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P_p-Open Sets and P_p-Continuous Functions

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Abstract

In this paper we introduce a new class of sets, called P_p-open sets, also using this set, we define and investigate some properties of the concept of P_p-continuity. In particular, P_p-open sets and P_p-continuity are defined to extend known results for preopen sets and pre-continuity.

Keywords: *P_p-open, preopen, P_p-g.closed sets, pre-continuous and P_p-continuous.*

1 Introduction

In 1982, Mashhour et al [15] defined a new class of sets called preopen sets. He proved that the union of any family of preopen sets is also preopen set and introduced two types of continuity called precontinuous and weak precontinuous functions. In 1987, Popa [16] defined pre-neighbourhood of a point x in a space X. El-Deeb et al [9] defined the preclosure of a subset A as the intersection of all preclosed sets containing A and the preinterior of A is the union of all preopen sets contained in A. In the present paper we introduce a new type of preopen sets called P_p-open, this type of sets lies strictly between the pre-θ-open sets and preopen sets. We also study its fundamental properties and then we define further topological properties such as, P_p-neighborhood, P_p-interior, P_p-closure, P_p-derived set and P_p-boundary of a set. Mashhour et al [15] defined a function $f : X \rightarrow Y$ to be pre-continuous if $f^{-1}(V)$ is preopen set in X for every open set V of Y. Long and Herrington [14] have introduced a new class of functions called strongly θ-continuous function. We also introduce and investigate the concept of P_p-continuous functions. It will be shown that P_p-continuity is weaker than quasi θ-continuity mentioned in [20], but it is stronger than pre-continuity [15].

2 Preliminaries

Throughout the present paper, a space x always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let A be a subset of a space X.The closure and interior of A with respect to X are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a space X is said to be preopen [15] (resp., semi-open [13], α-open [18], β-open [1] and regular open [23]), if $A \subseteq Int(Cl(A))$ (resp., $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(Int(A)))$, $A \subseteq Cl(Int(Cl(A)))$ and $A = Int(Cl(A))$). The complement of a preopen (resp., regular open) set is said to be preclosed [9] (resp., regular closed[23]). The family of all preopen (resp., semi-open, α-open, β-open and regular open) subsets of X is denoted by $PO(X)$ (resp., $SO(X)$, $\alpha O(X)$, $\beta O(X)$ and $RO(X)$. The intersection of all preclosed sets of X containing A is called the preclosure [9] of A. The union of all preopen sets of X contained in A is called the preinterior. A subset A of a space X is called preclopen [11], if A is both preopen and preclosed while it is called pre-regular open [16], if $PIntPCl(A) = A$. In 1968, Velicko [24] defined the concepts of δ-open and θ-open sets in X denoted by ($\delta O(X)$ and $\theta O(X)$ respectively). A subset A of a space X is called δ-open (resp., θ-open) set if for each $x \in A$, there exists an open set G such that $x \in G \subseteq Int(Cl(G)) \subseteq A$ (resp., $x \in G \subseteq Cl(G) \subseteq A$). Joseph and Kwack [10] (resp., Di Maio and T. Noiri [12]), defined a subset A of a space X to be θ-semi-open (resp., semi-θ-open), if for each $x \in A$, there exists a semi-open set G such that $x \in G \subseteq Cl(G) \subseteq A$ (resp., $x \in G \subseteq SCl(G) \subseteq A$). The family of all θ-semi-open (resp., semi-θ-open) subsets of X is denoted by $\theta SO(X)$ (resp., $S\theta O(X)$). We recall that a topological X locally indiscrete [8] if every open subset of X is closed.

Definition 2.1 *[22] A space (X, τ) is said to have the property P if the closure is preserved under finite intersection or equivalently, if the closure of intersection of any two subsets equals the intersection of their closures.*

From the above definition Paul and Bhattacharyya [22] pointed out the following remark:

Remark 2.2 *If a space X has the property P, then the intersection of any two preopen sets is preopen, as a consequence of this, $PO(X, \tau)$ is a topology for X and it is finer than τ.*

Definition 2.3 *The point $x \in X$ is said to be a pre-θ-cluster [19] point of a subset A, if $PCl(U) \cap A \neq \phi$ for every $U \in PO(X)$.*
The set of all pre-θ-cluster points of A is called the pre-θ-closure of A and is denoted by $PCl_\theta(A)$.
A subset A of a topological space (X, τ) is said to be pre-θ-closed [5] if $PCl_\theta(A) = A$. The complement of a pre-θ-closed set is called pre-θ-open and it is denoted by $P\theta O(X)$.

Lemma 2.4 *[6] A subset U of a space X is pre-θ-open in X if and only if for each $x \in U$, there exists a preopen set V with $x \in V$ such that $PCl(V) \subseteq U$.*

Lemma 2.5 *[8] Let (X,τ) be a topological space. If $A \in \alpha O(X)$ and $B \in PO(X)$, then $A \cap B \in PO(X)$.*

The following results also can be found in [2].

