Gen. Math. Notes, Vol. 26, No. 2, February 2015, pp.48-58
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
www.i-csrs.org
Available free online at http://www.geman.in

An Integral Operator which Preserves the Univalence

Horiana Tudor

Department of Mathematics
Faculty of Mathematics and Computer Science
"Transilvania" University of Braşov, 500091, Braşov, Romania
E-mail: horianatudor@yahoo.com

(Received: 18-12-14 / Accepted: 4-2-15)

Abstract

We obtain a sufficient condition for the analyticity and the univalence of a class of functions defined by an integral operator. This integral operator preserves the class of univalent functions.

Keywords: *Löewner chain, univalent functions, integral operator.*

1 Introduction

Let $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r\}$, $0 < r \leq 1$, be the disk of radius r centered at 0, let $\mathcal{U} = \mathcal{U}_1$ be the unit disk and let $I = [0, \infty)$.

Denote by $\mathcal{A}$ the class of analytic functions in $\mathcal{U}$ which satisfy the usual normalization $f(0) = f'(0) - 1 = 0$ and let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$, f univalent in $\mathcal{U}$.

An important problem in the theory of univalent functions is to find integral operators which preserve the class of univalent functions. We mention the well known integral operators due to Kim and Merkes [3], Pfaltzgraff [9], Moldoveanu and Pascu [6] and the recently generalization of these results obtained by author in [13].

A function $f \in \mathcal{S}$ is said to be in the class of φ-spirallike functions of order ρ in $\mathcal{U}$, denoted by $\mathcal{S}^*(\varphi, \rho)$, if

$$Re\left(e^{i\varphi}\frac{zf'(z)}{f(z)}\right) > \rho\cos\varphi, \quad z\in\mathcal{U}\ ,$$

where $\varphi\in(-\pi/2,\ \pi/2)$, $\rho\in[0,1)$.

The class $\mathcal{S}^*(\varphi,\rho)$ was studied by Libera [4] and Keogh and Merkes [2]. Note that $\mathcal{S}^*(\varphi,0)$ is the class of spirallike functions introduced by Špaček [11], $\mathcal{S}^*(0,\rho)=\mathcal{S}^*(\rho)$ is the class of starlike functions of order ρ and $\mathcal{S}^*(0,0)=\mathcal{S}^*$ is the familiar class of starlike functions.

Before proving our main result we need a brief summary of theory of Löewner chains.

A function $L(z,t):\mathcal{U}\times I\to\mathbb{C}$ is said to be a *Löewner chain* or a *subordination chain* if $L(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t\in I$ and $L(z,t)\prec L(z,s)$ for all $0\le t\le s<\infty$, where the symbol " $\prec$ " stands for subordination.

The following result due to Pommerenke is often used to obtain univalence criteria.

Theorem 1.1. ([10]). *Let $L(z,t)=a_1(t)z+a_2(t)z^2+\ldots$, $a_1(t)\neq 0$, be an analytic function in $\mathcal{U}_r$ for all $t\in I$, locally absolutely continuous in I, locally uniform with respect to $\mathcal{U}_r$. For almost all $t\in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z}=p(z,t)\frac{\partial L(z,t)}{\partial t}\ , \qquad \forall z\in\mathcal{U}_r\ , \tag{1}$$

where $p(z,t)$ is analytic in $\mathcal{U}$ and satisfies the condition $\Re p(z,t)>0$ for all $z\in\mathcal{U}$, $t\in I$. If $\lim\limits_{t\to\infty}|a_1(t)|=\infty$ and $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\ge 0}$ forms a normal family in $\mathcal{U}_r$, then for each $t\in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$.

At the end of this section we formulate lemmas which will be used in the following sections.

Lemma 1.2. ([1]). *Let $f\in S$. Then*

$$\left|\frac{zf'(z)}{f(z)}\right|\le\frac{1+|z|}{1-|z|}\ , \qquad \forall z\in\mathcal{U}, \tag{2}$$

and

$$\left|-2|z|^2+(1-|z|^2)\frac{zf''(z)}{f'(z)}\right|\le 4|z|\ , \qquad \forall z\in\mathcal{U}. \tag{3}$$

Lemma 1.3. ([5]). *If $f\in\mathcal{S}^*(\varphi,\rho)$ and a is a fixed point from the unit disk $\mathcal{U}$, then the function h,*

$$h(z)=\frac{a\cdot z}{f(a)(z+a)(1+\bar{a}z)^{\psi}}\cdot f\left(\frac{z+a}{1+\bar{a}z}\right) \tag{4}$$

where

$$\psi = e^{-2i\varphi} - 2\rho e^{-i\varphi}\cos\varphi \tag{5}$$

is a function of the class $\mathcal{S}^*(\varphi,\rho)$.

