Gen. Math. Notes, Vol. 23, No. 1, July 2014, pp. 45-58
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Study of Variants of Cantor Sets Using Iterated Function System

Ashish[1], Mamta Rani[2] and Renu Chugh[3]

[1]Department of Mathematics, Central University of Haryana
Jant-Pali, Mahendergarh, 123029, India
E-mail: drashishkumar108@gmail.com
[2]Department of Computer Science
Central University of Rajasthan, Kishangarh, 305801, India
E-mail: mamtarsingh@gmail.com
[3]Department of Mathematics, Maharshi Dayanand University,
Rohtak 124001, India
E-mail: chugh.r1@gmail.com

(Received: 22-11-13 / Accepted: 17-4-14)

Abstract

The classical Cantor set discovered and introduced by the famous mathematicians Henery Smith and George Cantor has many interesting properties in the field of set theory, Topology and fractal geometry. One of the way in which the behavior of Cantor sets can be specified is through the use of Iterated Function System. In (2008), Gerald Edgar in his book gave a systematic study of classical Cantor ternary set in iterated function system and introduced some beautiful properties. Our goal in this paper is to present two new examples of Cantor variants using iterated function system.

Keywords: *Cantor one-fifth set, Cantor middle one-half set, Iterated function system (IFS), Self- similarity, Invariant sets.*

1 Introduction

Cantor set is a classical example of perfect subset of the closed interval [0, 1], which has the same cardinality as the real line but whose Lebesgue measure is zero [12]. It was discovered in 1875 by Henry John Stephen Smith [27] and first introduced by German mathematician George Cantor (1845 - 1918) that became known as Cantor ternary set [4-9]. Later on, Devil's and other researchers gave graphical representation of Cantor set in the form of staircases [17, 18, 23].

In the words of the mathematician David Hilbert, "No one shall expel us from the paradise that Cantor has created for us" [12]. For detailed study on Cantor set and on its applications, one may refer to Beardon [3], Devaney [10], Falconer [13] and Peitgen, Jurgen & Saupe [21]. The Cantor set has many interesting properties and consequences in the field of set theory, topology, and fractal theory [15, 16, 17, 18, 28]. In 1999, P. Mendes, showed that the sum of two homogenous Cantor set is often a uniformly contracting self-similar set under some conditions [20]. Dovgoshey et. al. [11] gave a systematic survey on the properties of the Cantor ternary map. In the recent years, Michael Barnsley [1, 2] gave the concept of iterated function system that is an efficient way of specifying many of the sets that we will be interested in. Also, for more applications of Cantor set in discrete dynamical system and mathematical analysis, one may refer to [14, 19, 22].

In 2008, Gerald Edgar in his book "Measure, Topology, and Fractal Geometry" [12] presented various properties of Cantor set using iterated function system. Furthermore, Shaver [26] studied many other general Cantor sets that were constructed by removing different parts of different lengths from the initiator and also presented some properties using iterated function system. The interesting point here is that some of the Cantor sets given by Rani and Prasad [23] are common to Cantor sets given by Shaver [26]. Recently, Rani, Ashish and Chugh [25] studied the variants of Cantor sets using mathematical feedback system and in [24] they introduced new examples of fractal string.

In this paper we study some properties of variants of Cantor sets using iterated function system. In Section 2, we deal with some basic definitions pertaining to the notion of fractal string with several new ones that have been taken into sequel. In Section 3 and 4, the main results of our study have been presented, followed by the concluding remarks in Section 5.

2 Preliminaries

Throughout this paper we study the various properties of Cantor one-fifth set and Cantor middle one-half set using the iterated function system. In 2008, Gerald Edgar presented the following definitions of ratio list, iterated function system, invariants sets or attractors etc.

