_{CC_5}(f)$$

$$=\frac{h}{15}\left[f(\alpha-h)+8f\left(\alpha-\frac{1}{\sqrt{2}}h\right)+12f(\alpha)+8f\left(\alpha+\frac{1}{\sqrt{2}}h\right)+f(\alpha+h)\right] \tag{2.2}$$

where $\alpha=\frac{1}{2}(a+b)$, $h=\frac{1}{2}(b-a)$

Each of the rules $R_{L_4}(f)$ and $R_{CC_5}(f)$ is of precision 5. Let $E_{L_4}(f)$ and $E_{CC_5}(f)$ denote the errors in approximating the integral $I(f)$ by the rules $R_{L_4}(f)$ and $R_{CC_5}(f)$ respectively. Using Maclaurin's expeansion of functions in Eqs. (2.1) and (2.2), we get

$$I(f)=R_{L_4}(f)+E_{L_4}(f) \tag{2.3}$$

$$I(f)=R_{CC_5}(f)+E_{CC_5}(f) \tag{2.4}$$

where

$$E_{L_4}(f)=\frac{-32}{7!\times 75}h^7 f^{(vi)}(\alpha)-\frac{128}{9!\times 125}h^9 f^{(viii)}(\alpha)$$
$$-\frac{3136}{11!\times 1875}h^{11} f^{(x)}(\alpha)-\frac{7296}{13!\times 3125}h^{13} f^{(xii)}(\alpha)-\cdots$$

$$E_{CC_5}(f)=\frac{2}{7!\times 15}h^7 f^{(vi)}(\alpha)+\frac{1}{9!\times 5}h^9 f^{(viii)}(\alpha)$$
$$+\frac{1}{11!\times 6}h^{11} f^{(x)}(\alpha)+\frac{1}{13!\times 20}h^{13} f^{(xii)}(\alpha)+\cdots$$

Now multiplying Eq (2.4) by $\frac{16}{5}$ and then adding it with Eq (2.3), we obtain

$$I(f)=\frac{1}{21}\left[5R_{L_4}(f)+16R_{CC_5}(f)\right]+\frac{1}{21}\left[5E_{L_4}(f)+16E_{CC_5}(f)\right]$$

or $$I(f)=R_{L_4CC_5}(f)+E_{L_4CC_5}(f) \tag{2.5}$$

where $$R_{L_4CC_5}(f)=\frac{1}{21}\left[5R_{L_4}(f)+16\,R_{CC_5}(f)\right] \tag{2.6}$$

or $$R_{L_4CC_5}(f)=\frac{h}{630}\left\{57\left[f(\alpha-h)+f(\alpha+h)\right]+256\left[f\left(\alpha-\frac{h}{\sqrt{2}}\right)+f\left(\alpha+\frac{h}{\sqrt{2}}\right)\right]\right.$$
$$\left.+125\left[f\left(\alpha-\frac{h}{\sqrt{5}}\right)+f\left(\alpha+\frac{h}{\sqrt{5}}\right)\right]+384f(\alpha)\right\} \tag{2.7}$$

This is the desired mixed quadrature rule of precision 7 for the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$E_{L_4CC_5}(f)=\frac{1}{21}\left[5E_{L_4}+16E_{CC_5}(f)\right] \tag{2.8}$$

or

$$E_{L_4CC_5}(f)=\frac{-16}{9!\times 175}h^9 f^{(viii)}(\alpha)-\frac{712}{11!\times 2625}h^{11}f^{(x)}(\alpha)-\frac{6796}{13!\times 13125}h^{13}f^{(xii)}(\alpha)-\cdots$$

or $$\left|E_{L_4CC_5}(f)\right|\cong\frac{16}{9!\times 175}h^9\left|f^{(viii)}(\alpha)\right| \tag{2.9}$$

The Rule $R_{L_4CC_5}(f)$ is called a mixed type rule of precision 7, as it is constructed from two different types of rules of the same precision.