Theorem 2.6

1. *Let (X,τ) be a topological space. If $G \in \tau$ and $Y \in PO(X)$, then $G \cap Y \in PO(X)$.*
2. *Let (Y,τ_Y) be a subspace of a space (X,τ). If $A \in PO(X, \tau)$ and $A \subseteq Y$, then $A \in PO(Y, \tau_Y)$. Moreover, if Y is an α-open subspace of X, $F \in PC(X, \tau)$ and $F \subseteq Y$, then $F \in PC(Y, \tau_Y)$.*
3. *Let (Y,τ_Y) be a subspace of a space (X,τ). If $A \in PO(Y, \tau_Y)$ and $Y \in PO(X, \tau)$, then $A \in PO(X, \tau)$.*

Theorem 2.7 *[7] For any subset A of a space (X, τ). The following statements are equivalent:*

1. *A is clopen.*
2. *A is α-open and closed .*
3. *A is preopen and closed.*

Theorem 2.8 *[2] For any spaces X and Y. If $A \subseteq X$ and $B \subseteq Y$ then,*

1. *$PInt_{X\times Y}(A \times B) = PInt_X(A) \times PInt_Y(B)$.*
2. *$PCl_{X\times Y}(A \times B) = PCl_X(A) \times PCl_Y(B)$.*

Theorem 2.9 *[3] Let (X,τ) be any space, then $PO(X,\tau) = PO(X, \tau_\alpha)$.*

Theorem 2.10 *[8] A topological space (X, τ) is locally indiscrete if and only if every subset of X is preopen.*

Theorem 2.11 *[2] A topological space (X, τ) is s^{**}-normal if and only if for every semi-closed set F and every semi-open set G containing F, there exists an open set H such that $F \subseteq H \subseteq Cl(H) \subseteq G$.*

Theorem 2.12 *[25] If X is s^{**}-normal, then $S\theta O(X) = \theta O(X)$.*

Definition 2.13 *[9] A topological space X is said to be P-regular if for each closed subset F of X and each point $x \notin F$ there exist $U, V \in PO(X)$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.*

Definition 2.14 *[21] A space X is said to be pre-regular if for each preclosed set F and each point $x \notin F$, there exist disjoint preopen sets U and V such that $x \in U$ and $F \subseteq V$*

Lemma 2.15 *[21] A space X is pre-regular if and only if for each $x \in X$ and each $H \in PO(X)$ there exists $G \in PO(X)$ such that $x \in G \subseteq PCl(G) \subseteq H$.*

Theorem 2.16 *[4] A space X is pre-regular if and only if $PCl(A) = PCl_\theta(A)$ for each subset A of X.*

Theorem 2.17 *[17] A space X is pre-T_1 if and only if the singleton set $\{x\}$ is preclosed for each point $x \in X$.*

3 P_p-Open Sets

Definition 3.1 *A subset A of a space X is called P_p-open, if for each $x \in A \in PO(X)$, there exists a preclosed set F such that $x \in F \subseteq A$.*

A subset B of a space X is called P_p-closed, if $X \setminus B$ is P_p-open. The family of all P_p-open (P_p-closed) subsets of a topological space (X, τ) is denoted by $P_pO(X, \tau)$ or $P_pO(X)$ ($P_pC(X, \tau)$ or $P_pC(X)$).

Proposition 3.2 *A subset A of a space X is P_p-open if and only if A is preopen set and is a union of preclosed sets.*

proof. Obvious.

It is clear from the definition that every P_p-open subset of a space X is preopen, but the converse is not true in general as shown in the following example:

Example 3.3 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, then $\{a\} \in PO(X)$ but $\{a\} \notin P_pO(X)$.*

The following result shows that any union of P_p-open sets in a topological space (X, τ) is P_p-open.

Proposition 3.4 *Let $\{A_\lambda : \lambda \in \Delta\}$ be a collection of P_p-open sets in a topological space X, then $\bigcup\{A_\lambda \in \Delta\}$ is P_p-open.*

proof. Let A_λ is P_p-open set for each λ, then A_λ is preopen and hence $\bigcup\{A_\lambda : \lambda \in \Delta\}$ is preopen. Let $X \in \bigcup\{A_\lambda : \lambda \in \Delta\}$, there exist $\lambda \in \Delta$ such that $X \in A_\lambda$. Since A_λ is P_p-open for each λ, there exists a preclosed set F such that $x \in F \subset A_\lambda \subseteq \bigcup\{A_\lambda : \lambda \in \Delta\}$, so $x \in F \subseteq \bigcup\{A_\lambda : \lambda \in \Delta\}$.
Therefore, $\bigcup\{A_\lambda : \lambda \in \Delta\}$ is P_p-open set.

The following example shows that the intersection of two P_p-open sets need not be P_p-open.

Example 3.5 *Consider $X = \{a, b, c, d\}$ with the topology*
$\tau = \{\phi, \{b\}, \{a, d\}, \{a, b, d\}, X\}$.
Here $\{a, b, c\} \in P_pO(X)$ and $\{b, c, d\} \in P_pO(X)$, but $\{a, b, c\} \cap \{b, c, d\} = \{b, c\} \notin P_pO(X)$.

From the above example we notice that the family of all P_p-open sets need not be a topology on X.