Definition 2.1: *Ratio list: A finite list of positive numbers,* $(r_1, r_2, ..., r_n)$ *, where* $r_1, r_2, ..., r_n \in \mathbb{R}$ *denote the ratios, in which the interval [0, 1] can be divided is called the ratio list [12].*

Definition 2.2: *Iterated Function System: A system realizing a ratio list* $(r_1, r_2, ..., r_n)$ *, in a metric space* X *in to a list* $(f_1, f_2, ..., f_n)$ *, where* $f_i : X \to X$ *is a similarity contraction mapping with ratio* r_i *is called iterated function system [12].*

Definition 2.3: *Invariant set or attractor: A non-empty compact set* $Y \subseteq X$ *is called an invariant set or attractor for the iterated function system* $(f_1, f_2, ..., f_n)$ *if and only if* $Y = f_1[Y] \cup f_2[Y] \cup ..... \cup f_n[Y]$ *for* $n \in \mathbb{N}$ *[12].*

For example, the triadic Cantor dust is invariant set for an iterated function system realizing the ratio list $(1/3, 1/3)$. The Sierpinski gasket is an invariant set for an iterated function system realizing the ratio list $(1/2, 1/2, 1/2)$.

Definition 2.4: *Similarity value of ratio list* $(r_1, r_2, ..., r_n)$ *, is the positive number* $'s'$ *such that* $r_1^s + r_2^s + ... + r_n^s = 1$ *[12].*

Definition 2.5: *The number* $'s'$ *is called the similarity dimension of a non-empty compact set* Y *if and only if* Y *satisfies a self-referential equation of the type*

$$Y = \bigcup_{i=1}^{n} f_i[Y],$$

where $(f_1, f_2, ..., f_n)$ *is a hyperbolic iterated function system of similarities whose ratio list has similarity value* $'s'$.

For example, let us take the Sierpinski gasket. The ratio list is $(1/2, 1/2, 1/2)$. So, we get

$$(1/2)^s + (1/2)^s + (1/2)^s = 1$$

Thus, the similarity dimension is $\log 3 / \log 2 \approx 1.585$.

Proposition 2.1: [12, p. 6] *The triadic Cantor dust* C *satisfies the self – referential equation* $C = f_1[C] \cup f_2[C]$.

Throughout this paper, F be a Cantor one-fifth set and P a Cantor middle one-half set.

3 Cantor One-Fifth Set

In this section we consider, F be a Cantor one-fifth set having ratio $r = 1/5$, center $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a similarity contraction mapping on $\mathbb{R}$ with ratio 'r', defined by $f(x) = rx + (1-r)a$. Also, throughout this section, we call the ordered pair (f_1, f_2, f_3) as iterated function system.

The Cantor one-fifth set is a subset of the interval [0, 1]. That is constructed by, dividing the initiator [0, 1] into five line segments with dividing ratio 1/5 and by dropping the second and fourth line segment i.e. (1/5, 2/5) and (3/5, 4/5), we call them Holes (i.e. the parts that are removed). Then, the set F_1 is obtained by leaving $[0,1/5] \cup [2/5, 3/5] \cup [4/5, 1]$. Again, after repeating the same process with the remaining closed line segments, we get the set of holes that have been removed during the process. Then, the set F_2 is obtained by leaving

$$F_2 = \underbrace{[0,1/25] \cup [2/25, 3/25] \cup [4/25, 1/5]}_{f_1(x)}$$

$$\cup \underbrace{[2/5, 11/25] \cup [12/25, 13/25] \cup [14/25, 3/5]}_{f_2(x)}$$

$$\cup \underbrace{[4/5, 21/25] \cup [22/25, 23/25] \cup [24/25, 1]}_{f_3(x)}$$

Further, repeating the same process over and over again, by removing the holes of second and fourth position at each step from each closed interval, we obtain a sequence H_n of holes. The number of holes H_n consists of 2.3^{n-1} for n = 1, 2, 3,… open interval and F_n consists of 3^n disjoint closed interval. Thus, the Cantor one-fifth set would be the limit $'F'$ of the sequence F_n of sets. We also analyze that $F_0 \supseteq F_1 \supseteq F_2 \supseteq ......$ So, we define limit $'F'$ as the intersection of the sets F_n i.e.

$$F = \bigcap_{n \in \mathbb{N}} F_n .$$

Fig. 1 shows the graphical representation of Cantor one-fifth set constructed as above.