3 Construction of the Mixed Quadrature Rule of Precision Nine

We choose the mixed rule $\left(R_{L_4CC_5}(f)\right)$:

$$I(f)=\int_{\alpha-h}^{\alpha+h}f(x)\,dx\approx R_{L_4CC_5}(f)=\frac{h}{630}\left[57\left\{f(\alpha-h)+f(\alpha+h)\right\}+\right.$$
$$+256\left\{f\left(\alpha-\frac{h}{\sqrt{2}}\right)+f\left(\alpha+\frac{h}{\sqrt{2}}\right)\right\}$$
$$\left.+125\left\{f\left(\alpha-\frac{h}{\sqrt{5}}\right)+f\left(\alpha+\frac{h}{\sqrt{5}}\right)\right\}+384f(\alpha)\right] \tag{3.1}$$

and the Lobatto 5-point rule $\left(R_{L_5}(f)\right)$:

$$I(f)=\int_{\alpha-h}^{\alpha+h} f(x)\,dx \approx R_{L_5}(f)=\frac{h}{90}\Big[9\left\{f(\alpha-h)+f(\alpha+h)\right\}$$

$$+49\left\{f\left(\alpha-\frac{\sqrt{3}}{\sqrt{7}}h\right)+f\left(\alpha+\frac{\sqrt{3}}{\sqrt{7}}h\right)\right\}+64\,f(\alpha)\Big] \quad (3.2)$$

Each of the rules $R_{L_4CC_5}(f)$ and $R_{L_5}(f)$ is of precision 7. Let $E_{L_4CC_5}(f)$ and $E_{L_5}(f)$ denote the errors is approximating the integral $I(f)$ by the rules $R_{L_4CC_5}(f)$ and $R_{L_5}(f)$ respectively. Using Maclaurin's expansion of functions in Eqs (3.1) and (3.2), we get

$$I(f)=R_{L_4CC_5}(f)+E_{L_4CC_5}(f) \quad (3.3)$$

and $$I(f)=R_{L_5}(f)+E_{L_5}(f) \quad (3.4)$$

where

$$E_{L_4CC_5}(f)=\frac{-16}{9!\times 175}h^9 f^{(viii)}(\alpha)-\frac{712}{11!\times 2625}h^{11} f^{(x)}(\alpha)-\frac{6796}{13!\times 13125}h^{13} f^{(xii)}(\alpha)-\cdots$$

$$E_{L_5}(f)=\frac{-32}{9!\times 245}h^9 f^{(viii)}(\alpha)-\frac{128}{11!\times 343}h^{11} f^{(x)}(\alpha)-\frac{8256}{13!\times 12005}h^{13} f^{(xii)}(\alpha)-\cdots$$

Now multiplying the equations (3.3) and (3.4) by $\frac{2}{7}$ and $-\frac{1}{5}$ respectively, and then adding the resulting equations we obtain,

$$I(f)=\frac{1}{3}\Big[10R_{L_4CC_5}(f)-7R_{L_5}(f)\Big]+\frac{1}{3}\Big[10E_{L_4CC_5}(f)-7E_{L_5}(f)\Big]$$

or $$I(f)=R_{L_4CC_5L_5}(f)+E_{L_4CC_5L_5}(f) \quad (3.5)$$

where $$R_{L_4CC_5L_5}(f)=\frac{1}{3}\Big[10R_{L_4CC_5}(f)-7R_{L_5}(f)\Big] \quad (3.6)$$

or $$R_{L_4CC_5L_5}(f)=\frac{h}{1890}\Big\{129\big[f(\alpha-h)+f(\alpha+h)\big]+2560\left[f\left(\alpha-\frac{h}{\sqrt{2}}\right)+f\left(\alpha+\frac{h}{\sqrt{2}}\right)\right]$$

$$-2401\left[f\left(\alpha-\frac{\sqrt{3}}{\sqrt{7}}h\right)+f\left(\alpha+\frac{\sqrt{3}}{\sqrt{7}}h\right)\right]$$

$$+1250\left[f\left(\alpha-\frac{h}{\sqrt{5}}\right)+f\left(\alpha+\frac{h}{\sqrt{5}}\right)\right]+704f(\alpha)\Bigg\} \tag{3.7}$$