Lemma 1.4. ([8]). *Let* f *be an analytic function in* $\mathcal{U}$, $f(0)=0$ *and* M *a positive real number . If* $\Re f(z) \le M$ *for all* $z\in\mathcal{U}$, *then*

$$|\, f(z)\,| \le \frac{2M\cdot|z|}{1-|z|}\,, \qquad \forall z\in\mathcal{U}.$$

2 Main Result

Making use of Theorem 1.1, the essence of which is the construction of suitable Löewner chain, we can prove our main result.

Theorem 2.1. *Let* α, c *be complex numbers,* n *natural number,* $n\ge 1$, *such that*

$$\left|\,\alpha - 2 + \frac{1}{n}\,\right| < 1 \qquad and \qquad |\,c\,| < 1. \tag{6}$$

For $f\in\mathcal{A}$, *if the inequality*

$$\left|\, c|z|^{2n} + (1-|z|^{2n})\left[\,(\alpha-1)\frac{z^n f'(z^n)}{f(z^n)} + \frac{1-n}{n}\,\right]\,\right| \le 1 \tag{7}$$

is true for all $z\in\mathcal{U}$, *then the function*

$$F_{n,\alpha}(z) = \left[(n(\alpha-1)+1)\int_0^z f^{\alpha-1}(u^n)du\right]^{\frac{1}{n(\alpha-1)+1}}\,, \tag{8}$$

where the principal branch is intended, is analytic and univalent in $\mathcal{U}$.

Proof. Let us prove that there exists a real number $r\in(0,1]$ such that the function $L:\mathcal{U}_r\times I \longrightarrow \mathbb{C}$, defined formally by

$$L(z,t) = \left[\int_0^{e^{-t}z} f^{\alpha-1}(u^n)du \;+\; \frac{e^{2nt}-1}{n(1+c)}e^{-t}z\cdot f^{\alpha-1}(e^{-nt}z^n)\right]^{\frac{1}{n(\alpha-1)+1}} \tag{9}$$

is analytic in $\mathcal{U}_r$ for all $t\in I$.

From the analyticity of the function f in $\mathcal{U}$ it follows that the function $h_1(z)=\frac{f(z^n)}{z^n}$ is analytic in $\mathcal{U}$ and since $h_1(0)=1$ there is a disk $\mathcal{U}_{r_1}$, $r_1\in(0,1]$, in which $h_1(z)\neq 0$. Therefore we can choose the uniform branch of $(h_1(z))^{\alpha-1}$ equal to 1 at the origin, denoted by h_2. It is easy to see that the function

$$h_3(z,t) = \int_0^{e^{-t}z} u^{n(\alpha-1)}h_2(u)du$$

can be written as $h_3(z,t) = z^{n(\alpha-1)+1} h_4(z,t)$, where h_4 is also analytic in $\mathcal{U}_{r_1}$. The function

$$h_5(z,t) = h_4(z,t) \ + \ \frac{e^{2nt}-1}{n(1+c)} e^{-[n(\alpha-1)+1]t} \cdot h_2(e^{-t}z)$$

is analytic in $\mathcal{U}_{r_1}$ and

$$h_5(0,t) = e^{-[n(\alpha-1)+1]t} \left[\frac{e^{2nt}}{n(1+c)} + \frac{1}{n(\alpha-1)+1} - \frac{1}{n(1+c)} \right] .$$

Let us now prove that $h_5(0,t) \neq 0$ for any $t \in I$. We have $h_5(0,0) = \frac{1}{n(\alpha-1)+1}$. Assume that there exists $t_0 > 0$ such that $h_5(0,t_0) = 0$. Then $e^{2nt_0} = 1 - \frac{n(1+c)}{n(\alpha-1)+1}$ and because $\frac{n(1+c)}{n(\alpha-1)+1}$ is a real number only in the case

$$\frac{\Re\, n(1+c)}{\Re\, n(\alpha-1)+1} = \frac{\Im\, n(1+c)}{\Im\, n(\alpha-1)+1} = \mu \ \ ,$$

