Gen. Math. Notes, Vol. 33, No. 2, April 2016, pp.72-77
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Left (Right) Centralizer of $\sigma-$Square Closed Lie Ideals of $\sigma-$Prime Rings

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(Received: 4-3-16 / Accepted: 10-4-16)

Abstract

Let R be a $\sigma-$prime ring and F be a nonzero left (right) centralizer of R. This work includes two parts. In the first part, when I is a nonzero $\sigma-$ideal of R we prove that (i) if F commutes with σ on I and $[x,R]IF(x)=(0)$ for all $x\in I$, then R is commutative. (ii) If $r\in Sa_{\sigma}(R)$ or F commutes with σ on I and $[F(x),r]=0$ for all $x\in I$, then $r\in Z(R)$. (iii) If $r\in Sa_{\sigma}(R)$ such that $F([x,r])=0$ for all $x\in R$, then $r\in Z(R)$. (iv) If R is a $2-$torsion free σ-prime ring and $F([x,y])=0$ for all $x,y\in R$, then R is a commutative ring. In the second part, when R is a $2-$torsion free and U is a nonzero $\sigma-$square closed Lie ideal of R such that $U\nsubseteq Z(R)$ we prove that: (i) if $r\in U\cap Sa_{\sigma}(R)$ and $[F(x),r]=0$ for all $x\in U$, then $r\in Z(R)$. (ii) If $r\in U\cap Sa_{\sigma}(R)$ and $F([x,r])=0$ for all $x\in U$, then $r\in Z(R)$.

Keywords: *$\sigma-$prime ring, $\sigma-$ideal, centralizer.*

1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. R is said to be $2-$*torsion free* if whenever $2x=0$, then $x=0$. An additive mapping $\sigma:R\rightarrow R$ is called an *involution* if σ is an anti-homomorphism and $\sigma(\sigma(x))=x$ for all $x\in R$. R is called $\sigma-$*prime ring* where σ is an involution of R if $aRb=aR\sigma(b)=(0)$ implies that $a=0$ or $b=0$. A nonempty subset A of R is called $\sigma-$invariant if $\sigma(A)\subseteq A$. An ideal I of R is a $\sigma-$*ideal* if I

is a $\sigma-$invariant. A Lie ideal U of R is a $\sigma-$*Lie ideal* if U is a $\sigma-$invariant. U is called a $\sigma-$square closed Lie ideal of R if U is a $\sigma-$Lie ideal and $u^2 \in U$ for all $u \in U$. In all that follows $Sa_\sigma(R)$ denote the set of all symmetric or skew symmetric elements of R; i.e., $Sa_\sigma(R) = \{x \in R \mid \sigma(x) = \pm x\}$. For any $x, y \in R$, $xy - yx$ will be denoted by $[x, y]$. An additive mapping $F : R \to R$ is called a *left (right) centralizer* in case $F(xy) = F(x)y$ $(F(xy) = xF(y))$ for all $x, y \in R$.

An additive mapping $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. When R is a $\sigma-$prime ring, I is a $\sigma-$ideal of R and d is a nonzero derivation of R, in [2] and [3], Oukhitite and Salhi show that (i) if d commutes with σ on I and $[x, R]Id(x) = (0)$ for all $x \in I$, then R is commutative. (ii) If $r \in Sa_\sigma(R)$ satisfies $[d(x), r] = 0$ for all $x \in I$, then $r \in Z(R)$. Furthermore, if $a \in Sa_\sigma(R)$ and $d([R, a]) = (0)$, then $a \in Z(R)$. In particular, if $d(xy) - d(yx) = 0$ for all $x, y \in R$, then R is commutative ring.

In this paper, we tackle the hypothesis of [2] and [3] for a nonzero left (right) centralizer of R on a $\sigma-$ideal of R. Moreover, we get some results under the same conditions for a nonzero $\sigma-$square closed Lie ideal of R.

In all that follows, we assume that R is a $\sigma-$prime ring, I is a nonzero $\sigma-$ideal of R, U is a nonzero $\sigma-$square closed Lie ideal of R such that $U \nsubseteq Z(R)$ and F is a nonzero left (right) centralizer of R.