This is the desired mixed quadrature rule of precision 9 for the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$E_{L_4CC_5L_5}(f)=\frac{1}{3}\left[10E_{L_4CC_5}(f)-7E_{L_5}(f)\right] \tag{3.8}$$

or $$E_{L_4CC_5L_5}(f)=-\frac{-368}{11!\times 11025}h^{11}f^{(x)}(\alpha)-\frac{46808}{13!\times 385875}h^{13}f^{(xii)}(\alpha)-\cdots$$

or $$\left|E_{L_4CC_5L_5}(f)\right|\cong\frac{368}{11!\times 11025}h^{11}\left|f^{(x)}(\alpha)\right| \tag{3.9}$$

The rule $R_{L_4CC_5L_5}(f)$ is called a mixed type rule of precision 9, as it is constructed from two different types of rules of the same precision.

4 Construction of the Mixed Quadrature Rule of Precision Eleven

We choose the mixed rule $\left(R_{L_4CC_5L_5}(f)\right)$:

$$I(f)=\int_{\alpha-h}^{\alpha+h}f(x)\,dx\approx R_{L_4CC_5L_5}(f)$$

$$=\frac{h}{1890}\Bigg\{129\left[f(\alpha-h)+f(\alpha+h)\right]+2560\left[f\left(\alpha-\frac{h}{\sqrt{2}}\right)+f\left(\alpha+\frac{h}{\sqrt{2}}\right)\right]$$

$$-2401\left[f\left(\alpha-\frac{\sqrt{3}}{\sqrt{7}}h\right)+f\left(\alpha+\frac{\sqrt{3}}{\sqrt{7}}h\right)\right]+1250\left[f\left(\alpha-\frac{h}{\sqrt{5}}\right)+f\left(\alpha+\frac{h}{\sqrt{5}}\right)\right]+704\ f(\alpha)\Bigg\}$$

(4.1)

And the kronrod extension of the Lobatto 4- point rule $\left(R_{KEL_4}(f)\right)$:

$$I(f)=\int_{\alpha-h}^{\alpha+h}f(x)\,dx\approx R_{KEL_4}(f)$$

$$= \frac{h}{1470}\left\{77\left[f(\alpha-h)+f(\alpha+h)\right] \; +432\left[f\left(\alpha-\frac{\sqrt{2}}{\sqrt{3}}h\right)+f\left(\alpha+\frac{\sqrt{2}}{\sqrt{3}}h\right)\right]\right.$$

$$\left. +625\left[f\left(\alpha-\frac{h}{\sqrt{5}}\right)+f\left(\alpha+\frac{h}{\sqrt{5}}\right)\right]+672\; f(\alpha)\right\} \tag{4.2}$$

Each of the rules $R_{L_4CC_5L_5}(f)$ and $R_{KEL_4}(f)$ is of precision 9. Let $E_{L_4CC_5L_5}(f)$ and $E_{KEL_4}(f)$ denote the errors in approximating the integral $I(f)$ by the rules $R_{L_4CC_5L_5}(f)$ and $R_{KEL_4}(f)$ respectively. Using Maclaurin's expansion of functions in Eqs (4.1) and (4.2), we get

$$I(f)=R_{L_4CC_5L_5}(f)+E_{L_4CC_5L_5}(f) \tag{4.3}$$

$$I(f)=R_{KEL_4}(f)+E_{KEL_4}(f) \tag{4.4}$$

where

$$E_{L_4CC_5L_5}(f)=\frac{-368}{11!\times 11025}h^{11}f^{(x)}(\alpha)-\frac{46808}{13!\times 385875}h^{13}f^{(xii)}(\alpha)-\cdots$$

$$E_{KEL_4}(f)=-\frac{-32}{11!\times 4725}h^{11}f^{(x)}(\alpha)-\frac{2368}{13!\times 70875}h^{13}f^{(xii)}(\alpha)-\cdots$$