Proposition 3.6 *If the family of all preopen sets of a space X is a topology on X, then the family of P_p-open is also a topology on X.*

proof. It is enough to show that the finite intersection of P_p-open sets is also P_P-open set. Let A and B be two P_p-open sets, then A and B are preopen sets. Since $PO(X)$ is a topology on X. Then $A \cap B \in PO(X)$. Let $x \in A \cap B$, then $x \in A$ and $x \in B$, there exists preclosed sets E and F such that $x \in E \subseteq A$ and $x \in F \subseteq B$, this implies that $x \in E \cap F \subseteq A \cap B$. Since any intersection of preclosed sets is preclosed, then $A \cap B$ is P_p-open set. This completes the proof.

Corollary 3.7 *Let (X, τ) be a topological space. If X has a property P, then $P_pO(X)$ forms a topology on X.*

proof. Follows from Proposition 3.6.

Proposition 3.8 *If a space X is pre-T_1-space, then $PO(X) = P_pO(X)$.*

proof. Since the space X is pre-T_1, then by Theorem 2.17 every singleton is preclosed set and hence $x \in \{x\} \subseteq A$. Therefore, $A \in P_pO(X)$. Thus $PO(X) = P_pO(X)$.

Proposition 3.9 *For any subset A of a space X. If $A \in P\theta O(X)$, then $A \in P_pO(X)$.*

proof. Let $A \in P\theta O(X)$. If $A = \phi$, then $A \in P_pO(X)$. If $A \neq \phi$, then for each $x \in A$, there exists a preopen set G such that $X \in G \subseteq PCl(G) \subseteq A$ implies that $x \in PCl(G) \subseteq A$. Since $A \in P\theta O(X)$ and $P\theta O(X) \subseteq PO(X)$ in general, then $A \in PO(X)$. Therefore, $A \in P_pO(X)$.

Remark 3.10 *It clear that each preregular (preclopen or θ-open) set are P_p-open.*

The following example shows that a P_p-open set need not be θ-open.

Example 3.11 *Consider $X = \{a, b, c, d\}$, with the topology*
$\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$
Then the set $\{a, b, c\} \in P_pO(X)$, but $\{a, b, c\} \notin \theta O(X)$

Theorem 3.12 *For any space X, $PCl(PInt(\{x\})) = \{x\}$ if and only if $\{x\}$ is P_p-open.*

proof. Let $PCl(PInt(\{x\})) = \{x\}$, implies that $\{x\}$ is preopen and preclosed, then $\{x\}$ is preregular open, therefore $\{x\} \in P_pO(X)$.
Conversely, let $\{x\}$ be P_p-open, this implies that $x \in \{x\} \subseteq \{x\}$. Since $\{x\} \in P_pO(X)$, then $\{x\} \in PC(X)$ and hence $\{x\}$ is pre-open and pre-closed. Therefore, $PCl(PInt(\{x\})) = \{x\}$.

Proposition 3.13 *Let (X, τ) be a topological space, then $\{x\}$ is $P_pO(X)$ if and only if it is preclopen for every $x \in X$.*

proof. Obvious.

Proposition 3.14 *A subset A of a space (X, τ) is P_p-open if and only if for each $x \in A$, there exists a P_p-open set B such that $x \in B \subseteq A$.*

proof. Suppose that for each $x \in A$, there exists a P_p-open set B such that $x \in B \subseteq A$. Thus $A = \cup B_\lambda$ where $B_\lambda \in P_pO(X)$ for each λ, and by Proposition 3.4, A is P_p-open.
The other part is obvious.

Proposition 3.15 *Let (X, τ) be a pre-regular space, then $\tau \subseteq P_p(X)$.*

proof. Let A be any open subset of a space X. This implies that A is preopen. If $A = \phi$, then $A \in P_pO(X)$. If $A \neq \phi$, since X is pre-regular, by Lemma 2.15, for each $x \in A \subseteq X$, there exists a pre-open set G such that $x \in G \subseteq PCl(G) \subseteq A$. Thus we have $x \in PCl(G) \subseteq A$. Therefore, $\tau \subseteq P_pO(X)$.

Proposition 3.16 *Let (X, τ) be a topological space, and $A, B \subseteq X$. If $A \in P_pO(X)$ and B is both α-open and preclosed, then $A \cap B \in P_pO(X)$.*

proof. Let $A \in P_pO(X)$ and B is α-open, then A is preopen set, and by Lemma 2.5, $A \cap B \in PO(X)$. Let $x \in A \cap B$, $x \in A$ and $x \in B$, there exists a preclosed set F such that $x \in F \subseteq A$. Since B is preclosed, implies that $F \cap B$ is preclosed, then $x \in F \cap B \subseteq A \cap B$. Thus $A \cap B$ is P_p-open set in X.

Corollary 3.17 *If a space X is locally indiscrete, then $PO(X) = P_pO(X)$.*

proof. Follows from Theorem 2.10.

Proposition 3.18 *If a topological space (X, τ) is locally indiscrete, then $\tau \subseteq P_pO(X)$.*

proof. Since X is locally indiscrete, then by Corollary 3.17, $PO(X) = P_pO(X)$. Therefore, $\tau \subseteq P_pO(X)$.

The following example shows that the converse of Proposition 3.18 is not true.

Example 3.19 *Consider $X = \{a, b, c, d\}$, with the topology*
$\tau = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}$,
then $\tau \subseteq P_pO(X)$, but X is not locally indiscrete.

Proposition 3.20 *If (X, τ) is indiscrete topology, then $P_pO(X)$ is discrete topology in X.*

proof. Let X be indiscrete topology, then every subset of X is preopen, then $PO(X) = P_pO(X)$. Therefore, $P_pO(X)$ is discrete topology on X.