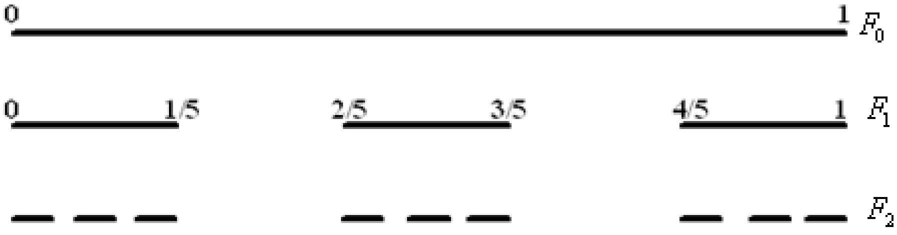

Fig. 1: Cantor One-Fifth Set

In the following theorem, we state without proof the well-known result of quinary Cantor one-fifth set (see [24]).

Theorem 3.1: *Let* $x \in [0,1]$, *then x belongs to the Cantor one-fifth set* F *if and only if x has a base 5 expansion using the digits 0, 2 and 4.*

Now, we prove the main theorem of this section which satisfies the self-referential equation of Cantor one-fifth set using functions f_1, f_2 and f_3.

Theorem 3.2: *Let* f_1, f_2 *and* f_3 *be the similarity contraction mappings on* $\mathbb{R}$ *defined by*

$$f_1(x) = x/5, \; f_2(x) = (x+2)/5, \; f_3(x) = (x+4)/5, \tag{1}$$

where all the mappings have the ratio $1/5$. Then, the Cantor one-fifth set F satisfies the self - referential equation

$$F = f_1[F] \cup f_2[F] \cup f_3[F] \tag{2}$$

for the iterated function system (f_1, f_2, f_3).

Proof: In the starting of this section, we study the Cantor one-fifth set by simply removing the one-fifth open intervals (holes) from the initiator [0, 1]. Now, using the iterated function system (f_1, f_2, f_3), we generate the Cantor one-fifth set which is quite different from above said method. The geometrical representation of Cantor one-fifth set using IFS is as follows:

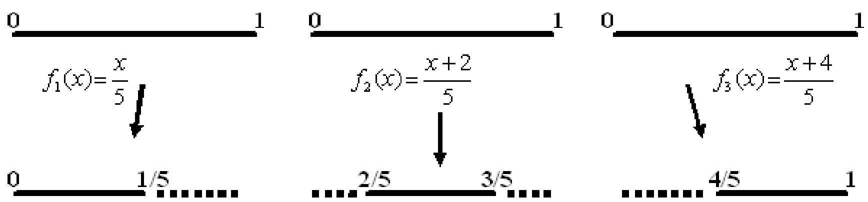

Fig. 2: Representation of IFS (f_1, f_2, f_3)

In Fig. 2, by using the mappings f_1, f_2 and f_3 on the initiator $F_0=[0,1]$, we study the Cantor one-fifth set in the following way, such that

$$f_1\left[[0,1]\right]=[0,1/5],\ f_2\left[[0,1]\right]=[2/5,3/5],\ f_3\left[[0,1]\right]=[4/5,1]$$

Thus,

$$F_1=[0,1/5]\cup[2/5,3/5]\cup[4/5,1]=f_1[F_0]\cup f_2[F_0]\cup f_3[F_0] \qquad (3)$$

Further, repeating the same process and substituting the value of F_1 in the mappings f_1, f_2 and f_3, we obtain Table 1, 2 and 3.