Now multiplying the equations (4.3) and (4.4) by $\frac{-1}{3}$ and $\frac{23}{14}$ respectively and then adding the resulting equations we obtain,

$$I(f)=\frac{1}{55}\left[69R_{KEL_4}(f)-14R_{L_4CC_5L_5}(f)\right]$$

$$+\frac{1}{55}\left[69E_{KEL_4}(f)-14E_{L_4CC_5L_5}(f)\right]$$

or $$I(f)=R_{L_4CC_5L_5KEL_4}(f)+E_{L_4CC_5L_5KEL_4}(f) \tag{4.5}$$

Where $$R_{L_4CC_5L_5KEL_4}(f)=\frac{1}{55}\left[69R_{KEL_4}(f)-14R_{L_4CC_5L_5}(f)\right] \tag{4.6}$$

or $$R_{L_4CC_5L_5KEL_4}(f)=\frac{h}{727650}\left\{35175\left[f(\alpha-h)+f(\alpha+h)\right]\right.$$

$$+268272\left[f\left(\alpha-\frac{\sqrt{2}}{\sqrt{3}}h\right)+f\left(\alpha+\frac{\sqrt{2}}{\sqrt{3}}h\right)\right]-250880\left[f\left(\alpha-\frac{h}{\sqrt{2}}\right)+f\left(\alpha+\frac{h}{\sqrt{2}}\right)\right]$$

$$+235298\left[f\left(\alpha-\frac{\sqrt{3}}{\sqrt{7}}h\right)+f\left(\alpha+\frac{\sqrt{3}}{\sqrt{7}}h\right)\right]$$

$$\left.+265625\left[f\left(\alpha-\frac{h}{\sqrt{5}}\right)+f\left(\alpha+\frac{h}{\sqrt{5}}\right)\right]+348320\ f(\alpha)\right\} \tag{4.7}$$

This is the desired mixed quadrature rule of precision 11 for the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$E_{L_4CC_5L_5KEL_4}(f)=\frac{1}{55}\left[69E_{KEL_4}(f)-14E_{L_4CC_5L_5}(f)\right] \tag{4.8}$$

or $$E_{L_4CC_5L_5KEL_4}(f)=-\frac{4016}{13!\times 363825}h^{13}f^{(xii)}(\alpha)-\cdots$$

or $$\left|E_{L_4CC_5L_5KEL_4}(f)\right|=\frac{4016}{13!\times 363825}h^{13}\left|f^{(xii)}(\alpha)\right| \tag{4.9}$$

The rule $R_{L_4CC_5L_5KEL_4}(f)$ is called a mixed type rule of precision 11, as it is constructed from two different types of rules of the same precision.