The following example shows that the converse of Proposition 3.20 is not true.

Example 3.21 *Consider $X = \{a, b, c\}$, with the topology*
$\tau = \{\phi, \{a\}, \{b, c\}, X\}$, *then $P_pO(X)$ is discrete topology in X, but (X, τ) is not indiscrete topology.*

Proposition 3.22 *For any topological space (X, τ), we have:*

1. *$P_pO(X)$ is discrete if and only if $PO(X)$ is discrete.*
2. *τ is discrete if and only if $P_pO(X)$ is discrete.*

proof. Obvious.

Proposition 3.23 *For any subset A of a space (X, τ). The following statements are equivalent:*

1. *A is clopen.*
2. *A is P_p-open and closed .*
3. *A is preopen and closed.*

proof. Follows from Theorem 2.7.

Proposition 3.24 *For a topological space (X, τ), the following conditions are equivalent:*

1. *X is locally indiscrete.*
2. *Every subset of X is P_p-open.*
3. *Every singleton in X is P_p-open.*
4. *Every closed subset of X is P_p-open.*

proof. Follows from Theorem 2.10.

Proposition 3.25 *Let X and Y be two topological spaces and $X \times Y$ be the product topology. If $A \in P_pO(X)$ and $B \in P_pO(Y)$, then $A \times B \in P_pO(X \times Y)$.*

proof. Let $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. Since $A \in P_pO(X)$ and $B \in P_pO(Y)$, there exist preclosed sets F and E in X and Y respectively, such that $x \in F \subseteq A$ and $y \in E \subseteq B$. Therefore, $(x, y) \in F \times E \subseteq A \times B$. Since $A \in \mathrm{PO}(X)$ and $B \in \mathrm{PO}(Y)$. Then by Theorem 2.8(1), $A \times B \in \mathrm{PO}(X \times Y)$. Since F is preclosed in X and E is preclosed in Y and by Theorem 2.8 (2), $F \times E$ is preclosed in $(X \times Y)$. Therefore, $A \times B \in P_pO(X \times Y)$.

Proposition 3.26 *Topological spaces (X, τ) and (X, τ_α) have the same class of P_p-open sets.*

proof. Let A be any subset of a space X and $A \in P_pO(X, \tau)$. If $A{=}\phi$, then $A \in P_pO(X, \tau_\alpha)$. If $A \neq \phi$, since $A \in P_pO(X, \tau)$, then $A \in PO(X, \tau)$ and $A{=}\cup F_\lambda$, where F_λ is preclosed for each λ Since $A \in PO(X, \tau)$, then by Theorem 2.9, $A \in PO(X, \tau_\alpha)$. Again since $F_\lambda \in PC(X, \tau)$ for each λ, then by Theorem 2.9, $F_\lambda \in PC(X, \tau_\alpha)$ for each λ. Therefore, by Proposition 3.2, $A \in P_pO(X, \tau_\alpha)$. Hence $P_pO(X, \tau) \subseteq P_pO(X, \tau_\alpha)$. On the other hand, we can prove similarly $P_pO(X, \tau_\alpha) \subseteq P_pO(X, \tau)$. Therefore, we get $P_pO(X, \tau_\alpha) = P_pO(X, \tau)$.

Proposition 3.27 *Let (X, τ) be any s^{**}-normal space. If $A \in S\theta O(X)$, then $A \in P_pO(X)$.*

proof. Let $A \in S\theta O(X)$, if $A = \phi$, then $A \in P_pO(X)$. If $A \neq \phi$. Since the space X is s^{**}-normal, then by Theorem 2.12, $S\theta O(X) = \theta O(X)$. Hence $A \in \theta O(X)$. But $\theta O(X) \subseteq P_pO(X)$ in general. Therefore, $A \in P_pO(X)$

Corollary 3.28 *Let (X, τ) be any s^{**}-normal space. If $A \in \theta SO(X)$, then $A \in P_pO(X)$.*

proof. Follows from Proposition 3.27, and the fact that $\theta SO(X) \subseteq S\theta O(X)$.

Proposition 3.29 *Let Y be an α-open subspace of a space (X,τ). If $A \in P_pO(X, \tau)$ and $A \subseteq Y$, then $A \in P_pO(Y, \tau_Y)$.*

proof. Let $A \in P_pO(X, \tau)$, then $A \in PO(X, \tau)$ and for each $x \in A$, there exists a preclosed set F in X such that $x \in F \subseteq A$. Since $A \in PO(X, \tau)$ and $A \subseteq Y$. Then by Theorem 2.6, $A \in PO(Y, \tau_Y)$. Since F preclosed set in X and $A \subseteq Y$. Then by Theorem 2.6, F preclosed set in Y. Hence $A \in P_pO(Y, \tau_Y)$.