$f_1(x)=\dfrac{x}{5}, x\in F_1$	
$f_1(0)$	0
$f_1(1/5)$	$1/25$
$f_1(2/5)$	$2/25$
$f_1(3/5)$	$3/25$
$f_1(4/5)$	$4/25$
$f_1(1)$	$1/5$

Table 1

$f_2(x)=\dfrac{x+2}{5}, x\in F_1$	
$f_2(0)$	$2/5$
$f_2(1/5)$	$11/25$
$f_2(2/5)$	$12/25$
$f_2(3/5)$	$13/25$
$f_2(4/5)$	$14/25$
$f_2(1)$	$3/5$

Table 2

$f_3(x)=\dfrac{x+4}{5}, x\in F_1$	
$f_3(0)$	$4/5$
$f_3(1/5)$	$21/25$
$f_3(2/5)$	$22/25$
$f_3(3/5)$	$23/25$
$f_3(4/5)$	$24/25$
$f_3(1)$	1

Table 3

Thus,

$$F_2=\underbrace{[0,1/25]\cup[2/25,3/25]\cup[4/25,1/5]}_{f_1(x)}$$

$$\cup\underbrace{[2/5,11/25]\cup[12/25,13/25]\cup[14/25,3/5]}_{f_2(x)}$$

$$\cup\underbrace{[4/5,21/25]\cup[22/25,23/25]\cup[24/25,1]}_{f_3(x)}$$

$$F_2=f_1[F_1]\cup f_2[F_1]\cup f_3[F_1] \qquad (4)$$

Now, by using induction on [0, 1], we obtain

$$F_{n+1} = f_1[F_n] \cup f_2[F_n] \cup f_3[F_n] \tag{5}$$

for n = 0, 1, 2, …. .

To prove Eq. (2), we will first prove that $F \subseteq f_1[F] \cup f_2[F] \cup f_3[F]$ and then converse. Let $x \in F$, which implies $x \in F_1$. Then, either $x \in [0, 1/5]$, $x \in [2/5, 3/5]$ or $x \in [4/5, 1]$. In order to prove the above inequality $F \subseteq f_1[F] \cup f_2[F] \cup f_3[F]$, we study these three cases one by one.

First, let $x \in [4/5, 1]$. Now, using Eq. (5) for any n, $x \in F_{n+1} = f_1[F_n] \cup f_2[F_n] \cup f_3[F_n]$.

But we know that

$f_1[F_n] \subseteq f_1\big[[0, 1]\big] = [0, 1/5]$ and $f_2[F_n] \subseteq f_2\big[[0, 1]\big] = [2/5, 3/5]$.

So that, it implies $x \in f_3[F_n]$ or $5x - 4 \in F_n$, for each n = 1, 2, 3,…

Hence, $5x - 4 \in \bigcap_{n \in \mathbb{N}} F_n = F$. Thus, we get

$$x \in f_3[F] \tag{6}$$

Secondly, let $x \in [2/5, 3/5]$ and using Eq. (5) for any n,

$$x \in F_{n+1} = f_1[F_n] \cup f_2[F_n] \cup f_3[F_n] .$$

But we know that

$f_1[F_n] \subseteq f_1\big[[0, 1]\big] = [0, 1/5]$ and $f_3[F_n] \subseteq f_3\big[[0, 1]\big] = [4/5, 1]$

So that, it implies that $x \in f_2[F_n]$ or $5x - 2 \in F_n$, for each n = 1, 2, 3, …,

whence, $5x - 2 \in \bigcap_{n \in \mathbb{N}} F_n = F$. Thus, we get

$$x \in f_2[F] \tag{7}$$

Last, consider $x \in [0, 1/5]$ and using Eq. (5) for any n,

$$x \in F_{n+1} = f_1[F_n] \cup f_2[F_n] \cup f_3[F_n]$$

But we know that

$f_2[F_n] \subseteq f_2\left[[0,1]\right] = [2/3, 3/5]$ and $f_3[F_n] \subseteq f_3\left[[0,1]\right] = [4/5, 1]$

So that, it implies $x \in f_1[F_n]$ or $5x \in F_n$, for each n = 1, 2, …, hence, $5x \in \bigcap_{n\in\mathbb{N}} F_n = F$. Thus, we get

$$x \in f_1[F] \tag{8}$$

Hence, using Eq. (6), (7) and (8) the inequality $F \subseteq f_1[F] \cup f_2[F] \cup f_3[F]$ holds true.

