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On a Class of Meromorphic Univalent Functions Defined by Hypergeometric Function

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Abstract

In this paper, we introduce the class of meromorphic univalent functions defined by hypergeometric function. We obtain some interesting properties like coefficient inequality, distortion and growth theorems, Hadamard product(or convolution), radii of starlikeness and convexity for the functions in the class $H^*(\beta, \alpha, k)$.

Keywords: *Meromorphic univalent function, Hypergeometric functions, Hadamard product, Starlike function, Convex function.*

1 Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic and meromorphic univalent in the punctured unit disk $^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = -\{0\}$.

Let A be a subclass of $\sum$ of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, (a_n \geq 0, n \in \mathbb{N}). \quad (1)$$

A function $f \in A$ is meromorphic starlike function of order $\rho, (0 \leq \rho < 1)$if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \quad (z \in).$$

The class of all such functions is denoted by $A^*(\rho)$. A function $f \in A$ is meromorphic convex function of order $\rho, (0 \leq \rho < 1)$ if

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \quad (z \in).$$

The Hadamard product (or convolution) of two functions, f given by (1) and

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, (b_n \geq 0, n \in \mathbb{N}),$$

is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n,$$

see [9], let us define the function $\tilde{\phi}(a, c; z)$ defined by

$$\tilde{\phi}(a, c; z) = \frac{1}{z} + \sum_{n=0}^{\infty} |\frac{(a)_{n+1}}{(c)_{n+1}}| a_n z^n,$$

for $c \neq 0, -1, -2, ...,$ and $a \in \mathbb{C} - \{0\}$, where $(\lambda)_n = \lambda(\lambda+1)_{n+1}$ is the Puchhammer symbol. We note that

$$\tilde{\phi}(a, c; z) = \frac{1}{z} \ {}_2F_1(1, a, c; z),$$

where

$${}_2F_1(b, a, c; z) = \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(c)_n} \frac{z^n}{n!},$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\tilde{\phi}(a, c; z)$, using the Hadamard product for $f \in \sum$, we define a new linear operator $L^*(a, c)$ on $\sum$ by

$$L^*(a, c) f(z) = \tilde{\phi}(a, c; z) * f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |\frac{(a)_{n+1}}{(c)_{n+1}}| a_n z^n. \quad (2)$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [2],[3], Liu [5], Liu and Srivastava [6], [7], [8], Cho and Kim [1]. For a function $f \in L^*(a,c)f(z)$ we define

$$I^0(L^*(a,c)f(z)) = L^*(a,c)f(z),$$

and for $k = 1, 2, 3, ...,$

$$I^k(L^*(a,c)f(z)) = z(I^{k-1}L^*(a,c)f(z))' + \frac{2}{z},$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} n^k |\frac{(a)_{n+1}}{(c)_{n+1}}| a_n z^n, \tag{3}$$

We note that $I^k(L^*(a,c)f(z))$ studied by Frasin and Darus [4].
It follows from (2) that

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a+1)L(a,c)f(z). \tag{4}$$

Also, from (3) and (4) we get

$$z(I^kL(a,c)f(z))' = aI^kL(a+1,c)f(z) - (a+1)I^kL(a,c)f(z).$$

Now we define the following:
Definition 1.1 : *A function $f \in A$ of the form (1) is in the class $H^*(\beta, \alpha, k)$ if it satisfies the following condition:*

$$|\frac{\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'} + 2}{\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'} + 2\alpha}| < \beta, \tag{5}$$

where $\lambda > -1, 0 \le \alpha < 1, 0 < \beta \le 1, k = 1, 2, 3, ...$

In this paper we obtain coefficient estimates for the class $H^*(\beta, \alpha, k)$, Hadamard product, growth and distortion theorems, radii of starlikeness and convexity.

2 Coefficient Inequality

Theorem 2.1 : *A function f defined by (1) is in the class $H^*(\beta, \alpha, k)$, if and only if*

$$\sum_{n=1}^{\infty} n^{k+1} |\frac{(a)_{n+1}}{(c)_{n+1}}| [n(1+\beta) + (1 + \beta(2\alpha - 1))] a_n \le 2\beta(1-\alpha). \tag{6}$$

The result is sharp for the function

$$f_n(z)=\frac{1}{z}+\frac{2\beta(1-\alpha)}{n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|[n(1+\beta)+(1+\beta(2\alpha-1))]}z^n, n\geq 1 \tag{7}$$