Proposition 3.30 *Let (Y,τ_Y) be a subspace of a space (X,τ) and $A \subseteq Y$. If $A \in P_pO(Y, \tau_Y)$ and Y is preclopen, then $A \in P_pO(X, \tau)$.*

proof. Let $A \in P_PO(X, \tau_Y)$, then $A \in PO(X, \tau_Y)$ and for each $x \in A$, there exists a preclosed set F in Y such that $x \in F \subseteq A$. Since Y is preclopen, then $Y \in PO(X, \tau)$ and since $A \in PO(X, \tau_Y)$, then by Theorem 2.6, $A \in PO(X, \tau)$. Again since Y is preclopen implies Y is preclosed set in X and since F is preclosed set in Y, therefore by Theorem 2.6, F is preclosed set in X. Hence $A \in P_PO(X, \tau)$.

From Propositions 3.29 and 3.30, we obtain the following result:

Corollary 3.31 *Let (X,τ) be a topological space and A, Y subsets of X such that $A \subseteq Y \subseteq X$ and Y is preclopen. Then $A \in P_pO(Y)$ if and only if $A \in P_pO(X)$.*

Proposition 3.32 *Let (Y,τ_Y) be a subspace of a space (X,τ). If $A \in P_pO(Y, \tau_Y)$ and $Y \in PR(X, \tau)$, then $A \in P_pO(X, \tau)$.*

proof. Obvious.

Corollary 3.33 *Let (X,τ) be a topological space and A, Y subsets of X such that $A \subseteq Y \subseteq X$ and $Y \in PR(X)$. Then $A \in P_pO(Y)$ if and only if $A \in P_pO(X)$.*

Corollary 3.34 *Let A and Y be any subsets of a space X. If $A \in P_PO(X)$ and Y is both α-open preclosed subset of X, then $A \cap Y \in P_pO(Y)$.*

proof. Follows from Proposition 3.16 and Proposition 3.29.

The following diagram shows that the relations among $P_PO(X)$, $\theta O(X)$, $P\theta O(X)$, $\delta O(X)$, τ, $\alpha O(X)$ and $PO(X)$.

$$\begin{tikzcd}
\theta O(X) \arrow[r] \arrow[d] & P\theta O(X) \arrow[r] & P_PO(X) \arrow[dr] & \\
\delta O(X) \arrow[r] & \tau \arrow[r] & \alpha O(X) \arrow[r] & PO(X)
\end{tikzcd}$$

Diagram 1

Remark 3.35 *In Diagram 1, we notice the following statements:*

1. *τ is incomparable with $P_pO(X)$.*
2. *$\delta O(X)$ is incomparable with $P_pO(X)$.*
3. *$\alpha O(X)$ is incomparable with $P_pO(X)$.*

Definition 3.36 *Let A be a subset of a space X and $x \in X$, then:*

1. *A subset N of X is said to be P_p-neighborhood of x, if there exists a P_p-open set U in X such that $x \in U \subseteq N$.*
2. *P_p-interior of a set A (briefly, $P_pInt(A)$) is the union of all P_p-open sets which are contained in A.*
3. *A point $x \in X$ is said to be P_p-limit point of A if for each P_p-open set U containing x, $U \cap (A \setminus \{x\}) \neq \phi$. The set of all P_p-limit points of A is called a P_p-derived set of A and is denoted by $P_pD(A)$.*
4. *A point $x \in X$ is said to be in P_p-closure of A if for each P_p-open set U containing x such that $U \cap A \neq \phi$.*
5. *P_p-closure of a set A (briefly, $P_pCl(A)$.) is the intersection of all P_p-closed sets containing A.*
6. *P_p-boundary of A is defined as $P_pCl(A) \setminus P_pInt(A)$ and is denoted by $P_pBd(A)$.*

The topological properties of P_p-neighborhood, P_p-interior, P_p-closure, P_p-derived and P_p-boundary are the same as in the supratopoology.

4 P_p-g.Closed Sets

Definition 4.1 *A subset A of X is said to be a P_p-generalized closed(briefly, P_p-g.closed) set, if $P_pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is P_p-open set in (X, τ). The family of all P_p-g.closed sets of a topological space (X, τ) is denoted by $P_pGC(X, \tau)$ or $P_pGC(X)$.*

It is clear that every P_p-closed set is P_p-g.closed set, but the converse is not true in general as it is shown in the following example.

Example 4.2 *Considering the space (X, τ) as defined in Example 3.5. Then we have $\{c\} \in P_pGC(X)$, but $\{c\} \notin P_pC(X)$.*

Proposition 4.3 *The intersection of a P_p-generalized closed set and a P_p-closed set is always P_p-generalized closed.*

proof. Let A be P_p-generalized closed and F be P_p-closed set. Assume that U be a P_p-open set such that $A \cap F \subseteq U$. Set $G = X \setminus F$, then $A \subseteq U \cup G$. Since G is P_p-open, $U \cup G$ is P_p-open and since A is P_p-generalized closed, then $P_pCl(A) \subseteq U \cup G$. Now, $P_pCl(A \cap F) \subseteq P_pCl(A) \cap P_pCl(F) = P_pCl(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \phi \subseteq U$.

Proposition 4.4 *If A is both P_p-open and P_p-generalized closed set in X, then A is P_p-closed set.*

proof. Since A is P_p-open and P_p-generalized closed set in X, $P_pCl(A) \subseteq A$, but $A \subseteq P_pCl(A)$. Therefore $A = P_pCl(A)$, and hence A is P_p-closed set.