Conversely,

To prove that $F \supseteq f_1[F] \cup f_2[F] \cup f_3[F]$, let x be any number, such that $x \in f_1[F] \cup f_2[F] \cup f_3[F]$. Then, either $x \in f_1[F]$, $x \in f_2[F]$ or $x \in f_3[F]$. We take the case $x \in f_3[F]$, other two cases are similar. Thus, $5x - 4 \in F$. Also, we can write $5x - 4 \in F_n$ or $x \in f_3[F_n] \subseteq F_{n+1}$. Thus,

$$x \in \bigcap_{n\in\mathbb{N}} F_{n+1} = \bigcap_{n\in\mathbb{N}} F_n = F$$

This implies that the inequality $F \supseteq f_1[F] \cup f_2[F] \cup f_3[F]$ holds true. Hence, this completes the proof of the theorem.

Remark 3.1: By using Proposition 2.1 and Theorem 3.2, the Cantor one-fifth set is unique non-empty compact invariant set for the iterated function system (f_1, f_2, f_3).

Proposition 3.1: *There also exits some sets $M \neq F$ that also satisfy the inequality $M = f_1[M] \cup f_2[M] \cup f_3[M]$.*

Self - Similarity

As discussed in Fig. 2, the first step of division of initiator [0, 1] is $F_1 = [0,1/5] \cup [2/5,3/5] \cup [4/5,1]$ corresponding to the ratio list $(1/5,1/5,1/5)$. Then from the Definition 2.4, the similarity value of a ratio list $(r_1, r_2, ..., r_n)$ is the value $'s'$ such that $r_1^s + r_2^s + ... + r_n^s = 1$. Thus,

$$(1/5)^s + (1/5)^s + (1/5)^s = 1$$

$$s = \frac{\log 1/3}{\log 1/5} \approx 0.6867$$

Thus, $s \approx 0.6867$. Hence, from Definition 2.5 and Theorem 3.2, the similarity dimension of Cantor one-fifth set is also $s \approx 0.6867$.

4 Cantor Middle One-Half Set

Throughout this section we consider, P be a Cantor middle one-half set having ratio $r = 1/2$, center $a \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be a mapping on $\mathbb{R}$ with ratio 'r', defined by $g(x) = rx + (1-r)a$. Here, we call the ordered pair (g_1, g_2) as iterated function system.

Under the construction of Cantor middle one-half set, we begin with the interval $0 \le x \le 1$. First, remove the middle one half interval $1/4 < x < 3/4$, call as holes, from the initiator that gives rise to two closed intervals $0 \le x \le 1/4$ and $3/4 \le x \le 1$, denote as P_1. Again, repeating the same process with the remaining closed intervals, the set P_2 is obtained by leaving

$$P_2 = \underbrace{[0,1/16] \cup [3/16,1/4]}_{g_1(x)} \cup \underbrace{[3/4,13/16] \cup [15/16,1]}_{g_2(x)}$$

Further, repeating the same process again, at each step removing the holes from the middle of each successive closed interval, we get the sequence S_n of holes. The number of holes S_n consists of 2^{n-1} open interval and P_n consists of 2^n disjoint closed intervals. Further, what is left in the limit is the Cantor middle one-half set denoted by P. Fig. 3 shows the geometrical representation of Cantor middle one-half set constructed as above.