We shall use basic commutator identities:

$$[xy, z] = x[y, z] + [x, z]y$$
$$[x, yz] = y[x, z] + [x, y]z.$$

The material in this work is a part of first author's Master's Thesis which is supervised by Prof. Dr. Neşet Aydın.

2 Results

Lemma 2.1. *[1, 3) of Teorem 2.2] For a $\sigma-$prime ring R, if I is a nonzero $\sigma-$ideal and $aIb = aI\sigma(b) = (0)$, then $a = 0$ or $b = 0$.*

Lemma 2.2. *[4, Lemma 4] If $U \nsubseteq Z(R)$ is a $\sigma-$Lie ideal of a $2-$torsion free $\sigma-$prime ring R and $a, b \in R$ such that $aUb = \sigma(a)Ub = (0)$, then $a = 0$ or $b = 0$.*

Lemma 2.3. *[5, Lemma 2.7] Let R be a $2-$torsion free $\sigma-$prime ring and U be a $\sigma-$Lie ideal of R. If $a \in R$ such that $[a, U] \subseteq Z(R)$ then either $U \subseteq Z(R)$ or $a \in Z(R)$.*

Lemma 2.4. *Suppose that F commutes with σ on I. If $[x,R]IF(x)=(0)$ for all $x\in I$, then R is commutative.*

Proof. Since $t=x-\sigma(x)\in I$ for $x\in I$, then $[t,r]IF(t)=(0)$ for all $r\in R$. Since $t\in Sa_{\sigma}(R)\cap I$ and F commutes with σ on I, we obtain $[t,r]IF(t)=[t,r]\,I\sigma(F(t))=(0)$ for all $r\in R$. According to Lemma 2.1 we obtain,

$$[t,r]=0 \text{ or } F(t)=0,\ \forall r\in R$$

If $[t,r]=0$ for all $r\in R$, we have $[x,r]=[\sigma(x),r]$ for all $r\in R,\ x\in I$. Replacing x by $\sigma(x)$ in hypothesis and using the last equation, we get $[x,r]IF(x)=[x,r]I\sigma(F(x))=(0)$ for all $r\in R,\ x\in I$. Using again Lemma 2.1, consequently either

$$x\in Z(R) \text{ or } F(x)=0,\ \forall x\in I$$

If $F(t)=0,$ we get $F(x)=\sigma(F(x))$ since F commutes with σ on I. Thus, we have $[x,r]IF(x)=[x,r]I\sigma\left(F(x)\right)=(0)$ for all $r\in R$. Once again using Lemma 2.1, we get $x\in Z(R)$ or $F(x)=0$. So, in both cases

$$x\in Z(R) \text{ or } F(x)=0,\ \forall x\in I$$

Let us consider that $A=\{x\in I\mid F(x)=0\}$ and $B=\{x\in I\mid x\in Z(R)\}$. It is clear that A and B are additive subgroups of I such that $I=A\cup B$. But a group can not be union of two its proper subgroups and therefore $I=A$ or $I=B$. If $I=A$, then $F(x)=0$ for all $x\in I$. Since I is a $\sigma-$ideal, we obtain $F=0$. It is a contradiction. Hence, $I=B$ so that $I\subseteq Z(R)$. Thus, $[x,r]=0$ for all $x\in I$ and $r\in R$. If we replace x by yx where $y\in R$, to obtain $[y,r]I=(0)$. Since I is a $\sigma-$ideal, consequently we get $[y,r]=0$ for all $y,r\in R$. Therefore, R is a commutative ring. □

Lemma 2.5. *Let $r\in Sa_{\sigma}(R)$ or F commutes with σ on I. If $[F(x),r]=0$ for all $x\in I$, then $r\in Z(R)$.*