The union of two P_p-g.closed sets need not be P_p-g.closed set in general. It is shown by the following example:

Example 4.5 *In Example 3.5, $\{a\} \in P_pGC(X)$ and $\{d\} \in P_pGC(X)$, but $\{a\} \cup \{d\} = \{a, d\} \notin P_pGC(X)$.*

The intersection of two P_p-g.closed sets need not be P_p-g.closed set in general. It is shown by the following example:

Example 4.6 *In Example 3.5, $\{a, c, d\} \in P_pGC(X)$ and $\{a, b, d\} \in P_pGC(X)$, but $\{a, c, d\} \cup \{a, b, d\} = \{a, d\} \notin P_pGC(X)$.*

Proposition 4.7 *If a subset A of X is P_p-g.closed set and $A \subseteq B \subseteq P_pCl(A)$, then B is a P_p-g.closed set in X.*

proof. Let A be a P_p-g.closed set such that $A \subseteq B \subseteq P_pCl(A)$. Let U be a P_p-open set of X such that $B \subseteq U$. Since A is P_p-g.closed, we have $P_pCl(A) \subseteq U$. Now $P_pCl(A) \subseteq P_pCl(B) \subseteq P_pCl\left[P_pCl(A)\right] = P_pCl(A) \subseteq U$. That is $P_pCl(B) \subseteq U$, where U is P_p-open. Therefore, B is a P_p-g.closed set in X.

The converse of Proposition 4.7 is not true in general as it can be seen from the following example:

Example 4.8 *In Example 3.5. Let $A = \{a\}$ and $B = \{a, b\}$, then A and B are P_p-g.closed sets in (X, τ), but $A \subseteq B \nsubseteq P_pCl(A)$.*

Proposition 4.9 *For each $x \in X$, $\{x\}$ is P_p-closed or $X \setminus \{x\}$ is P_p-g.closed in (X, τ).*

proof. Suppose that $\{x\}$ is not P_p-closed, then $X \setminus \{x\}$ is not P_p-open. Let U be any P_p-open set such that $X \setminus \{x\} \subseteq U$, implies $U = X$. Therefore $P_pCl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is P_p-g.closed.

Proposition 4.10 *A subset A of X is P_p-g.closed if and only if $P_pCl(\{x\}) \cap A \neq \phi$, holds for every $x \in P_pCl(A)$.*

proof. Let U be a P_p-open set such that $A \subseteq U$ and let $x \in P_pCl(A)$. By assumption, there exists a point $z \in P_pCl(\{x\})$ and $z \in A \subseteq U$. Then $U \cap \{x\} \neq \phi$, hence $x \in U$, this implies that $P_pCl(A) \subseteq U$. Therefore, A is P_p-g.closed.
Conversely, suppose that $x \in P_pCl(A)$ such that $P_pCl(\{x\}) \cap A = \phi$. Since $P_pCl(\{x\})$ is P_p-closed. Therefore, $X \setminus P_pCl(\{x\})$ is a P_p-open set in X. Since $A \subseteq X \setminus P_pCl(\{x\})$ and A is P_p-g.closed implies that $P_pCl(A) \subseteq X \setminus P_pCl(\{x\})$ holds, and hence $x \notin P_pCl(A)$. This is a contradiction. Therefore, $P_pCl(\{x\}) \cap A \neq \phi$.

Proposition 4.11 *A subset A of a space X is P_p-g.closed if and only if $P_pCl(A) \setminus A$ does not contain any non-empty P_p-closed set.*

proof. Necessity. Suppose that A is a P_p-g.closed set in X. We prove the result by contradiction. Let F be a P_p-closed set such that $F \subseteq P_pCl(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since A is P_p-g.closed and $X \setminus F$ is P_p-open, therefore, $P_pCl(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus P_pCl(A)$. Hence $F \subseteq P_pCl(A) \cap (X \setminus P_pCl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $P_pCl(A) \setminus A$ does not contain any non-empty P_p-closed set in X.
Sufficiency. Let $A \subseteq U$, where U is P_p-open in X. If $P_pCl(A)$ is not contained in U, then $P_pCl(A) \cap X \setminus U \neq \phi$. Now, since $P_pCl(A) \cap X \setminus U \subseteq P_pCl(A) \setminus A$ and $P_pCl(A) \cap X \setminus U$ is a non-empty P_p-closed set, then we obtain a contradiction. Therefore, A is P_p-g.closed.

Proposition 4.12 *If A is a P_p-g.closed set of a space X, then A is P_p-closed if and only if $P_pCl(A) \setminus A$ is P_p-closed.*

proof. Necessity. If A is a P_p-g.closed set which is also P_p-closed, then by Proposition 4.11, $P_pCl(A) \setminus A = \phi$, which is P_p-closed.
Sufficiency. Let $P_pCl(A) \setminus A$ be a P_p-closed set and A be P_p-g.closed. Then by Proposition 4.11, $P_pCl(A) \setminus A$ does not contain any non-empty P_p-closed subset. Since $P_pCl(A) \setminus A$ is P_p-closed and $P_pCl(A) \setminus A = \phi$, this shows that A is P_p-closed.