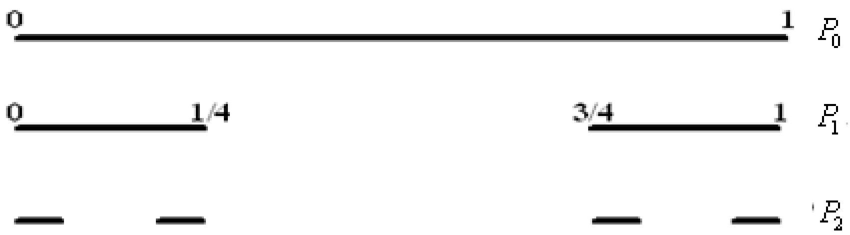

Fig. 3: Cantor Middle One-Half Set

In the following results, we state without proof the well-known result of quarternary Cantor middle one-half set (see [25]).

Proposition 4.1: *Let* $x \in [0,1]$. *Then x belongs to the Cantor middle one-half set* P *if and only if x has a quaternary expansion using the digits 0 and 3.*

Now, we prove the main theorem of this section, which satisfies the self referential equation of Cantor middle one-half set using functions g_1 and g_2.

Theorem 4.2: *Let* g_1 *and* g_2 *be the similarity contraction mappings on* $\mathbb{R}$ *defined by*

$$g_1(x) = x/4 \text{, and } g_2(x) = (x+3)/4, \tag{9}$$

where all the mappings have the ratio 1/4. Then, the Cantor middle one-half set P satisfies the self – referential equation

$$P = g_1[P] \cup g_2[P] \tag{10}$$

for the iterated function system (g_1, g_2).

Proof. First, let us consider the geometrical representation of the Cantor middle one-half set for the iterated functions system (g_1, g_2) shown in Fig. 4.

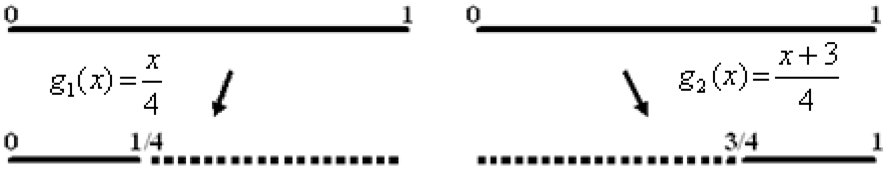

Fig. 4: Representation of IFS (g_1, g_2)

In Fig. 4, using the mappings g_1 and g_2, we study the Cantor middle one-half set in the following way, such that

$$g_1[[0,1]] = [0,1/4] \text{ and } g_2[[0,1]] = [3/4,1]$$

Thus,

$$P_1 = [0,1/4] \cup [3/4,1] = g_1[P_0] \cup g_2[P_0] \tag{11}$$

Again, repeating the process and substituting the value of P_1 in the given iterated function system (g_1, g_2), we obtain Table 4 and 5.

$g_1(x) = \frac{x}{4}, x \in P_1$	
$g_1(0)$	0
$g_1(1/4)$	1/16
$g_1(3/4)$	3/16
$g_1(1)$	1/4

Table 4

$g_2(x) = \frac{x+3}{4}, x \in P_1$	
$g_2(0)$	3/4
$g_2(1/4)$	13/16
$g_2(3/4)$	15/16
$g_2(1)$	1

Table 5

Thus,

$$P_2 = \underbrace{[0,1/16]\cup[3/16,1/4]}_{g_1(x)} \cup \underbrace{[3/4,13/16]\cup[15/16,1]}_{g_2(x)}$$

$$P_2 = g_1[P_1]\cup g_2[P_1] \tag{12}$$

Now, by using induction on [0, 1], we obtain

$$P_{n+1} = g_1[P_n]\cup g_2[P_n] \tag{13}$$

for n = 0, 1, 2, … .

To prove Eq. (10), we first prove that $P\subseteq g_1[P]\cup g_2[P]$ and then converse. Let $x\in P$, which implies $x\in P_1$. Then, either $x\in[0,1/4]$ or $x\in[3/4,1]$. In order to prove the above inequality, we prove two cases one by one.

