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Power Type α-Centroidal Mean and Its Dual

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Abstract

The paper defines the power type α-centroidal mean and its dual form in two variables. Some interesting results related to monotonicities as well have been obtained.

Keywords: *Monotonicity, inequality, contra harmonic mean, centroidal mean.*

1 Introduction

Mathematical means defined by pythagorean school are considered as the foremost contribution from ancient Greeks ([1], [11]). On the basis of propositions, four fundamental named means are specified as arithmetic mean, geometric mean, harmonic mean and contra harmonic mean.

Among the new means, an important mean which has engrossed the attention to explore, is the power mean.

Let $a, b > 0$ be positive real numbers. The power mean of order $r \in \Re$ of a and b is defined by

$$M_r = M_r(a,b) = (\tfrac{a^r+b^r}{2})^{1/r}$$

for some particular value of r, we can get primary means as given below.

- $A = M_1(a,b) = \frac{a+b}{2}$,
- $G = G(a,b) = M_0(a,b) = \underset{k\to 0}{Lim} M_r(a,b) = \sqrt{ab}$
- $H = H(a,b) = M_{-1}(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}$

which are named as arithmetic, geometric and harmonic mean of a and b respectively.

For $a, b > 0$

$$L(a,b) = \begin{cases} \frac{a-b}{\ln a - \ln b} & a \neq b \\ a & a = b \end{cases} \tag{1.1}$$

$$I(a,b) = \begin{cases} e^{\left(\frac{a \ln a - b \ln b}{a-b} - 1\right)} & a \neq b \\ a & a = b \end{cases} \tag{1.2}$$

$$H_e(a,b) = \frac{a + \sqrt{ab} + b}{3} \tag{1.3}$$

$$H_p(a,b) = \left(\frac{a^p + \sqrt{a^p b^p} + b^p}{3}\right)^{\frac{1}{p}} \tag{1.4}$$

are respectively called logarithmic mean, identric mean, heron mean and power $-$type generalized heron mean.

In [11], the definition of contra-harmonic mean on the basis of proportions is given by;

$$C(a,b) = \frac{a^2 + b^2}{a+b}. \tag{1.5}$$

Many researchers have explored various means and their properties through the above said fundamental means(refer [3] - [9]), obtained some remarkable inequalities and identities. The mean inequalities collections are mentioned in [2]. In ([4], [6], [10]), the authors has defined oscillatory mean, r^{th} oscillatory mean and obtained some new inequalities. Further, obtained the best possible values of these means with logarithmic mean, identric mean and power mean.

Definition 1.1. *[10] For $a, b > 0$ and $\alpha \in (0,1)$, the oscillatory mean and its dual form are as follows;*

$$O(a,b;\alpha) = \alpha G(a,b) + (1-\alpha)A(a,b) \tag{1.6}$$

and

$$O^{(d)}(a,b;\alpha) = G^{\alpha}(a,b)A^{1-\alpha}(a,b). \tag{1.7}$$

Definition 1.2. *[1] For $a, b > 0$, the extended mean is defined as*

$$E_{s,t}(a,b) = \begin{cases} \left(\frac{t(a^s-b^s)}{s(a^t-b^t)}\right)^{\frac{1}{s-t}}, & if\ (s-t)st \neq 0,\ a \neq b \\ exp\left(-\frac{1}{s} + \frac{a^s loga - b^s logb}{a^s-b^s}\right), & if\ s = t \neq 0,\ a \neq b \\ exp\left(\frac{a^s-b^s}{s(a^s loga - b^s logb)}\right)^{\frac{1}{s}}, & if\ s \neq 0, t = 0,\ a \neq b \\ \sqrt{ab}, & if\ s = t = 0 \\ a, & if\ a = b. \end{cases} \tag{1.8}$$

Definition 1.3. *For $a > b > 0$ centroidal means is defined as*

$$E_{2,3}(a,b) = \frac{2}{3}\left(\frac{a^2+ab+b^2}{a+b}\right) \tag{1.9}$$

In [8] K.M. Nagaraja et. al. have introduced the α-centroidal mean and its dual as follows.

Definition 1.4. *For $a, b >$ 0 and $\alpha \in (0,1)$, α-centroidal mean and its dual form are respectively defined as follows:*

$$CT(a,b;\alpha) = \alpha H(a,b) + (1-\alpha)C(a,b) \tag{1.10}$$

and

$$CT^{(d)}(a,b;\alpha) = H^{\alpha}(a,b)C^{1-\alpha}(a,b). \tag{1.11}$$

This motivates us to study power type α-centroidal mean and its dual. Also we have established some fascinating results and inter-related inequalities.