Proposition 4.13 *Every subset of a space X is P_p-g.closed if and only if $P_pO(X, \tau) = P_pC(X, \tau)$.*

proof. Let $U \in P_pO(X, \tau)$. Then by hypothesis, U is P_p-g.closed which implies that $P_pCl(U) \subseteq U$, then $P_pCl(U) = U$, therefore $U \in PC(X, \tau)$. Also let $V \in P_pC(X, \tau)$. Then $X \setminus V \in P_pO(X, \tau)$, hence by hypothesis $X \setminus V$ is P_p-g.closed and then $X \setminus V \in P_pC(X, \tau)$, thus $V \in P_pO(X, \tau)$ according to the above we have $P_pO(X, \tau) = P_pC(X, \tau)$.
Conversely, if A is a subset of a space X such that $A \subseteq U$ where $U \in P_pO(X, \tau)$, then $U \in P_pC(X, \tau)$ and therefore, $P_pCl(U) \subseteq U$ which shows that A is P_p-g.closed.

5 P_p-Continuous Functions

Definition 5.1 *A function $f : X \to Y$ is called P_p-continuous at a point $x \in X$, if for each open set V of Y containing $f(x)$, there exists a P_p-open set U of X containing x such that $f(U) \subseteq V$. If f is P_p-continuous at every point x of X, then it is called P_p-continuous.*

We recall the following definitions.

Definition 5.2 *A function $f : X \to Y$ is called:*

1. *precontinuous [15], if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in PO(X, x)$ such that $f(U) \subseteq V$.*

2. *strongly θ-continuous [14], if the inverse image of each open subset of Y is θ-open in X.*

3. *quasi θ-continuous [20] at a point $x \in X$, if for each θ-open set V of Y containing $f(x)$, there exists a θ-open set U of X containing x such that $f(U) \subseteq V$.*

Proposition 5.3 *A function $f : X \to Y$ is P_p-continuous if and only if the inverse image of every open set in Y is a P_p-open in X.*

proof. It is clear.

The proof of the following corollaries follows directly from their definitions and are thus omitted.

Corollary 5.4 *Every P_p-continuous function is precontinuous.*

Corollary 5.5 *Every quasi θ-continuous is P_p-continuous.*

The examples are given below demonstrate that the converses of the previous corollaries are false.

Example 5.6 *Consider $X = \{a, b, c\}$ with the topology $\tau = \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is precontinuous, but it is not P_p-continuous, because $\{a\}$ is an open set in (X, σ) containing $f(a) = a$, there exists no P_p-open set U in (X, τ) containing a such that $a \in f(U) \subseteq \{a\}$.*

Example 5.7 *Consider $X = \{a, b, c, d\}$ with the two topologies*
$\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ *and*
$\sigma = \{\phi, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, X\}$.
Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is P_p-continuous, but it is not quasi θ-continuous.

Proposition 5.8 *A function $f : X \to Y$ is P_p-continuous if and only if f is pre-continuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a preclosed set F of X containing x such that $f(F) \subseteq V$.*

proof. Let $f : X \to Y$ be a P_p-continuous and also let $x \in X$ and V be any open set of Y containing $f(x)$. By hypothesis, there exists a P_p-open set U of X containing x such that $f(U) \subseteq V$. Since U is P_p-open set. Then for each $x \in U$, there exists a preclosed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. Hence P_p-continuous always implies pre-continuous.
Conversely, let V be any open set of Y. We have to show that $f^{-1}(V)$ is P_p-open set in X. Since f is pre-continuous, then $f^{-1}(V)$ is preopen set in X. Let $x \in f^{-1}(V)$, then $f(x) \in V$. By hypothesis, there exists a preclosed set F of X containing x such that $f(F) \subseteq V$, which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is P_p-open set in X. Hence by Proposition 5.3, f is P_p-continuous.

Here, we begin with the following characterizations of P_p-continuous functions.

Proposition 5.9 *For a function $f : X \to Y$, the following statements are equivalent:*

1. *f is P_p-continuous.*

2. *$f^{-1}(V)$ is a P_p-open set in X, for each open set V of Y.*

3. *$f^{-1}(F)$ is a P_p-closed set in X, for each closed set F of Y.*

4. *$f(P_pCl(A)) \subseteq Cl(f(A))$, for each subset A of X.*

5. *$P_pCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for each subset B of Y.*

6. *$f^{-1}(Int(B)) \subseteq P_pInt(f^{-1}(B))$, for each subset B of Y.*

7. *$Int(f(A)) \subseteq f(P_pInt(A))$, for each subset A of X.*

proof. Straightforward.

Proposition 5.10 *Let $f : X \to Y$ be a function and X is locally indiscrete space. Then f is P_p-continuous if and only if f is Pre-continuous.*

proof. Follows from Corollary 3.17.

Proposition 5.11 *Let $f : X \to Y$ be a function and X is pre-T_1 space. Then f is P_p-continuous if and only if f is Pre-continuous.*

proof. Follows from Proposition 3.8.

Proposition 5.12 *Let $f : X \to Y$ be a P_p-continuous function. If Y is any subset of a topological space Z, then $f : X \to Z$ is P_p-continuous.*

proof. Let $x \in X$ and V be any open set of Z containing $f(x)$, then $V \cap Y$ is open in Y. But $f(x) \in Y$ for each $x \in X$, then $f(x) \in V \cap Y$. Since $f : X \to Y$ is P_p-continuous, then there exists a P_p-open set U containing x such that $f(U) \subseteq V \cap Y \subseteq V$. Therefore, $f : X \to Z$ is P_p-continuous.