Proof. Taking xy with $y\in I$ instead of x in hypothesis, we conclude $0=[F(xy),r]=[F(x)y,r]=F(x)[y,r]+[F(x),r]y$ for all $x,y\in I$. By using $[F(x),r]=0$ for all $x\in I$ in the last equation, it follows that $F(x)[y,r]=0$ for all $x,y\in I$. If we replace y by yk where $k\in R$ in the last equality, we get

$$0=F(x)[yk,r]=F(x)y[k,r]+F(x)[y,r]k \text{ for all } x,y\in I \text{ and all } k\in R.$$

In this equation if we use $F(x)[y,r]=0$ for all $x,y\in I$, we obtain $F(x)y[k,r]=0$ for all $x,y\in I$ and all $k\in R$. Therefore

$$F(x)I[k,r]=(0)$$

First of all assume that $r \in Sa_\sigma(R)$. We obtain that $F(x)I\sigma([k,r]) = (0)$. Using Lemma 2.1, we get $F(x) = 0$ for all $x \in I$ or $r \in Z(R)$. Assume that $F(x) = 0$ for all $x \in I$. If we replace x by tx where $t \in R$, to obtain $F(t)x = 0$ for all $t \in R$ and $x \in I$. Since I is a $\sigma-$ideal, we conclude that $F = 0$, a contradiction. So that, $r \in Z(R)$. In the second case, if F commutes with σ on I, we get $F(x)I[k,r] = \sigma(F(x))I[k,r] = (0)$. Using Lemma 2.1, we get $F(x) = 0$ or $r \in Z(R)$ for all $x \in I$. If $F(x) = 0$ for $x \in I$, then $F = 0$, a contradiction. Therefore, $r \in Z(R)$. □

Lemma 2.6. *If $r \in Sa_\sigma(R)$ such that $F([x,r]) = 0$ for all $x \in R$, then $r \in Z(R)$.*

Proof. Assume that $r \notin Z(R)$. If $F(r) = 0$, we have $F(x)r = 0$ for all $x \in R$. In this equation, replace x by xy where $y \in R$, we get $F(x)yr = 0$ for all $y \in R$. Since $r \in Sa_\sigma(R)$, it yields $F(x)Rr = F(x)R\sigma(r) = (0)$. Since R is a $\sigma-$prime ring, we get $r = 0$, a contradiction. Thus, $F(r) \neq 0$. For any $x \in R$, we obtain $F([rx,r]) = 0$ from the hypothesis. Consequently, we have $F(r)[x,r] = 0$ for all $x \in R$. If we replace x by sx where $s \in R$ in last equality, we get $F(r)s[x,r] = 0$ for all $x, s \in R$. Using by $r \in Sa_\sigma(R)$ and the σ-primeness of R yields $[x,r] = 0$ which proves $r \in Z(R)$, a contradiction. So, $r \in Z(R)$. □

From this point on, R is a $2-$torsion free $\sigma-$prime ring.

Theorem 2.7. *If $F([x,y]) = 0$ for all $x, y \in R$, then R is a commutative ring.*

Proof. For $y \in Sa_\sigma(R)$, we have $F([x,y]) = 0$ for all $x \in R$ from the hypothesis. Applying the Lemma 2.6, we conclude $y \in Z(R)$. For any $r \in R$, $r + \sigma(r)$ and $r - \sigma(r)$ are elements of $Sa_\sigma(R)$, yields $r + \sigma(r) \in Z(R)$ and $r - \sigma(r) \in Z(R)$. So that, $2r \in Z(R)$. Since R is $2-$torsion free, yields $r \in Z(R)$ for all $r \in R$. Therefore, R is a commutative ring. □

Lemma 2.8. *If $r \in U \cap Sa_\sigma(R)$ and $[F(u), r] = 0$ for all $u \in U$, then $r \in Z(R)$.*