Proof : To proof the sufficient part, let the inequality (6) holds true and let $|z|=1$,
by (5), we have

$$|\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'}+2|-\beta|\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'}+2\alpha|<0,$$

$$|z(I^kL^*(a,c)f(z))''+2(I^kL^*(a,c)f(z))'|-\beta|z(I^kL^*(a,c)f(z))''+2\alpha((I^kL^*(a,c)f(z))')|,$$

$$=|\sum_{n=1}^{\infty}(n+1)n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|a_nz^{n-1}-\beta|\frac{2(1-\alpha)}{z^2}+\sum_{n=1}^{\infty}(n-1+2\alpha)n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|a_nz^{n-1}|,$$

$$\leq\sum_{n=1}^{\infty}n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|[n(1+\beta)+(1+\beta(2\alpha-1))]a_n-2\beta(1-\alpha)\leq 0.$$

Thus by the maximam modulus theorem, we have $f\in H^*(\beta,\alpha,k)$.
Conversely, suppose that f of the form (1) is in the class $H^*(\beta,\alpha,k)$, then by (5) we have

$$|\frac{\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'}+2}{\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'}+2\alpha}|<\beta,$$

$$|\frac{\sum_{n=1}^{\infty}(n+1)n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|a_nz^{n-1}}{\frac{2(1-\alpha)}{z^2}+\sum_{n=1}^{\infty}(n-1+2\alpha)n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|a_nz^{n-1}}|<\beta.$$

Since $|\Re(z)|\leq|z|$ for all z, we get

$$\Re\{\frac{\sum_{n=1}^{\infty}(n+1)n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|a_nz^{n-1}}{\frac{2(1-\alpha)}{z^2}+\sum_{n=1}^{\infty}(n-1+2\alpha)n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|a_nz^{n-1}}\}<\beta. \tag{8}$$

Now by choosing the value of z on the real axis so that the value of $\frac{z(I^kL^*(a,c)f(z))''}{(I^kL^*(a,c)f(z))'}$ is real, then clearing the denominator of (8) and letting $z\to 1^-$ through real values, we get the inequality (6).
The result is sharp for the function

$$f_n(z)=\frac{1}{z}+|\frac{(c)_{n+1}}{(a)_{n+1}}|\frac{2\beta(1-\alpha)}{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}z^n, n\geq 1$$

Corollary 2.1 : *If* $f\in H^*(\beta,\alpha,k)$, *then*

$$a_n\leq|\frac{(c)_{n+1}}{(a)_{n+1}}|\frac{2\beta(1-\alpha)}{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]},$$

where $\lambda>-1, 0\leq\alpha<1, 0<\beta\leq 1, k=1,2,3,....$

3 Growth and Distortion Theorems

A growth and distortion property for the function $f \in H^*(\beta, \alpha, k)$ is given as follows:

Theorem 3.1 : *A function f defined by (1) is in the class $H^*(\beta, \alpha, k)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{r} - |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)}r \leq |f(z)| \leq \frac{1}{r} + |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)}r,$$

with equality for

$$f(z) = \frac{1}{z} + |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)}z.$$

Proof : Since $f \in H^*(\beta, \alpha, k)$, we have from Theorem 2.1 the inequality

$$\sum_{n=1}^{\infty} n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|[n(1+\beta)+(1+\beta(2\alpha-1))]a_n \leq 2\beta(1-\alpha),$$

Then

$$|f(z)| \leq |\frac{1}{z}| + \sum_{n=1}^{\infty} a_n|z|^n,$$

for $0 < |z| = r < 1$, we have

$$|f(z)| \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n,$$

$$\leq \frac{1}{r} + |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)}r.$$

Also

$$|f(z)| \geq |\frac{1}{z}| - \sum_{n=1}^{\infty} a_n|z|^n,$$

$$\geq \frac{1}{r} - |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)}r, |z| = r.$$

Hence the proof is complete.

Theorem 3.2 : *A function f defined by (1) is in the class $H^*(\beta, \alpha, k)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{r^2} - |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)} \leq |f'(z)| \leq \frac{1}{r^2} + |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)},$$

with equality for

$$f(z) = \frac{1}{z} + |\frac{(c)_2}{(a)_2}|\frac{\beta(1-\alpha)}{(1+\alpha\beta)}z.$$

Proof : From Theorem 2.1 we have

$$\sum_{n=1}^{\infty} n^{k+1}|\frac{(a)_{n+1}}{(c)_{n+1}}|[n(1+\beta)+(1+\beta(2\alpha-1))]a_n \leq 2\beta(1-\alpha).$$