In first case, let $x\in[3/4,1]$ and using Eq. (13) for any n, $x\in P_{n+1} = g_1[P_n]\cup g_2[P_n]$. But we know that

$$g_1[P_n]\subseteq g_1\big[[0,1]\big] = [0,1/4]$$

So that, it implies that $x\in g_2[P_n]$ or $4x-3\in P_n$, for each n = 1, 2, 3, …, hence, $4x-3\in \bigcap_{n\in\mathbb{N}} P_n = P$. Thus, we get

$$x\in g_2[P] \tag{14}$$

Secondly, let $x\in[0,1/4]$, and using Eq. (13) for any n, $x\in P_{n+1} = g_1[P_n]\cup g_2[P_n]$. But we know that

$$g_2[P_n]\subseteq g_2\big[[0,1]\big] = [3/4,1] \ .$$

So that, it implies that $x\in g_1[P_n]$ or $4x\in P_n$, this holds for each n = 1, 2, 3, …, whence, $4x\in \bigcap_{n\in\mathbb{N}} P_n = P$. Thus, we get

$$x\in g_1[P] \tag{15}$$

Hence, from Eq. (14) and (15) the inequality $P\subseteq g_1[P]\cup g_2[P]$ holds true. Conversely,

To prove that $P\supseteq g_1[P]\cup g_2[P]$, let x be any number, such that $x\in g_1[P]\cup g_2[P]$. Then, either $x\in g_1[P]$, or $x\in g_2[P]$. Let us take the case $x\in g_2[P]$, other is similar. Thus, $4x-3\in P$. Also, we can say $4x-3\in P_n$ or $x\in g_2[P_n]\subseteq P_{n+1}$. Thus,

$$x \in \bigcap_{n\in\mathbb{N}} P_{n+1} = \bigcap_{n\in\mathbb{N}} P_n = P .$$

This completes the proof $P \supseteq g_1[P] \cup g_2[P]$. Hence, the theorem is proved.

Remark 4.1: By using Proposition 2.1, and Theorem 4.2, the Cantor middle one-half set is unique non-empty compact invariant set for the iterated function system (g_1, g_2) .

Self - Similarity

As discussed in Fig. 3, the first step of division of initiator [0, 1] is $P_1 = [0,1/4] \cup [3/4,1]$ corresponding to the ratio list $(1/4,1/4)$. Then, from the Definition 2.4, the similarity value of a ratio list $(1/4,1/4)$ is as follows:

$$(1/4)^s + (1/4)^s = 1$$
$$s = \frac{\log 1/2}{\log 1/4} \approx 0.5$$

Thus $s \approx 0.5$. By using Definition 2.5 and Theorem 4.2, the Similarity dimension of Cantor middle one half set is also $s \approx 0.5$.

4 Conclusion

The Cantor one-fifth set and Cantor middle one-half set both are examples of fractal sets. In this paper, properties of both the Cantor sets have been studied and following results and conclusions were drawn:

1. The Cantor one-fifth set and Cantor middle one-half set have been defined.
2. After studying the iterated function system $(f_1, f_2, ..., f_n)$, a new approach to construct the Cantor one-fifth set and Cantor middle one-half set have been analyzed which is quite different from previous methods in the literature.
3. In Section 3, using iterated function system (f_1, f_2, f_3) , we proved that Cantor one-fifth set satisfies the self-referential equation for the ratio list $(1/5,1/5,1/5)$.
4. In Section 4, using the iterated function system (g_1, g_2) , we proved that Cantor one-fifth set satisfies the self-referential equation for the ratio list $(1/4,1/4)$.
5. For both the Cantor sets, we also established that self similarity value is equivalent to the similarity dimension.

References

[1] M.F. Barnlsey, *Fractal Everywhere (Second Edition)*, Academic Press, Boston, (1993).

[2] M.F. Barnlsey, *Superfractals*, Cambridge University Press, Cambridge, (2006).