Proof. For all $u, v \in U$, $(u+v)^2 \in U$ together with $[u,v] \in U$ yields $2uv \in U$. Taking $2uv$ with $u, v \in U$ instead of u in hypothesis, we obtain $2[F(uv), r] = 0$. Since R is $2-$torsion free, consequently we have $F(u)[v,r] = 0$ for all $u, v \in U$. Replace v by $2wv$ where $w \in U$ in this equation and by using the hypothesis, we get $F(u)w[v,r] = 0$ for all $u, v, w \in U$. Since $r \in Sa_\sigma(R)$ and by using Lemma 2.2, we have either $F(u) = 0$ or $[v,r] = 0$ for all $u, v \in U$. If $F(u) = 0$ for all $u \in U$, then $F = 0$, a contradiction. So that, $[U, r] = (0)$. By using Lemma 2.3, we conclude $r \in Z(R)$. □

Lemma 2.9. *If $r \in U \cap Sa_\sigma(R)$ and $F([u,r]) = 0$ for all $u \in U$, then $r \in Z(R)$.*

Proof. Assume that $r \notin Z(R)$. From the hypothesis, we have $F([u,r]) = 0$ for all $u \in U$. If $F(r) = 0$, then we get $F(u)r = 0$ for all $u \in U$. Replacing u by $2vu$ where $v \in U$ in last equality and by using R is 2−torsion free, we get $F(v)ur = 0$ for all $u, v \in U$. Since $r \in Sa_\sigma(R)$, it yields $F(v)Ur = F(v)U\sigma(r) = (0)$. From the Lemma 2.2, we get $F = 0$ or $r = 0$, a contradiction. Therefore $F(r) \neq 0$. For any $u \in U$, from the hypothesis, we obtain $F([2ru, r]) = 0$. Since R is 2-torsion free ring, we have $F(r)[u,r] = 0$ for all $u \in U$. Replacing u by $2vu$ where $v \in U$ in last equality, we get $F(r)v[u,r] = 0$ for all $u, v \in U$. By using $r \in Sa_\sigma(R)$ and Lemma 2.2, we have $[u,r] = 0$ for all $u \in U$. By using Lemma 2.3, we get a contradiction. So that, $r \in Z(R)$. □

Theorem 2.10. *Let F be a left (right) centralizer and commute with σ on U. If $[u,R]UF(u) = (0)$ for all $u \in U$, then $F = 0$.*

Proof. Since $t = u - \sigma(u) \in U$ for all $u \in U$, then $[t,r]UF(t) = (0)$ for all $r \in R$. Since $t \in Sa_\sigma(R) \cap U$ and F commutes with σ on U, we obtain $[t,r]UF(t) = [t,r]U\sigma(F(t)) = (0)$ for all $r \in R$. According to Lemma 2.2, we get

$$[t,r] = 0 \text{ or } F(t) = 0, \ \forall r \in R$$

First of all, if $[t,r] = 0$ for all $r \in R$, we have $[u,r] = [\sigma(u), r]$ for all $r \in R$, $u \in U$. Replacing u by $\sigma(u)$ in hypothesis and using the last equation, we get $[u,r]UF(u) = [u,r]U\sigma(F(u)) = (0)$ for all $r \in R$, for all $u \in U$. By using Lemma 2.2, we see that either

$$u \in Z(R) \text{ or } F(u) = 0, \ \forall u \in U$$

In the second, if $F(t) = 0$, we get $F(u) = \sigma(F(u))$ since F commutes with σ on U. Thus, we have $[u,r]UF(u) = [u,r]U\sigma(F(u)) = (0)$ for all $r \in R$. Once again, using Lemma 2.2, we get $[u,r] = 0$ or $F(u) = 0$. So, in both cases

$$[u,r] = 0 \text{ or } F(u) = 0, \ \forall u \in U$$

Let us consider that $A = \{u \in U \mid u \in Z(R)\}$ and $B = \{u \in U \mid F(u) = 0\}$. It is clear that A and B are additive subgroups of U such that $U = A \cup B$. But a group can not be union of two its proper subgroups and therefore $U = A$ or $U = B$. If $U = A$, then $U \subseteq Z(R)$, a contradiction. Hence, $U = B$. So, $F(u) = 0$ for all $u \in U$. If replace u by $2uv$ where $v \in U$, we have $2F(u)v = 0$. Since R is 2−torsion free and using Lemma 2.2, we get $F = 0$. □

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