from hypothesis (6) we obtain $\mu > 0$ and then we conclude that $h_5(0,t) \neq 0$ for all $t \in I$. Therefore there is a disk $\mathcal{U}_{r_2}$, $r_2 \in (0, r_1]$, in which $h_5(z,t) \neq 0$ for all $t \in I$. So, we can choose an uniform branch of $[h_5(z,t)]^{1/[n(\alpha-1)+1]}$ analytic in $\mathcal{U}_{r_2}$, denoted by $h_6(z,t)$, that is equal to

$$a_1(t) = e^{\frac{2n-n(\alpha-1)-1}{n(\alpha-1)+1}t} \left[\frac{1}{n(1+c)} + \left(\frac{1}{n(\alpha-1)+1} - \frac{1}{n(1+c)} \right) e^{-2nt} \right]^{\frac{1}{n(\alpha-1)+1}}$$

at the origin and for $a_1(t)$ we fix a determination. From these considerations , it follows that relation (9) may be written as $L(z,t) = z \cdot h_6(z,t) = a_1(t)z + \ldots$ and then function $L(z,t)$ is analytic in $\mathcal{U}_{r_2}$.

Under the assumption of the theorem, $|(\ \alpha - 1 + \frac{1}{n} \ ) - 1| < 1$, which is equivalent to $\Re \ \frac{1}{\alpha - 1 + \frac{1}{n}} > \frac{1}{2}$ we have $\Re \ \frac{2n-n(\alpha-1)-1}{n(\alpha-1)+1} > 0$ and then $\lim\limits_{t\to\infty} |a_1(t)| = \infty$. We saw also that $a_1(t) \neq 0$ for all $t \in I$. From the analyticity of $L(z,t)$ in $\mathcal{U}_{r_2}$, it follows that there exists a number $r_3 \in (0, r_2\ ]$ and a constant $k = k(r_3)$ such that

$$|\ L(z,t)/a_1(t)\ | < k, \qquad z \in \mathcal{U}_{r_3},$$

and hence $\{L(z,t)/a_1(t)\}$ forms a normal family in $\mathcal{U}_{r_3}$. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and r_4, $0 < r_4 \leq r_3$, there exists a constant $K_1 > 0$ (that depends on T and r_4) such that $\left| \ \frac{\partial L(z,t)}{\partial t} \ \right| < K_1, \qquad \forall z \in \mathcal{U}_{r_4}, \quad t \in [0,T]$. It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to $z \in \mathcal{U}_{r_4}$. The function $p(z,t)$ defined by (1) is analytic in a disk $\mathcal{U}_r$, $r \in (0, \ r_4]$, for all $t \in I$. In order to prove that function $p(z,t)$ is analytic and has positive real part in $\mathcal{U}$, we will show that function $w(z,t) = (p(z,t)-1)/(p(z,t)+1)$, $z \in \mathcal{U}_r$, $t \in I$, is analytic in $\mathcal{U}$, and

$$| \, w(z,t) \, | \; < \; 1, \qquad \forall z \in \mathcal{U}, \quad t \in I \, .$$

Elementary calculation gives

$$w(z,t) = c \cdot e^{-2nt} + (1-e^{-2nt}) \left[\, (\alpha-1) \frac{e^{-nt} z^n f'(e^{-nt} z^n)}{f(e^{-nt} z^n)} + \frac{1-n}{n} \right] \quad (10)$$

We have $w(z,0) = c$ and $w(0,t) = ce^{-2nt} + (1-e^{-2nt})(\alpha - 1 + \frac{1-n}{n})$. In view of (6) we obtain that

$$| \, w(z,0) \, | < 1 \quad \text{and also} \quad | \, w(0,t) \, | < 1 \qquad (11)$$

From (7) we deduce that $f(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$ and taking into account (11) we get that $w(z,t)$ is analytic in unit disk $\mathcal{U}$. Let t be a fixed positive number, $z \in \mathcal{U}, \; z \neq 0$. Since $|e^{-t} z| \leq e^{-t} < 1$ for all $z \in \overline{\mathcal{U}} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ we conclude that function $w(z,t)$ is analytic in $\overline{\mathcal{U}}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|. \qquad (12)$$

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (10) we get

$$w(e^{i\theta},t) = c|u|^{2n} + (1-|u|^{2n}) \left[\, (\alpha-1) \frac{u^n f'(u^n)}{f(u^n)} + \frac{1-n}{n} \, \right]$$

Since $u \in \mathcal{U}$, the inequality (7) implies $|w(e^{i\theta},t)| \leq 1$ and from (11) and (12) we conclude that $|w(z,t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$.