5 ‘C’ Program for Adaptive Quadrature Routines Involving Mixed Quadrature Rules

```
#include"stdio.h"
#include"conio.h"
#include"math.h"
#include"stdlib.h"
    long double Integrand(long double E,long double A,long double B);
    long double REC_Integrand_R1(long double a,long double b);
    long double REC_Integrand_R2(long double a,long double b);
    long double REC_Integrand_R3(long double a,long double b);
    long double Integrand(long double E,long double A,long double B)
       {
          long double m,integral1,integral2,integral3,I,I1,sum_register=0.0f,E1;
          static int c=1;
          printf("\n\t\t\t%d->PRESS ENTER TO PROCEED\n\n\n",c);
          c=c++;
          getch();
          m=(A+B)/2;
          integral2=REC_Integrand_R2(A,m);
          printf("\nValue of i2:%.20Lf\n",integral2);
          integral3=REC_Integrand_R3(m,B);
          printf("\nValue of i3:%.20Lf\n",integral3);
```

```
        I=integral2+integral3;
        printf("\nValue of I:%.20Lf\n",I);
        integral1=REC_Integrand_R1(A,B);
        I1=integral1;
        printf("\nValue of I1:%.20Lf\n",I1);
        E1=fabsl(I-I1);
        printf("\nValue of E1:%.20Lf\n",E1);
        printf("\nValue of ME:%.20Lf\n",E/2);
        if(fabsl(I-integral1) < E/2)
        {
            printf("\nError critririon satisfied\n");
            return I;
        }
            else
                {
                  printf("\nError critririon doesnt get satisfied\n");
                  sum_register=sum_register+Integrand(E,A,m);
                  return (sum_register+Integrand(E,m,B));
                }
    }
    long double REC_Integrand_R2(long double a,long double b)
        {
            long double h,m,p,q,r;
            long double x1,x2,x5,x6,x7,x8,x9,x10,x11,RL4CC5L5;
            printf("\na:%.20Lf\n",a);
            printf("\nb:%.20Lf\n",b);
            h=(b-a)/2;
            m=(a+b)/2;
            p=1/sqrt(2);
            q=sqrt(3)/sqrt(7);
            r=1/sqrt(5);
            x1=sin(a)*exp(a/10);
            x2=sin(b)*exp(b/10);
            x5=sin(m-p*h)*exp((m-p*h)/10);
            x6=sin(m+p*h)*exp((m+p*h)/10);
            x7=sin(m)*exp(m/10);
            x8=sin(m-q*h)*exp((m-q*h)/10);
            x9=sin(m+q*h)*exp((m+q*h)/10);
            x10=sin(m-r*h)*exp((m-r*h)/10);
            x11=sin(m+r*h)*exp((m+r*h)/10);
            RL4CC5L5=(long double)h/1890*(129*(x1+x2)+2560*(x5+x6)
                                       -2401*(x8+x9)+1250*(x10+x11)+704*x7);
            return RL4CC5L5;
        }
        long double REC_Integrand_R3(long double a,long double b)
        {
            long double h,m,o,p,q,r;
```

```
        long double x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,RL4CC5L5KEL4;
        printf("\na:%.20Lf\n",a);
        printf("\nb:%.20Lf\n",b);
        h=(b-a)/2;
        m=(a+b)/2;
        o=sqrt(2)/sqrt(3);
        p=1/sqrt(2);
        q=sqrt(3)/sqrt(7);
        r=1/sqrt(5);
        x1=sin(a)*exp(a/10);
        x2=sin(b)*exp(b/10);
        x3=sin(m-o*h)*exp((m-o*h)/10);
        x4=sin(m+o*h)*exp((m+o*h)/10);
        x5=sin(m-p*h)*exp((m-p*h)/10);
        x6=sin(m+p*h)*exp((m+p*h)/10);
        x7=sin(m)*exp(m/10);
        x8=sin(m-q*h)*exp((m-q*h)/10);
        x9=sin(m+q*h)*exp((m+q*h)/10);
        x10=sin(m-r*h)*exp((m-r*h)/10);
        x11=sin(m+r*h)*exp((m+r*h)/10);
        RL4CC5L5KEL4=(long
double)h/727650*(35175*(x1+x2)+268272*(x3+x4)
            -250880*(x5+x6)+235298*(x8+x9)

  +265625*(x10+x11)+348320*x7);
        return RL4CC5L5KEL4;
      }
      long double REC_Integrand_R1(long double a,long double b)
      {
        long double h,m,p,r,RL4CC5;
        long double x1,x2,x5,x6,x7,x10,x11;
        printf("\na:%.20Lf\n",a);
        printf("\nb:%.20Lf\n",b);
        h=(b-a)/2;
        m=(a+b)/2;
        p=1/sqrt(2);
        r=1/sqrt(5);
        x1=sin(a)*exp(a/10);
        x2=sin(b)*exp(b/10);
        x5=sin(m-p*h)*exp((m-p*h)/10);
        x6=sin(m+p*h)*exp((m+p*h)/10);
        x7=sin(m)*exp(m/10);
        x10=sin(m-r*h)*exp((m-r*h)/10);
        x11=sin(m+r*h)*exp((m+r*h)/10);
        RL4CC5=(long double)h/630*(57*(x1+x2)+ 256*(x5+x6)

  +125*(x10+x11)+384*x7);
```

```
    return RL4CC5;
}
void main()
{
    long double A,B,E,Ev,FErr;
    int ch;
    long double Ans;
    printf("\t\t\tProgram Adaptive quadrature:\n");
    clrscr();
    while(1)
    {
        clrscr();
        printf("\n\n\n\t 1.Enter the value of the nodes
                                A,B,PrescribedtoleranceEAndExact val");
        printf("\n\n\n\t\t 2.Integrating the given function");
        printf("\n\n\n\t\t 3.Quit");
        printf("\n\n\n\n\t\t\tEnter ur choice:");
        scanf("%d",&ch);
        clrscr();
        switch(ch)
          {
              case 1:printf("\n\t\t Enter the value of A:");
              scanf("%Lf",&A);
              printf("\n\t\t Enter the value of B:");
              scanf("%Lf",&B);
              printf("\n\t\t Enter the value of E:");
              scanf("%Lf",&E);
              printf("\n\t\tEnter the Exact Value Ev:");
              scanf("%Lf",&Ev);
              break;
              case 2:
              Ans=Integrand(E,A,B);
              printf("\n\nAns:%.15Lf\n",Ans);
              FErr=fabsl(Ev-Ans);
              printf("\n\nFinal Error:%.15Lf\n",FErr);
              break;
              case 3:
              exit(0);
              default :
              printf("\nInvalid choice");
              break;
          }
        printf("\n\t\t\t\PRESS ENTER");
        getch();
    }
}
```