2 Power Type α-Centroidal Mean and its Dual

In this section, the power type α-centroidal mean and its dual are introduced as follows.

Definition 2.1. *For $a, b >$ 0 and $\alpha \in (0,1)$, power type α-centroidal mean and its dual form are respectively defined as follows:*

$$CT(a,b;\alpha,k) = \begin{cases} \left(\alpha H(a^k,b^k) + (1-\alpha)C(a^k,b^k)\right)^{\frac{1}{k}}, & for\ k \neq 0 \\ \sqrt{ab}, & for\ k = 0 \end{cases} \tag{2.1}$$

and

$$CT^{(d)}(a,b;\alpha,k) = \begin{cases} \Big(H^{\alpha}(a^k,b^k)C^{1-\alpha}(a^k,b^k)\Big)^{\frac{1}{k}}, & \textit{for } k \neq 0 \\ \sqrt{ab}, & \textit{for } k = 0 \end{cases} \tag{2.2}$$

Note : For $\alpha = \mathbf{\frac{1}{3}}$ and $k = 1$, the $CT(a,b;\alpha,k) = E_{2,3}(a,b)$ is called as centroidal mean.

For $\alpha \in (0,1)$ the power type α - centroidal mean and its dual satisfies the following properties.

Property 2.1. *Power type α - centroidal mean and its dual are means. That is*

$$Min\{a,b\} \leq \{CT(a,b;\alpha,k), CT^d(a,b;\alpha,k)\} \leq Max\{a,b\}$$

Property 2.2. *The means $CT(a,b;\alpha,k)$ and $CT^{(d)}(a,b;\alpha,k)$ are symmetric and homogeneous;*

1. **Symmetric** :
   $CT(a,b;\alpha,k) = CT(b,a;\alpha,k)$ *and* $CT^{(d)}(a,b;\alpha,k) = CT^{(d)}(b,a;\alpha,k)$.

2. **Homogeneous** :
   $CT(at,bt;\alpha,k) = tCT(a,b;\alpha,k)$ *and* $CT^{(d)}(at,bt;\alpha,k) = tCT^{(d)}(a,b;\alpha,k)$.

Proposition 2.1. *According to definition 2.1, the following characteristic properties for $CT(a,b;\alpha,k)$ and $CT^{(d)}(a,b;\alpha,k)$ are straightforward.*

For a real number $\alpha \in (0,1)$,

1. $min(a,b) \leq CT^{(d)}(a,b;\alpha,k) \leq CT(a,b;\alpha,k) \leq Max(a,b)$.
2. $H(a,b) \leq CT^{(d)}(a,b;\alpha,k) \leq CT(a,b;\alpha,k) \leq C(a,b)$.
3. $CT(a,b;\alpha,1) = CT(a,b;\alpha)$.
4. $CT(a,b;\alpha,0) = G(a,b)$.
5. $CT(a,b;\frac{1}{2},r) = M_r(a,b)$.
6. $CT(a,b;\frac{1}{2},k) = \frac{A(a,b)}{G(a,b)} = \frac{1}{H(a,b)}$.
7. $CT(a,b;\frac{1}{2},1) = A(a,b)$.
8. $CT(a,b;\frac{1}{3},1) = \frac{1}{3}(4A(a,b) - H(a,b))$.
9. $CT(a,b;\frac{2}{3},1) = \frac{2}{3}(C(a,b) + H(a,b))$.

10. $CT^{(d)}(a,b;\alpha,1) = CT^{(d)}(a,b;\alpha)$.

11. $CT^{(d)}(a,b;\frac{1}{2},1) = \frac{G(a,b)}{A(a,b)}\sqrt{A(a^2,b^2)}$.

12. $CT^{(d)}(a,b;\frac{1}{2},\frac{1}{2}) = \frac{2G(a,b)A(a,b)}{A(\sqrt{a},\sqrt{b})}$.

13. $CT^{(d)}(a,b;\frac{1}{3},1) = (H(a,b)C^2(a,b))^{\frac{1}{3}}$.

14. $CT^{(d)}(a,b;\frac{2}{3},-1) = \frac{1}{(H(a,b)C^2(a,b))^{\frac{1}{3}}}$.

3 Monotonic Results

In this section, the monotonic results of the power type α-centroidal mean and its dual are studied.

