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Properties for a Subclass of Convex Functions

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Abstract

Let S be the class of functions which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and a_n a complex number. Let T denote the class consisting of functions f of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ where a_n is a non negative real number. In this paper, we introduce a new subclass of S by adopting the idea of Ramesha et al. *[3] and Sudharsan* et al. *[5]. We also determine coefficient estimates, growth and extreme points for f belonging to this class.*

Keywords: *Analytic, Univalent, Coefficient Estimates.*

1 Introduction

Let S be the class of functions f which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where a_n is a complex number. In [3], Ramesha *et al.* introduced the class $H(\alpha)$ as follows:

Definition 1.1 *Let f be given by (1). Then, $f \in A$ for which*

$$Re\left\{\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0 \tag{2}$$

for $\alpha \geq 0$ and $\frac{f(z)}{z} \neq 0, z \in \mathcal{D}$.

Sudharsan et al. [5] introduced $S_s^*(\alpha,\beta)$ of functions analytic and univalent in D given by (1) and satisfying the condition $\left|\frac{zf'(z)}{f(z)-f(-z)} - 1\right| < \beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)} + 1\right|$ for some $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$ and $z \in D$. By developing the idea of Ramesha *et al.* [3], we now introduce a new subclass of S denote as $C(\alpha)$ as follows:

Definition 1.2 *Let f be given by (1). Then, $f \in A$ for which*

$$Re\left\{\frac{\alpha(z^2 f''(z))'}{f'(z)} + \frac{(zf'(z))'}{f'(z)}\right\} > 0, \tag{3}$$

for $0 \leq \alpha < 1$.

However, for this paper we consider a subclass of T where T denotes the class consisting of functions f of the form as

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \tag{4}$$

where a_n is a non negative real number. For $f \in T$, we define the class $CT(\alpha)$ with α satisfying the condition $0 \leq \alpha < 1$.

Definition 1.3 *A function $f \in CT(\alpha)$ if and only if it satisfies*

$$\left|\left(\frac{\alpha(z^2 f''(z))'}{f'(z)} + \frac{(zf'(z))'}{f'(z)}\right) - 1\right| < \left|\left(\frac{\alpha(z^2 f''(z))'}{f'(z)} + \frac{(zf'(z))'}{f'(z)}\right) + 1\right| \tag{5}$$

for $z \in D$.

We note that the above condition on α, is necessary to ensure this class form subclass of S.

2 Preliminary Result

The following preliminary lemma is required for proving the main results.

Lemma 2.1 *Let $f \in T$, then*

$$\sum_{n=2}^{\infty} n(n(n-1)\alpha + (n+1))a_n|z|^{n-1} < 2 \tag{6}$$

Proof: Since $f \in T$, $\sum_{n=2}^{\infty} n|a_n||z|^{n-1} < 1$, $\sum_{n=2}^{\infty} n^2|a_n||z|^{n-1} < 1$ and $\sum_{n=2}^{\infty} n^3|a_n||z|^{n-1} < 1$ (see Silverman [4]). Thus, we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n(n-1)\alpha + (n+1))a_n|z|^{n-1} \\
&= \sum_{n=2}^{\infty} n(n(n-1)\alpha)a_n|z|^{n-1} + \sum_{n=2}^{\infty} n(n+1)a_n|z|^{n-1} \\
&= \alpha \sum_{n=2}^{\infty} (n^3 - n^2)a_n|z|^{n-1} + \sum_{n=2}^{\infty} n^2 a_n|z|^{n-1} + \sum_{n=2}^{\infty} n a_n|z|^{n-1} \\
&= \alpha \sum_{n=2}^{\infty} n^3 a_n|z|^{n-1} - \alpha \sum_{n=2}^{\infty} n^2 a_n|z|^{n-1} + \sum_{n=2}^{\infty} n^2 a_n|z|^{n-1} + \sum_{n=2}^{\infty} n a_n|z|^{n-1} \\
&< \alpha - \alpha + 1 + 1 \\
&= 2
\end{aligned}$$

3 Main Results

In this section, we give results concerning the coefficient estimates, growth and extreme points for the functions $f \in CT(\alpha)$.

Theorem 3.1 *$f \in CT(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n^2((n-1)\alpha + 1)a_n \le 1 \tag{7}$$

Proof: We adopt the method used by Clunie and Keogh [1] and also by Owa [2]. First we prove the 'if' part. According to Definition 1.3

$$\begin{aligned}
&\left|\alpha(z^2f''(z))' + (zf'(z))' - f'(z)\right| - \left|\alpha(z^2f''(z))' + (zf'(z))' + f'(z)\right| \\
&= \left|-\sum_{n=2}^{\infty}(n(n-1)(n\alpha+1))a_nz^{n-1}\right| - \left|2-\sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_nz^{n-1}\right| \\
&\le \sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_nr^{n-1} - 2 + \sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_nr^{n-1} \\
&< \sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_n - 2 + \sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_n \\
&= \sum_{n=2}^{\infty}\Big(n(n-1)(n\alpha+1) + n(n(n-1)\alpha+(n+1))\Big)a_n - 2 \\
&= \sum_{n=2}^{\infty}2n^2((n-1)\alpha+1)a_n - 2 \\
&\le 0 \text{ by } (7)
\end{aligned}$$