6 Numerical Examples

Table 6.1: Comparison among the rule $\left(\boldsymbol{R}_{CC_5}(\boldsymbol{f}), \boldsymbol{R}_{L_4}(\boldsymbol{f}), \boldsymbol{R}_{L_5}(\boldsymbol{f}), \boldsymbol{R}_{K.E.L_4}(\boldsymbol{f})\right)$ for approximation of the integrals in the whole interval method

Integrals	Exact value (I)	Approximate value (P)			
		$R_{CC_5}(f)$ Precision -5	$R_{L_4}(f)$ Precision -5	$R_{L_5}(f)$ Precision -7	$R_{KEL_4}(f)$ Precision -9
$I_1 = \int_0^{10\pi} \sin x \; e^{x/10} dx$	-21.92147785	108.4224	-64.7978	75.9365	-51.9994
$I_2 = \int_0^4 13\left(x - x^2\right) e^{-3x/2} dx$	-1.548788372527	-1.4074	-2.1567	-1.6008	-1.5492
$I_3 = \int_0^{2\pi} x \sin 30x \; \cos x \, dx$	-0.20967247	-2.7992	1.1727	-2.9743	4.5687
$I_4 = \int_0^1 \frac{2 \quad dx}{\left(2 + \sin 10\pi x\right)}$	1.15470054	1.1748	1.1072	1.0910	1.0596
$I_5 = \int_0^1 x^{16} \cos\left(x^{16}\right) dx$	0.0491217295	0.0391	0.0473	0.0401	0.0461
$I_6 = \int_0^1 \sqrt{x} \; dx$	0.6666666…	0.6645	0.6568	0.6621	0.6649
$I_7 = \int_0^1 \sin\left(\sqrt{\pi x}\right) dx$	0.8497263254	0.8460	0.8319	0.8416	0.8466
$I_8 = \int_0^2 \sin^{-1}\left(\sqrt{\frac{x}{2+x}}\right) dx$	1.14159265358	1.1374	1.1216	1.1324	1.1381
$I_9 = \int_0^2 \frac{1}{4} \pi x^4 \cos \frac{1}{4} \pi x \; dx$	1.25952595…	1.2584	1.2631	1.25955	1.2595258
$I_{10} = \int_0^1 \left[\sec h^2 10(x - 0.2) + \sec h^4 100(x - 0.4) + \sec h^6 1000(x - 0.6) \right] dx$	0.2108027354	0.2090	0.2501	0.2599	0.1829
$I_{11} = \int_0^5 \frac{50}{\pi\left(1 + 2500x^2\right)} dx$	0.49872676724581	2.6709	6.6395	3.9928	2.1118
$I_{12} = \int_0^2 e^x \sin\left(x^2 \cos e^x\right) dx$	-1.115957990..	1.3254	-1.7165	1.9906	-2.3216