Theorem 3.1. *For $\alpha \in (0,1)$ a real number and for $a,b > 0$,*

$$CT^{(d)}(a,b;\alpha,k) \leq CT(a,b;\alpha,k).$$

Proof. The proof of theorem 3.1 follows from well known power mean inequality:

$$M_r(a,b) = \begin{cases} \left(\dfrac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{3.1}$$

□

Theorem 3.2. *For $a,b > 0$ and $\alpha \in (0,1)$, the power type α-centroidal mean $CT(a,b;\alpha,k)$ is an decreasing function with respect to α.*

$$CT(a,b;\alpha+1,k) \leqslant CT(a,b;\alpha,k) \tag{3.2}$$

Proof. From definition 2.1,

$$\begin{aligned}
CT(a,b;\alpha+1,k) =& \left[(\alpha+1)H(a^k,b^k) + [1-(1+\alpha)]C(a^k,b^k)\right]^{\frac{1}{k}} \\
=& \left[(\alpha+1)H(a^k,b^k) + (-\alpha)C(a^k,b^k)\right]^{\frac{1}{k}} \\
=& \left[(\alpha)H(a,b) + H(a^k,b^k) + (-\alpha)C(a^k,b^k)\right]^{\frac{1}{k}} \\
\leqslant& \left[\alpha H(a^k,b^k) + C(a^k,b^k) + (-\alpha)C(a^k,b^k)\right]^{\frac{1}{k}} \\
=& \left[\alpha H(a^k,b^k) + (1-\alpha)C(a^k,b^k)\right]^{\frac{1}{k}} \\
=& CT(a,b;\alpha,k).
\end{aligned}$$

□

Theorem 3.3. *For $a,b>0$ and $\alpha \in (0,1)$, the power type α-centroidal dual mean $CT^{(d)}(a,b;\alpha,k)$ is an decreasing function with respect to α.*

$$CT^{(d)}(a,b;\alpha+1,k) \leqslant CT^{(d)}(a,b;\alpha,k) \tag{3.3}$$

Proof. From definition 2.1,

$$\begin{aligned}
CT^{(d)}(a,b;\alpha+1,k) =& \left[H^{(\alpha+1)}(a^k,b^k)C^{[1-(1+\alpha)]}(a^k,b^k)\right]^{\frac{1}{k}} \\
=& \left[H^{(\alpha+1)}(a^k,b^k)C^{(-\alpha)}(a^k,b^k)\right]^{\frac{1}{k}} \\
=& \left[H(a^k,b^k)H^{(\alpha)}(a^k,b^k)C^{(-\alpha)}(a^k,b^k)\right]^{\frac{1}{k}} \\
\leqslant& \left[C(a^k,b^k)H^{(\alpha)}(a^k,b^k)C^{(-\alpha)}(a^k,b^k)\right]^{\frac{1}{k}} \\
=& \left[H^{(\alpha)}(a^k,b^k)C^{(1-\alpha)}(a^k,b^k)\right]^{\frac{1}{k}} \\
=& CT^{(d)}(a,b;\alpha,k).
\end{aligned}$$

□

4 Some Inequalities

In this section, we obtain the Taylor's series expansions of various means by replacing $a = t$ and $b = 1$ and inter-relate with known means with the best possible value for each relation.

$$A(a, b) = A(t, 1) = 1 + \frac{t}{2} \tag{4.1}$$

$$H(a, b) = H(t, 1) = 1 + \frac{t}{2} - \frac{1}{4}t^2 + ..., \tag{4.2}$$

$$G(a, b) = G(t, 1) = 1 + \frac{t}{2} - \frac{1}{8}t^2 + ..., \tag{4.3}$$

$$C(a, b) = C(t, 1) = 1 + \frac{t}{2} + \frac{1}{4}t^2 + ..., \tag{4.4}$$

$$O(t, 1; \alpha) = 1 + \frac{1}{2}t + \frac{-\alpha}{8}t^2 + ... \tag{4.5}$$

$$O^{(d)}(t, 1; \alpha) = 1 + \frac{1}{2}t + \frac{-\alpha}{8}t^2 + ... \tag{4.6}$$

$$L(a, b) = L(t, 1) = 1 + \frac{t}{2} - \frac{1}{12}t^2 + ..., \tag{4.7}$$

$$I(a, b) = I(t, 1) = 1 + \frac{t}{2} - \frac{1}{24}t^2 + ..., \tag{4.8}$$

$$H_p(a, b) = H(t^p, 1) = 1 + \frac{t}{2} + \frac{2p - 3}{24}t^2 + ..., \tag{4.9}$$

$$H_e(a, b) = H_e(t, 1) = 1 + \frac{t}{2} - \frac{1}{24}t^2 + ..., \tag{4.10}$$

$$M_r(a, b) = M_r(t, 1) = 1 + \frac{t}{2} + \frac{r - 1}{8}t^2 + ..., \tag{4.11}$$

$$CT(t, 1; \alpha) = 1 + \frac{1}{2}t - \frac{(1 - 2\alpha)}{4}t^2 + ..., \tag{4.12}$$

$$CT^{(d)}(t, 1; \alpha) = 1 + \frac{1}{2}t - \frac{(1 - 2\alpha)}{4}t^2 + ..., \tag{4.13}$$

$$CT(t, 1; \alpha, k) = 1 + \frac{1}{2}t - \frac{(3 - 4\alpha)k - 1}{8}t^2 + ..., \tag{4.14}$$

$$CT^{(d)}(t, 1; \alpha, k) = 1 + \frac{1}{2}t - \frac{(3 - 4\alpha)k - 1}{8}t^2 + ..., \tag{4.15}$$