[3] A.F. Beardon, On the Hausdorff dimension of general Cantor sets, *Proc. Camb. Phil. Soc.*, 61(1965), 679-694.

[4] G. Cantor, Uber unendliche lineare Punktmannichfaltigkeiten, Part 1, *Math. Ann.*, 15(1879), 1-7.

[5] G. Cantor, Uber unendliche lineare Punktmannichfaltigkeiten, Part 2, *Math. Ann.*, 17(1880), 355-358.

[6] G. Cantor, Uber unendliche lineare Punktmannichfaltigkeiten, Part 3, *Math. Ann.*, 20(1882), 113-121.

[7] G. Cantor, Uber unendliche lineare Punktmannichfaltigkeiten, Part 4, *Math. Ann.*, 21(1883), 51-58.

[8] G. Cantor, Uber unendliche lineare Punktmannichfaltigkeiten, Part 5, *Math. Ann.*, 21(1883), 545-591.

[9] G. Cantor, Uber unendliche lineare Punktmannichfaltigkeiten, Part 6, *Math. Ann.*, 23(1884), 453-488.

[10] R.L. Devaney, *A First Course in Chaotic Dynamical Systems*, Addison Wesley Pub. Company, Inc., (1992), 75-79.

[11] O. Dovgoshey, O. Martio, V. Ryazanov and M. Vaorinen, The Cantor function, *Expositions Mathematicae*, 24(1) (2006), 1-37.

[12] G. Edgar, *Measure Topology and Fractal Geometry*, Springer Verlag, New York, USA, (2008).

[13] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, (1985).

[14] J. Ferreirós, *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Math.*, Springer Verlag, Birkhauser, (1999), 162-165.

[15] J.F. Fleron, A note on the history of the Cantor set and Cantor functions, *Math. Magz.*, 67(1994), 136-140.

[16] R. Gutfraind, M. Sheintuch and D. Avnir, Multifractal scaling analysis of diffusion-limited reactions with Devil's staircase and Cantor set catalytic structures, *Chem. Phy. Lett.*, 174(1) (1990), 8-12.

[17] T. Horiguchi and T. Morita, Devil's staircase in one dimensional mapping, *Physica A*, 126(3) (1984), 328-348.

[18] T. Horiguchi and T. Morita, Fractal dimension related to Devil's staircase for a family of piecewise linear mappings, *Physica A*, 128(1-2) (1984), 289-295.

[19] J.S. Lee, Periodicity on Cantor sets, *Comm. Korean Math. Soc.*, 13(3) (1998), 595-601.

[20] P. Mendes, Sum of Cantor sets: Self-similarity and measure, *Proc. Amer. Math. Soc.*, 127(1999), 3305-3308.

[21] H.O. Peitgen, H. Jürgens and D. Saupe, *Chaos and Fractals: New Frontiers of Science (2nd ed.)*, Springer Verlag, (2004).

[22] A. Rahman, The Cantor set and its applications in nonlinear dynamics and chaos, http://web.njit.edu/~ar276/papers/Dynamical.pdf.

[23] M. Rani and S. Prasad, Superior Cantor sets and superior Devil's staircases, *Int. J. Artif. Life Res.*, 1(1) (2010), 78-84.

[24] M. Rani, Ashish and R. Chugh, New Cantor strings, (2014), (Accepted).

[25] M. Rani, Ashish and R. Chugh, Study of variants of Cantor sets, (2014), (Accepted).

[26] C. Shaver, An exploration of the Cantor set, *Mathematics Seminar*, http://www.rose-hulman.edu/mathjournal/archives/2010/vol11-n1/paper1/++v11n1-1pd.pdf.

[27] H.J.S. Smith, On the integration of discontinuous functions, *Proc. London Math. Soc.*, 6(1) (1875), 140-153.

[28] M. Tsuji, On the capacity of general Cantor sets, *J. Math. Soc. Japan*, 5(1953), 235-252.