$I_{13} = \int_0^1 \frac{30x^9\left(\cos x^6 - 1\right)}{1+x^{10}} e^{x^{15}} dx$	-0.7043797072	-0.7541	-1.5687	-1.0068	-0.7550
$I_{14} = \int_0^1 \frac{1}{x^4+1} dx$	0.866972987339	0.8672	0.8662	0.86699	0.8669728
$I_{15} = \int_{-1}^1 \frac{1}{x^4+x^2+0.9} dx$	1.5822329637	1.5813	1.5769	1.5791	1.582227
$I_{16} = \int_0^4 \cos\left(\cos t + 3\sin t + 2\cos 2t + 3\sin 2t + 3\cos 3t\right) dt$	0.966440320387	1.6148	-0.4494	0.1918	0.5107
$I_{17} = \int_0^{2\pi} x \cos 50x \sin x\, dx$	0.00251428	2.6001	-3.0751	4.1534	0.6316

Table 6.2: Comparison among the rules $\left(R_{L_4CC_5}(f), R_{L_4CC_5L_5}(f), R_{L_4CC_5L_5\,K.E.L_4}(f)\right)$ for approximation of the integrals in the whole interval method

Integrals	Exact value (I)	Approximate value (P)		
		$R_{L_4CC_5}(f)$ Precision -7	$R_{L_4CC_5L_5}(f)$ Precision -9	$R_{L_4CC_5L_5KEL_4}(f)$ Precision -11
$I_1 = \int_0^{10\pi} \sin x\, e^{x/10} dx$	-21.92147785	67.1795	46.7465	-77.1347
$I_2 = \int_0^4 13\left(x - x^2\right) e^{-3x/2} dx$	-1.548788372527	-1.5858	-1.5507	-1.5488
$I_3 = \int_0^{2\pi} x \sin 30x \cos x\, dx$	-0.20967247	-1.8535	0.7617	5.5377
$I_4 = \int_0^1 \frac{2\ dx}{\left(2+\sin 10\pi x\right)}$	1.15470054	1.1587	1.3168	0.9941
$I_5 = \int_0^1 x^{16} \cos\left(x^{16}\right) dx$	0.0491217295	0.0410	0.0433	0.0468
$I_6 = \int_0^1 \sqrt{x}\ dx$	0.6666666…	0.6627	0.6641	0.6651
$I_7 = \int_0^1 \sin\left(\sqrt{\pi x}\right) dx$	0.8497263254	0.8427	0.8452	0.8476

$I_8=\int_0^2 \sin^{-1}\left(\sqrt{\frac{x}{2+x}}\right)dx$	1.14159265358	1.1336	1.1365	1.1385
$I_9=\int_0^2 \frac{1}{4}\pi x^4 \cos\frac{1}{4}\pi x\ dx$	1.25952595...	1.25954	1.2595256	1.25952593
$I_{10}=\int_0^1 \left[\operatorname{sech}^2 10(x-0.2) + \operatorname{sech}^4 100(x-0.4) + \operatorname{sech}^6 1000(x-0.6)\right]dx$	0.2108027354	0.2188	0.1229	0.1982
$I_{11}=\int_0^5 \frac{50}{\pi\left(1+2500x^2\right)}dx$	0.49872676724581	3.6158	2.7361	1.9529
$I_{12}=\int_0^2 e^x \sin\left(x^2 \cos e^x\right)dx$	-1.115957990..	0.6011	-2.6409	-2.2403
$I_{13}=\int_0^1 \frac{30x^9\left(\cos x^6-1\right)}{1+x^{10}} e^{x^{15}} dx$	-0.7043797072	-0.9451	-0.8110	-0.7408
$I_{14}=\int_0^1 \frac{1}{x^4+1}dx$	0.866972987339	0.86698	0.866974	0.8669724
$I_{15}=\int_{-1}^1 \frac{1}{x^4+x^2+0.9}dx$	1.5822329637	1.5802	1.5829	1.5820
$I_{16}=\int_0^4 \cos\left(\cos t+3\sin t +2\cos 2t+3\sin 2t +3\cos 3t\right)dt$	0.966440320387	1.1233	3.2968	-0.1984
$I_{17}=\int_0^{2\pi} x\cos 50x \sin x\ dx$	0.00251428	1.2489	-5.5282	2.1995