Thus,

$$\left|\frac{\frac{\alpha(z^2f''(z))'+(zf'(z))'}{f'(z)} - 1}{\frac{\alpha(z^2f''(z))'+(zf'(z))'}{f'(z)} + 1}\right| < 1$$

and hence $f \in CT(\alpha)$.
To prove the 'only if' part, let

$$\left|\frac{\frac{\alpha(z^2f''(z))'+(zf'(z))'}{f'(z)} - 1}{\frac{\alpha(z^2f''(z))'+(zf'(z))'}{f'(z)} + 1}\right| = \left|\frac{-\sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_nz^{n-1}}{2-\sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_nz^{n-1}}\right| < 1.$$

We note that since f is analytic, continuous and non constant in D, the maximum modulus principle gives

$$\begin{aligned}
&\left|\frac{-\sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_nz^{n-1}}{2-\sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_nz^{n-1}}\right| \\
&= \frac{\left|\sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_nz^{n-1}\right|}{\left|2-\sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_nz^{n-1}\right|} \\
&\le \frac{\sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_n|z|^{n-1}}{2-\sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_n|z|^{n-1}} \\
&\le \frac{\sum_{n=2}^{\infty}n(n-1)(n\alpha+1)a_nr^{n-1}}{2-\sum_{n=2}^{\infty}n(n(n-1)\alpha+(n+1))a_nr^{n-1}}
\end{aligned}$$

where $\sum_{n=2}^{\infty} n(n(n-1)\alpha + (n+1))a_n|z|^{n-1} < 2$ from Lemma 2.1. Since $f \in CT(\alpha)$ and $0 < r < 1$, we obtain

$$\left(\frac{\sum_{n=2}^{\infty} n(n-1)(n\alpha+1)a_n r^{n-1}}{2-\sum_{n=2}^{\infty} n(n(n-1)\alpha+(n+1))a_n r^{n-1}} \right) < 1 \tag{8}$$

Now letting $r \to 1$, in (8) and using Lemma 2.1, we obtain

$$\sum_{n=2}^{\infty} n(n-1)(n\alpha+1)a_n \leq 2 - \sum_{n=2}^{\infty} n(n(n-1)\alpha+(n+1))a_n$$

and hence

$$\sum_{n=2}^{\infty} n^2((n-1)\alpha+1)a_n \leq 1$$

as required. Thus the proof of the Theorem 3.1 is completed.

The result is sharp for functions given by

$$f_n(z) = z - \frac{1}{n^2((n-1)\alpha+1)} z^n, n \geq 2.$$

Corollary 3.1 *If $f \in CT(\alpha)$ then*

$$a_n \leq \frac{1}{n^2((n-1)\alpha+1)}, \qquad n \geq 2.$$

Next, we give the result for growth of the class $CT(\alpha)$.

Theorem 3.2 *Let the functions f be defined by (4) and belongs to the class $CT(\alpha)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r - \frac{1}{4(\alpha+1)} r^2 \leq |f(z)| \leq r + \frac{1}{4(\alpha+1)} r^2$$

The result is sharp.

Proof: First, it is obvious that

$$4(\alpha+1)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n^2((n-1)\alpha+1)a_n$$

and as $f \in CT(\alpha)$, using inequality in Theorem 3.1 yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{4(\alpha+1)} \tag{9}$$

From (4) with $|z| = r(r < 1)$, we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^2$$

and

$$|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \geq r - \sum_{n=2}^{\infty} a_n r^2$$

Finally, using (9) in the above inequalities, gives the result in Theorem 3.2.

We note that result in Theorem 3.2 is sharp for the following function,

$$f_2(z) = z - \frac{1}{4(\alpha+1)} z^2$$

at $z = \pm r$.

Here, we consider extreme points for functions $f \in CT(\alpha)$.

Theorem 3.3 *Let $f_1(z) = z$ and $f_n(z) = z - \frac{1}{n^2((n-1)\alpha+1)} z^n$ for $n \geq 2$. Then $f \in CT(\alpha)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.*

Proof: Adopting the same technique used by Silverman [4], we first assume

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \lambda_n \left\{ \frac{1}{n^2((n-1)\alpha+1)} \right\} z^n. \end{aligned}$$

Next, since

$$\begin{aligned} &\sum_{n=2}^{\infty} \lambda_n \left\{ \frac{1}{n^2((n-1)\alpha+1)} \right\} \cdot \left\{ \frac{n^2((n-1)\alpha+1)}{1} \right\} \\ &= \sum_{n=2}^{\infty} \lambda_n \\ &= 1 - \lambda_1 \\ &\leq 1, \end{aligned}$$

therefore by Theorem 3.3, $f \in CT(\alpha)$.

Conversely, suppose $f \in CT(\alpha)$. Since

$$a_n \leq \frac{1}{n^2((n-1)\alpha+1)}, n \geq 2,$$

we may set $\lambda_n = n^2((n-1)\alpha+1)a_n, (n \geq 2)$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n f_n(z) &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\
&= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \left\{ z - \frac{1}{n^2((n-1)\alpha+1)} z^n \right\} \\
&= \left(1 - \sum_{n=2}^{\infty} \lambda_n\right) z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \lambda_n \frac{1}{n^2((n-1)\alpha+1)} z^n \\
&= z - \sum_{n=2}^{\infty} \lambda_n \frac{1}{n^2((n-1)\alpha+1)} z^n \\
&= z - \sum_{n=2}^{\infty} a_n z^n \\
&= f(z)
\end{aligned}$$

Hence, we complete the proof of Theorem 3.3.

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