Thus

$$|f'(z)| \leq |\frac{-1}{z^2}| + \sum_{n=1}^{\infty} n a_n |z|^{n-1},$$

for $0 < |z| = r < 1$, we get:

$$|f'(z)| \leq |\frac{1}{r^2}| + \sum_{n=1}^{\infty} n a_n,$$

$$\leq \frac{1}{r^2} + |\frac{(c)_2}{(a)_2}| \frac{2\beta(1-\alpha)}{(1+\alpha\beta)},$$

and

$$|f'(z)| \geq |\frac{-1}{z^2}| - \sum_{n=1}^{\infty} n a_n |z|^{n-1},$$

$$\geq |\frac{1}{r^2}| - \sum_{n=1}^{\infty} n a_n$$

$$\geq \frac{1}{r^2} - |\frac{(c)_2}{(a)_2}| \frac{2\beta(1-\alpha)}{(1+\alpha\beta)}.$$

This completes the proof.

4 Hadamard Product

Theorem 4.1 : *Let the functions $f, g \in H^*(\beta, \alpha, k)$. Then $(f*g) \in H^*(\delta, \alpha, k)$ for*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

and

$$(f*g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n,$$

where

$$\delta=\frac{2\beta^2(1-\alpha)(n+1)}{2\beta^2(1-\alpha)(n+2\alpha-1)-|\frac{(c)_{n+1}}{(a)_{n+1}}|n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]^2}$$

Proof : Since $f,g\in H^*(\beta,\alpha,k)$, then by Theorem 2.1 we have:

$$\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}a_n\le 1,$$

and

$$\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}b_n\le 1.$$

We must find the largest δ such that

$$\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\delta)+(1+\delta(2\alpha-1))]}{2\delta(1-\alpha)}a_nb_n\le 1.$$

By Cauchy- Schwarz inequality, we get

$$\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}\sqrt{a_nb_n}\le 1, \tag{9}$$

To prove the theorem it is enough to show that

$$\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\delta)+(1+\delta(2\alpha-1))]}{2\delta(1-\alpha)}a_nb_n$$

$$\le|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}\sqrt{a_nb_n},$$

which is equivalent to

$$\sqrt{a_nb_n}\le\frac{\delta[n(1+\beta)+(1+\beta(2\alpha-1))]}{\beta[n(1+\delta)+(1+\delta(2\alpha-1))]}.$$

From (9), we have

$$\sqrt{a_nb_n}\le|\frac{(c)_{n+1}}{(a)_{n+1}}|\frac{2\beta(1-\alpha)}{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}.$$

We must show that

$$|\frac{(c)_{n+1}}{(a)_{n+1}}|\frac{2\beta(1-\alpha)}{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}\le\frac{\delta[n(1+\beta)+(1+\beta(2\alpha-1))]}{\beta[n(1+\delta)+(1+\delta(2\alpha-1))]},$$

which gives

$$\delta \leq \frac{2\beta^2(\alpha-1)(n+1)}{2\beta^2(1-\alpha)(n+2\alpha-1)-|\frac{(c)_{n+1}}{(a)_{n+1}}|n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]^2}.$$

Hence the proof is complete.

Theorem 4.2 : *Let the functions $f_i(i=1,2)$ defined by*

$$f_i(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,i}z^n, (a_{n,i}\geq 0, i=1,2)$$

be in the class $H^*(\beta,\alpha,k)$. Then the function g defined by

$$g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n,$$

is in the class $H^*(\eta,\alpha,k)$, where

$$\eta=\frac{4\beta^2(\alpha-1)(n+1)}{4\beta^2(1-\alpha)(n+2\alpha-1)-|\frac{(a)_{n+1}}{(c)_{n+1}}|n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]^2}$$

Proof : Since $f_i\in H^*(\beta,\alpha,k),(i=1,2)$,then by Theorem 2.1 we have:

$$\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}a_{n,i}\leq 1, i=1,2.$$

Hence

$$\sum_{n=1}^{\infty}(|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)})^2a_{n,i}^2$$

$$\leq(\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}a_{n,i})^2\leq 1,(i=1,2.)$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{2}(|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)})^2(a_{n,1}^2+a_{n,2}^2)\leq 1,$$

To prove the theorem, we must find the largest η such that

$$\frac{n(1+\eta)+(1+\eta(2\alpha-1))]}{\eta}\leq\frac{|\frac{(a)_{n+1}}{(c)_{n+1}}n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]^2}{4\beta^2(1-\alpha)}, n\geq 1,$$

so that

$$\eta\leq\frac{4\beta^2(\alpha-1)(n+1)}{4\beta^2(1-\alpha)(n+2\alpha-1)-|\frac{(a)_{n+1}}{(c)_{n+1}}|n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]^2}.$$

Hence the proof is complete.