Table 6.3: Approximation of the integrals in the adaptive quadrature routines

Integrals	Exact value (I)	Approximate value (P)	No. of steps required	Error |P-I|	Prescribed tolerance (E)
$I_1=\int_0^{10\pi} \sin x\, e^{x/10}dx$	-21.92147785	-21.9214778542	47	4×10^{-9}	0.000001

$I_2 = \int_0^4 13\left(x - x^2\right) e^{-3x/2} dx$	-1.548788372527	-1.5478837265	5	$1x10^{-10}$	0.00001
$I_3 = \int_0^{2\pi} x \sin 30x \ \cos x \, dx$	-0.20967247	-0.209672479	161	$9x10^{-9}$	0.000001
$I_4 = \int_0^1 \frac{2 \ dx}{\left(2 + \sin 10\pi x\right)}$	1.15470054	1.154700538	47	$1x10^{-9}$	0.000001
$I_5 = \int_0^1 x^{16} \cos\left(x^{16}\right) dx$	0.0491217295	0.04912172951	7	$1x10^{-11}$	0.000001
$I_6 = \int_0^1 \sqrt{x} \ dx$	0.6666666…	0.66666658	19	$7x10^{-8}$	0.000001
$I_7 = \int_0^1 \sin\left(\sqrt{\pi x}\right) dx$	0.8497263254	0.84972618	19	$1x10^{-7}$	0.000001
$I_8 = \int_0^2 \sin^{-1}\left(\sqrt{\frac{x}{2+x}}\right) dx$	1.14159265358	1.141592599	21	$5x10^{-8}$	0.000001
$I_9 = \int_0^1 \Big[\sec h^2 10\left(x - 0.2\right) + \sec h^4 100\left(x - 0.4\right) + \sec h^6 1000\left(x - 0.6\right) \Big] dx$	0.2108027354	0.2108027355	103	$1x10^{-10}$	0.0000000001
$I_{10} = \int_0^5 \frac{50}{\pi\left(1 + 2500x^2\right)} dx$	0.49872676724581	0.49872676729	19	$4x10^{-11}$	0.000001
$I_{11} = \int_0^2 e^x \sin\left(x^2 \cos e^x\right) dx$	-1.115957990..	-1.1159579909	25	$9x10^{-10}$	0.000001
$I_{12} = \int_0^1 \frac{30x^9\left(\cos x^6 - 1\right)}{1 + x^{10}} e^{x^{15}} dx$	-0.7043797072	-0.70437970717	11	$2x10^{-11}$	0.000001
$I_{13} = \int_0^1 \frac{1}{x^4 + 1} dx$	0.866972987339	0.866972987333	3	$5x10^{-12}$	0.000001
$I_{14} = \int_{-1}^1 \frac{1}{x^4 + x^2 + 0.9} \ dx$	1.5822329637	1.58223296372	7	$2x10^{-11}$	0.000001
$I_{15} = \int_0^4 \cos\left(\cos t + 3\sin t + 2\cos 2t + 3\sin 2t + 3\cos 3t\right) dt$	0.966440320387	0.966440320376	33	$1x10^{-11}$	0.000001

$I_{16} = \int_0^{2\pi} x \cos 50x \sin x \, dx$	0.00251428	0.002514280138	249	1x10^{-10}	0.000001

7 Observation

From the table 6.1,we observed that for some integrals the results due to rules $R_{CC_5}(f)$, $R_{L_4}(f)$, $R_{L_5}(f)$, $R_{KEL_4}(f)$ when applied on whole interval are quite alarming. For those integrals, mixed quadrature on whole interval(table 6.2) gives somewhat better result. However when mixed quadratures are used in adaptive mode (Table 6.3) for all the integrals excitingly good results is obtained.

8 Conclusion

In this paper we have concentrated mainly on computation of definite integrals in the adaptive quadrature routines involving mixed quadrature rules. We observe that mixed rules so formed are very well used for evaluating real definite integrals in the adaptive quadrature routine.

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