From Theorem 1.1 it follows that $L(z,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$, for each $t \in I$. In particular, for $t = 0$ we conclude that the function

$$L(z,0) = \left[\, \int_0^z \; f^{\alpha-1}(u^n) du \, \right]^{\frac{1}{n(\alpha-1)+1}}$$

is analytic and univalent in $\mathcal{U}$, and also function $F_{n,\alpha}(z)$ defined by (8) is analytic and univalent in $\mathcal{U}$. □

Remark 2.2. *For $n = 1$, we get the result given in [7]. Every many-valued function, throughout in the sequel, is taken with the principal branch.*

3 Integral Operator Preserves the Univalence

In this section we study the integral operator (8) if f is univalent in $\mathcal{U}$ or belongs to some special subclasses of univalent functions and we see the important role played by the constant c in Theorem 2.1.

Theorem 3.1. *Let $f \in \mathcal{S}$, $\alpha \in \mathbb{C}$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If*

$$|\alpha - 1| \leq \frac{1}{4n}, \tag{13}$$

then the function $F_{n,\alpha}(z)$ defined by (8) and all the functions $F_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \leq k \leq n-1$ are analytic and univalent in $\mathcal{U}$.

Proof. For $f \in \mathcal{S}$ and taking $c = 0$, it is easy to check that inequality (13) implies inequalities (6) and (7) of Theorem 2.1. Indeed, applying (2) we find that

$$(1-|z|^{2n}) \left| (\alpha - 1) \frac{z^n f'(z^n)}{f(z^n)} + \frac{1-n}{n} \right| \leq |\alpha - 1|(1-|z|^{2n}) \frac{1+|z|^n}{1-|z|^n} + \frac{n-1}{n}(1-|z|^{2n})$$

$$\leq |\alpha - 1| \left((1+|z|^n)\right)^2 + \frac{n-1}{n} \leq 4|\alpha - 1| + \frac{n-1}{n} \leq 1 .$$

We have also

$$\left| \alpha - 2 + \frac{1}{n} \right| \leq |\alpha - 1| + \frac{n-1}{n} \leq \frac{1}{4n} + \frac{n-1}{n} < 1.$$

It follows that $F_{n,\alpha}(z)$ defined by (8) is analytic and univalent in $\mathcal{U}$. Since $\frac{1}{4n} < \frac{1}{4(n-k)}$, for k natural number, $1 \leq k \leq n-1$, inequality (13) implies $|\alpha - 1| \leq \frac{1}{4(n-k)}$, and then all the functions $F_{n-k,\alpha}(z)$ defined by (8) are analytic and univalent in $\mathcal{U}$.

□

In the next we consider the case when the function f belongs to some subsets of $\mathcal{S}$ and we expect that the hypothesis (13) of the Theorem 3.1 becomes larger.

Theorem 3.2. *Let $f \in \mathcal{S}^*(\varphi, \rho)$, $\alpha \in \mathbb{C}$, $n \in \mathbb{N}^*$. If*

$$|\alpha - 1| < \frac{1}{n[\, 1 + 2(1-\rho)\cos\varphi \,]} \tag{14}$$

then the function $F_{n,\alpha}(z)$ defined by (8) and all the functions $F_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \leq k \leq n-1$ are analytic and univalent in $\mathcal{U}$.

Proof. For $f \in \mathcal{S}^*(\varphi,\rho)$, and h be defined by (4), $h(z) = z + a_2z^2 + \ldots$, we obtain

$$a_2 = \frac{h''(0)}{2} = (1-|a|^2)\frac{f'(a)}{f(a)} - \frac{1+\psi|a|^2}{a} \ ,$$

where ψ is given by (5). It follows that

$$\frac{a\cdot f'(a)}{f(a)} = \frac{1+a\cdot a_2+\psi|a|^2}{1-|a|^2} \tag{15}$$

It is known that for $f \in \mathcal{S}^*(\varphi,\rho)$, $f(z) = z + a_2z^2 + \ldots$, we have (see [1])

$$|\ a_2\ | \le 2(1-\rho)\cos\varphi \ \ . \tag{16}$$

In view of (15), from (7) we get

$$c|z|^{2n} + (1-|z|^{2n})(\alpha-1)\frac{z^nf'(z^n)}{f(z^n)} + (1-|z|^{2n})\frac{1-n}{n} \tag{17}$$

$$= c|z|^{2n} + (\alpha-1)(1+a_2z^n+\psi|z|^{2n}) + (1-|z|^{2n})\frac{1-n}{n}$$

$$= \left[\ c + (\alpha-1)\psi + \frac{n-1}{n}\ \right]|z|^{2n} + (\alpha-1)(1+a_2z^n) + \frac{1-n}{n}$$