Theorem 4.3 : *If* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in H^*(\beta, \alpha, k)$, *and* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ *with* $|b_n| \leq 1$ *is in the class* $H^*(\beta, \alpha, k)$, *then* $f(z) * g(z) \in H^*(\beta, \alpha, k)$.

Proof : From Theorem 2.1 we have

$$\sum_{n=1}^{\infty} |\frac{(a)_{n+1}}{(c)_{n+1}}| n^{k+1}[n(1+\beta) + (1+\beta(2\alpha-1))]a_n \leq 2\beta(1-\alpha).$$

Since

$$\sum_{n=1}^{\infty} |\frac{(a)_{n+1}}{(c)_{n+1}}| \frac{n^{k+1}[n(1+\beta) + (1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} |a_n b_n|,$$

$$= \sum_{n=1}^{\infty} |\frac{(a)_{n+1}}{(c)_{n+1}}| \frac{n^{k+1}[n(1+\beta) + (1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} a_n |b_n|,$$

$$\leq \sum_{n=1}^{\infty} |\frac{(a)_{n+1}}{(c)_{n+1}}| \frac{n^{k+1}[n(1+\beta) + (1+\beta(2\alpha-1))]}{2\beta(1-\alpha)} a_n \leq 1.$$

Thus $f(z) * g(z) \in H^*(\beta, \alpha, k)$.
Hence the proof is complete.

Corollary 4.1 : *If* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in H^*(\beta, \alpha, k)$, *and* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ *with* $0 \leq b_n \leq 1$ *is in the class* $H^*(\beta, \alpha, k)$, *then* $f(z) * g(z) \in H^*(\beta, \alpha, k)$.

5 Radii of Starlikness and Convexity

Theorem 5.1 : *Let the function* $f(z)$ *defined by (1) be in the class* $H^*(\beta, \alpha, k)$. *Then* f *is meromorphically starlike of order* $\delta(0 \leq \delta < 1)$ *in the disk* $|z| < r_1(\beta, \alpha, k, \delta)$, *where*

$$r_1(\beta, \alpha, k, \delta) = \inf_n \{ |\frac{(a)_{n+1}}{(c)_{n+1}}| \frac{n^{k+1}[n(1+\beta) + (1+\beta(2\alpha-1))](1-\delta)}{2\beta(n+2-\delta)(1-\alpha)} \}^{\frac{1}{n+1}}$$

The result is sharp for the function given by (7).

Proof : It is enough to show that

$$|\frac{zf'(z)}{f(z)} + 1| \leq 1 - \delta,$$

$$|\frac{zf'(z)}{f(z)} + 1| = |\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{z^{-1} + \sum_{n=1}^{\infty} a_n z^n}| \leq \frac{\sum_{n=1}^{\infty}(n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}}.$$

This will be bounded by $1-\delta$,

$$\frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n|z|^{n+1}} \leq 1-\delta,$$

$$\sum_{n=1}^{\infty}(n+2-\delta)a_n|z|^{n+1} \leq 1-\delta,$$

by Theorem 2.1, we have

$$\sum_{n=1}^{\infty}|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(1-\alpha)}a_n \leq 1.$$

Hence

$$|z|^{n+1} \leq |\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))](1-\delta)}{2\beta(n+2-\delta)(1-\alpha)},$$

$$|z| \leq \{|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^{k+1}[n(1+\beta)+(1+\beta(2\alpha-1))](1-\delta)}{2\beta(n+2-\delta)(1-\alpha)}\}^{\frac{1}{n+1}}.$$

This completes the proof of the theorem.

Theorem 5.2 : *Let the function $f(z)$ defined by (1) be in the class $H^*(\beta,\alpha,k)$. Then f is meromorphically convex of order $\gamma(0 \leq \gamma < 1)$ in the disk $|z| < r_2(\beta,\alpha,k,\gamma)$, where*

$$r_2(\beta,\alpha,k,\gamma) = \inf_n\{|\frac{(a)_{n+1}}{(c)_{n+1}}|\frac{n^k(1-\gamma)[n(1+\beta)+(1+\beta(2\alpha-1))]}{2\beta(n+2-\gamma)(1-\alpha)}\}^{\frac{1}{n+1}}$$

The result is sharp for the function given by (7).

Proof : By using the same technique in the proof of Theorem 5.1 we can show that

$$|\frac{zf''(z)}{f'(z)}+2| \leq 1-\gamma, \ ((0 \leq \gamma < 1).$$

for $|z| < r_2$ with the aid of Theorem 2.1, we have the assertion of Theorem 5.2.

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