Proposition 5.13 *Let $f : X \to Y$ be P_p-continuous function. If A is α-open and preclosed subset of X, then $f|A : A \to Y$ is P_p-continuous in the subspace A.*

proof. Let V be any open set of Y. Since f is P_p-continuous. Then by Proposition 5.3, $f^{-1}(V)$ is P_p-open set in X. Since A is α-open and preclosed subset of X. By Corollary 3.34, $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is a P_p-open subset of A. This shows that $f|A : A \to Y$ is P_p-continuous.

Proposition 5.14 *A function $f : X \to Y$ is P_p-continuous, if for each $x \in X$, there exists a preclopen set A of X containing x such that $f|A : A \to Y$ is P_p-continuous.*

proof. Let $x \in X$, then by hypothesis, there exists a preclopen set A containing x such that $f|A : A \to Y$ is P_p-continuous. Let V be any open set of Y containing $f(x)$, there exists a P_p-open set U in A containing x such that $(f|A)(U) \subseteq V$. Since A is preclopen set, by Proposition 3.30, U is P_p-open set in X and hence $f(U) \subseteq V$. This shows that f is P_p-continuous.

Proposition 5.15 *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two P_p-continuous functions. If Y is Hausdorff, then the set $E = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = g(x_2)\}$ is P_p-closed in the product space $X_1 \times X_2$.*

proof. Let $(x_1, x_2) \notin E$. Then $f(x_1) \neq g(x_2)$. Since Y is Hausdorff, there exist open sets V_1 and V_2 of Y such that $f(x_1) \subseteq V_1$, $g(x_2) \subseteq V_2$ and $V_1 \cap V_2 = \phi$. Since f and g are P_p-continuous, then there exist P_p-open sets U_1 and U_2 of X_1 and X_2 containing x_1 and x_2 such that $f(U_1) \subseteq V_1$ and $g(U_2) \subseteq V_2$, respectively. Put $U = U_1 \times U_2$, then $(x_1, x_2) \in U$ and U is a P_p-open set in $X_1 \times X_2$, by Proposition 3.25, and $U \cap E = \phi$. Therefore, we obtain $(x_1, x_2) \notin P_pCl(E)$. Hence E is P_p-closed in the product space $X_1 \times X_2$.

Proposition 5.16 *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If f is P_p-continuous and g is continuous. Then the composition function $g \circ f : X \to Z$ is P_p-continuous.*

proof. Let V be any open subset of Z. Since g is continuous, $g^{-1}(V)$ is open subset of Y. Since f is P_p-continuous, then by Proposition 5.3, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is P_p-open subset in X. Therefore, by Proposition 5.3, $g \circ f$ is P_p-continuous.

Proposition 5.17 *Let $f : X \to Y$ be a P_p-continuous function and let $g : Y \to Z$ be a strongly θ-continuous function, then $g \circ f : X \to Z$ is P_p-continuous.*

proof. Let V be an open subset of Z. In view of strong θ-continuity of g, $g^{-1}(V)$ is a θ-open subset of Y. Again, since f is P_p-continuous, $(g \circ f)^{-1}(V)$=$F^{-1}(g^{-1}(V))$ is a P_p-open set in X. Hence $g \circ f$ is P_p-continuous.

Corollary 5.18 *Let $f : X \to Y$ be a P_p-continuous function and let $g : Y \to Z$ be a quasi θ-continuous function, then $(g \circ f) : X \to Z$ is P_p-continuous.*

Proposition 5.19 *If $f_i : X_i \to Y_i$ is P_p-continuous functions for i = 1, 2. Let $f : X_1 \times X_2 \to Y_1 \times Y_2$ be a function defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then f is P_p-continuous.*

proof. Let $R_1 \times R_2 \subseteq Y_1 \times Y_2$, where R_i is open set in Y_i for $i = 1, 2$. Then $f^{-1}(R_1 \times R_2) = f_1^{-1}(R_1) \times f_2^{-1}(R_2)$. Since f_i is P_p-continuous for $i = 1, 2$. By Proposition 5.3, and Proposition 3.25, $f^{-1}(R_1 \times R_2)$ is P_p-open set in $X_1 \times X_2$.

Proposition 5.20 *Let X, Y_1, Y_2 be topological spaces and $f_i : X \to Y_i$, for $i = 1, 2$, be functions. If a functions $g : X \to Y_1 \times Y_2$ defined as: $g(x)$ $= (x_1, x_2)$, where $f_i(x) = x_i$, for $i = 1, 2$ is P_p-continuous, then f_i is P_p-continuous for $i = 1, 2$.*

proof. Let $x \in X$ and V_1 be any open set in Y_1 containing $f_1(x) = x_1$, then $V_1 \times Y_2$ is open in $Y_1 \times Y_2$, which contain (x_1, x_2). Since g is P_p-continuous. Then by Proposition 5.3, $g^{-1}(V_1 \setminus Y_2)$ is P_p-open set in X. However, $f^{-1}(V_1) = g^{-1}(V_1 \setminus Y_2)$ $(f^{-1}Cl(V_1) = g^{-1}Cl(V_1 \setminus Y_2)$. Thus f_1 is P_p-continuous. Similarly, we can prove that f_2 is P_p-continuous. This completes the proof.

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