Let $c = -[\ (\alpha-1)\psi + \frac{n-1}{n}\ ]$. Then $|\ c\ | = \left|(\alpha-1)(\psi+1-1) + \frac{n-1}{n}\ \right| \le |\alpha-1|(|\psi+1|+1) + \frac{n-1}{n}$ and since $|\psi+1| = 2(1-\rho)\cos\varphi$, in view of (14), it is clear that $|c| < 1$. It is easy to check that inequality (14) implies also $|\ \alpha - 2 + \frac{1}{n}\ | <\ 1$. Taking into account (14) and (16), the relation (17) reduces to

$$\left|\ c|z|^{2n} + (1-|z|^{2n})(\alpha-1)\frac{z^nf'(z^n)}{f(z^n)} + (1-|z|^{2n})\frac{1-n}{n}\right|$$

$$\le |\alpha-1|(1+|a_2|) + \frac{n-1}{n} <\ 1$$

The conditions of Theorem 2.1 are verified. It follows that $F_{n,\alpha}(z)$ defined by (8) is analytic and univalent in $\mathcal{U}$. □

Corollary 3.3. *Let* $f \in \mathcal{S}^*$, $\alpha \in \mathbb{C}$, $n \in \mathbb{N}^*$. *If*

$$|\alpha-1|\ <\ \frac{1}{3n} \tag{18}$$

then the function $F_{n,\alpha}(z)$ *defined by (8) and all the functions* $F_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1\ \le\ k\ \le\ n-1$ *are analytic and univalent in* $\mathcal{U}$.

Example 3.4. Consider the function $f \in \mathcal{S}^*(\varphi,\rho)$ defined by

$$f(z) = z(1-z)^{-2(1-\rho)e^{-i\varphi}\cos\varphi}$$

For $\alpha \in \mathbb{C}$, $n \in \mathbb{N}^*$ such that $|\alpha - 1| < \frac{1}{n[\,1+2(1-\rho)\cos\varphi\,]}$, the function

$$z \cdot \left[\, \Omega\left(2(\alpha-1)(1-\rho)e^{-i\varphi}\cos\varphi,\ \alpha+\frac{1}{n}-1,\ \alpha+\frac{1}{n}\,;z^n\right)\right]^{\frac{1}{n(\alpha-1)+1}}$$

is analytic and univalent in $\mathcal{U}$, where by $\Omega(a,b,c;z)$ we denoted the hypergeometric function. The conditions of Theorem 3.2 are satisfied and from (8) we obtain that

$$F_{n,\alpha}(z) = \left[(n(\alpha-1)+1)\int_0^z u^{n(\alpha-1)}(1-u^n)^{-2(\alpha-1)(1-\rho)e^{-i\varphi}\cos\varphi}du\right]^{\frac{1}{n(\alpha-1)+1}}$$

By the change $u = t^{1/n}z$, we have

$$F_{n,\alpha}(z) = z\left[(\alpha+\frac{1}{n}-1)\int_0^1 t^{\alpha+\frac{1}{n}-2}(1-tz^n)^{-2(\alpha-1)(1-\rho)e^{-i\varphi}\cos\varphi}dt\right]^{\frac{1}{n(\alpha-1)+1}}$$

$$= z \cdot \left[\, \Omega\left(2(\alpha-1)(1-\rho)e^{-i\varphi}\cos\varphi,\ \alpha+\frac{1}{n}-1,\ \alpha+\frac{1}{n}\,;z^n\right)\right]^{\frac{1}{n(\alpha-1)+1}} ,$$

where

$$\Omega(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\ .$$

4 More about Main Result

If in a whole Löewner chain we replace a certain function by some expression, we do not obtain a new univalence criterion. We have the same criterion, but presented in another manner. Sometimes this is useful to study new integral operators as follows.