From Theorem 3.1 and Taylor's series expansion of various means from 4.2 to 4.15, the following inequalities for $a, b > 0$ and $\alpha \in (0, 1)$ is computed.

Proposition 4.1. *For $k_1 \leq \frac{1}{3(3-4\alpha)} \leq k_2$, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq L(a,b) \leq CT(a,b;\alpha,k_2). \tag{4.16}$$

Further, $k_1 = k_2 = \frac{1}{3(3-4\alpha)}$ is the best possible for (4.16).

Proof. From equations 4.2 to 4.15, we have

$$CT^{(d)}(a,b;\alpha,k_1) \leq L(a,b) \leq CT(a,b;\alpha,k_2)$$

holds whenever, $\frac{(3-4\alpha)k_1-1}{8} \leq \frac{-1}{12} \leq \frac{(3-4\alpha)k_2-1}{8}$

on rearrangement leads to $k_1 \leq \frac{1}{3(3-4\alpha)} \leq k_2$. □

Proposition 4.2. *For $k_1 \leq \frac{2}{3(3-4\alpha)} \leq k_2$, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq I(a,b) \leq CT(a,b;\alpha,k_2). \tag{4.17}$$

Further, $k_1 = k_2 = \frac{2}{3(3-4\alpha)}$ is the best possible for (4.17).

Proof. From equations 4.2 to 4.15, we have

$$CT^{(d)}(a,b;\alpha,k_1) \leq I(a,b) \leq CT(a,b;\alpha,k_2)$$

holds whenever, $\frac{(3-4\alpha)k_1-1}{8} \leq \frac{-1}{24} \leq \frac{(3-4\alpha)k_2-1}{8}$

on rearrangement leads to $k_1 \leq \frac{2}{3(3-4\alpha)} \leq k_2$. □

Proposition 4.3. *For $k_1 \leq \frac{3}{(3-4\alpha)} \leq k_2$, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq C(a,b) \leq CT(a,b;\alpha,k_2). \tag{4.18}$$

Further, $k_1 = k_2 = \frac{3}{3-4\alpha}$ is the best possible for (4.18).

Proof. From equations 4.2 to 4.15, we have

$$CT^{(d)}(a,b;\alpha,k_1) \leq C(a,b) \leq CT(a,b;\alpha,k_2)$$

holds whenever, $\frac{(3-4\alpha)k_1-1}{8} \leq \frac{1}{4} \leq \frac{(3-4\alpha)k_2-1}{8}$

on rearrangement leads to $k_1 \leq \frac{3}{3-4\alpha} \leq k_2$. □

Proposition 4.4. *For* $k_1 \leq \frac{2p}{3(3-4\alpha)} \leq k_2$*, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq H_p(a,b) \leq CT(a,b;\alpha,k_2). \tag{4.19}$$

Further, $k_1 = k_2 = \frac{2p}{3(3-4\alpha)}$ *is the best possible for (4.19).*

Proposition 4.5. *For* $k_1 \leq \frac{1-\alpha}{3-4\alpha} \leq k_2$*, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq O(a,b,\alpha) \leq CT(a,b;\alpha,k_2). \tag{4.20}$$

Further, $k_1 = k_2 = \frac{1-\alpha}{3-4\alpha}$ *is the best possible for (4.20).*

Proposition 4.6. *For* $k_1 \leq \frac{r}{3-4\alpha} \leq k_2$*, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq M_r(a,b) \leq CT(a,b;\alpha,k_2). \tag{4.21}$$

Further, $k_1 = k_2 = \frac{r}{3-4\alpha}$ *is the best possible for (4.21).*

Proposition 4.7. *For* $k_1 \leq \frac{4\alpha-1}{3-4\alpha} \leq k_2$*, the following double inequality holds*

$$CT^{(d)}(a,b;\alpha,k_1) \leq CT(a,b,\alpha) \leq CT(a,b;\alpha,k_2). \tag{4.22}$$

Further, $k_1 = k_2 = \frac{4\alpha-1}{3-4\alpha}$ *is the best possible for (4.22).*

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