Let $f,\ g \in \mathcal{A}$, $f(z) \neq 0,\ g(z) \neq 0,\ \forall z \in \mathcal{U}\setminus\{0\}$, $\alpha \in \mathbb{C}$. We can choose the principal branch of $\left(\frac{f(z)}{g(z)}\right)^{\alpha-1}$, analytic in $\mathcal{U}$, equal to 1 at the origin, denoted $h(z)$. Then $f^{\alpha-1}(z) = g^{\alpha-1}(z)h(z)$ and by logarithmic derivation, we get

$$(\alpha-1)\frac{z^n f'(z^n)}{f(z^n)} = (\alpha-1)\frac{z^n g'(z^n)}{g(z^n)} + \frac{z^n h'(z^n)}{h(z^n)} \tag{19}$$

Considering $f \in \mathcal{A}$ which verifies inequality (7) of Theorem 2.1, we deduce that $f(z) \neq 0,\ \forall z \in \mathcal{U}\setminus\{0\}$ and so are g and h.

In view of (19) we can give a new version of Theorem 2.1, namely

Theorem 4.1. *Let α, c be complex numbers, n natural number, $n \geq 1$, such that*

$$\left| \alpha - 2 + \frac{1}{n} \right| < 1 \qquad and \qquad |\, c \,| < 1. \tag{20}$$

For $g \in \mathcal{A}$ and h an analytic function in $\mathcal{U}$, $h(z) = 1 + c_1 z + \ldots$, if the inequality

$$\left| c|z|^{2n} + (1 - |z|^{2n}) \left[(\alpha - 1) \frac{z^n g'(z^n)}{g(z^n)} + \frac{z^n h'(z^n)}{h(z^n)} + \frac{1-n}{n} \right] \right| \leq 1 \tag{21}$$

is true for all $z \in \mathcal{U}$, then the function

$$G_{n,\alpha}(z) = \left[(n(\alpha - 1) + 1) \int_0^z g^{\alpha-1}(u^n) h(u^n) du \right]^{\frac{1}{n(\alpha-1)+1}}, \tag{22}$$

where the principal branch is intended, is analytic and univalent in $\mathcal{U}$.

Corollary 4.2. *Let α be a complex number, $n \in \mathbb{N}^*$, $\left| \alpha - 2 + \frac{1}{n} \right| < 1$ and h an analytic function in $\mathcal{U}$, $h(z) = 1 + c_1 z + \ldots$. If the inequality*

$$\Re \frac{zh'(z)}{h(z)} \leq \frac{1 - \left| \alpha - 2 + \frac{1}{n} \right|}{4} \tag{23}$$

is true for all $z \in \mathcal{U}$, then the function

$$H_{n,\alpha}(z) = \left[(n(\alpha - 1) + 1) \int_0^z u^{n(\alpha-1)} h(u^n) du \right]^{\frac{1}{n(\alpha-1)+1}}, \tag{24}$$

is analytic and univalent in $\mathcal{U}$.

Proof. Taking $g(z) \equiv z$ and $c = \alpha - 2 + \frac{1}{n}$, from (21) we get

$$\left| (1 - |z|^{2n}) \frac{z^n h'(z^n)}{h(z^n)} + \alpha - 2 + \frac{1}{n} \right| \leq 1 \tag{25}$$

and function $G_{n,\alpha}(z)$ from Theorem 4.1, denoted now by $H_{n,\alpha}(z)$, is defined by (24). Under the assumption (23), we can apply Lemma 1.4 to $\frac{zh'(z)}{h(z)}$ and it is easy to check that inequality (25) is true. □

Theorem 4.3. *Let f, $g \in \mathcal{S}$ and α, β, $\gamma \in \mathbb{C}$, $n \in \mathbb{N}^*$. If*

$$|\alpha - 1| + |\beta| + |\gamma| \leq \frac{1}{4n} \tag{26}$$

then the function

$$G_{n,\alpha}(z) = \left[(n(\alpha - 1) + 1) \int_0^z g^{\alpha-1}(u^n) \left(\frac{f(u^n)}{u^n} \right)^{\beta} (f'(u^n))^{\gamma} du \right]^{\frac{1}{n(\alpha-1)+1}} \tag{27}$$

is analytic and univalent in $\mathcal{U}$. Also the functions $G_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \leq k \leq n-1$ defined by (27) are analytic and univalent in $\mathcal{U}$.

Proof. Since f is univalent in $\mathcal{U}$ we can choose the analytic branch of $\left(\frac{f(u^n)}{u^n}\right)^\beta$ equal to 1 at the origin and also the analytic branch of $(f'(u^n))^\gamma$ equal to 1 at the origin. It results that function h,

$$h(u^n) = \left(\frac{f(u^n)}{u^n}\right)^\beta \cdot (f'(u^n))^\gamma$$

is analytic in $\mathcal{U}$, $h(0) = 1$ and $h(z) \neq 0$, $\forall z \in \mathcal{U}$. For this function h we shall establish if inequality (21) of Theorem 4.1 is true. By using (2) and (3), for $c = -2\gamma + \frac{1-n}{n}$ we obtain

$$\left| c|z|^{2n} + (1 - |z|^{2n}) \left[(\alpha - 1)\frac{z^n g'(z^n)}{g(z^n)} + \beta(\frac{z^n f'(z^n)}{f(z^n)} - 1) + \gamma\frac{z^n f''(z^n)}{f'(z^n)} + \frac{1-n}{n} \right] \right|$$

$$= \left| (\alpha - 1)(1 - |z|^{2n})\frac{z^n g'(z^n)}{g(z^n)} + \beta(1 - |z|^{2n}) \left(\frac{z^n f'(z^n)}{f(z^n)} - 1 \right) \right.$$

$$\left. + \gamma \left(-2|z|^{2n} + (1 - |z|^{2n})\frac{z^n f''(z^n)}{f'(z^n)} \right) + \frac{1-n}{n} \right| \leq 4|\alpha - 1| + 4|\beta| + 4|\gamma| + \frac{n-1}{n}.$$

In view of assumption (26), inequality (21) of Theorem 4.1 is true. Since $|\alpha - 1| \leq \frac{1}{4n}$ and $|\gamma| \leq \frac{1}{4n}$, it is easy to check that inequalities (20) are true and then $G_{n,\alpha}(z)$ defined by (27) is analytic and univalent in $\mathcal{U}$. □

Remark 4.4. *Theorems 4.1 and 4.3 include several various results for special values of the parameters α, β, γ and n. For example, taking $n = 1$ in Theorems 4.1 and 4.3 we get the results given in [12], respectively [13]. From Theorem 4.3, the special case $\alpha = 1$, $\gamma = 0$, $n = 1$ leads to the integral operator due to Kim and Merkes [3] and the case $\alpha = 1$, $\beta = 0$, $n = 1$ leads to the integral operator due to Pfaltzgraff [9]. The case $\beta = \gamma = 0$ represents a generalization of the integral operator due to Moldoveanu and Pascu [6].*

References

[1] A.W. Goodman, *Univalent Functions*, Mariner Publishing Company Inc., (1984).

[2] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proceedings of the American Mathematical Society*, 20(1969), 8-12.

[3] Y.J. Kim and E.P. Merkes, On an integral of power of a spirallike functions, *Kyungpook Math. Journal*, 12(2) (1972), 249-253.

[4] R.J. Libera, Univalent α-spiral functions, *Canadian Journal of Mathematics*, 19(1967), 725-733.

[5] R.J. Libera and M.R. Ziegler, Regular function f(z) for which zf'(z) is α-spiral, *Transactions of the American Math. Soc.*, 166(1972), 361-368.

[6] S. Moldoveanu and N.N. Pascu, Integral operator which preserves the univalence, *Mathematica Cluj-Napoca*, 32(55) (1990), 159-166.

[7] S. Moldoveanu and N.N. Pascu, On the univalence of an integral, *Proceeding of the Commemorative Session S. Stoillow*, Univ.of Braşov, (1987), 27-32.

[8] N.N. Pascu and D. Răducanu, *Funcţii Complexe*, Editura Univ. "Transilvania", Braşov, (1999).

[9] J. Pfaltzgraff, Univalence of the integral $(f'(z))^\lambda$, *Bulletin of the London Math. Soc.*, 7(3) (1975), 254-256.

[10] Ch. Pommerenke, *Univalent Functions*, Vandenhoech Ruprecht, Göttingen, (1975).

[11] L. Špaček, Contribution à la théorie des functions univalents, *Casopis Pro Pestovánì Matematiky A Fysiky*, 62(2) (1932), 12-19.

[12] H. Tudor, A univalence condition, *Bulletin of the Transilvania University of Braşov*, 3(52) (2010), 159-162.

[13] H. Tudor, An extension of Kim and Merkes' and of Pfaltzgraff's integral operators, *General Mathematics*, 19(4) (2011), 15-18.