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The n- Dimensional Generalized Weyl Fractional Calculus Containing to n- Dimensional $\overline{H}$-Transforms

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Abstract

The object of this paper is to establish a relation between the n-dimensional $\overline{\mathbf{H}}$*-transform involving the Weyl type n- dimensional Saigo operator of fractional integration.*

Keywords: *Fractional Integral, Riemann-Liouville Operator, Gauss Hypergeometric function,* $\overline{H}-$ *function , Fox's H-function, G-function.*

1 Introduction

Our purpose of this paper is to establish a theorem on n-dimensional $\overline{\mathbf{H}}$-transforms involving with Weyl type n-dimensional Saigo operators.
Further, a few interesting and elegant results as special cases of our main results has also been recorded.

2 Fractional Integrals and Derivative

An interesting and useful generalization of both the Riemann-Liouville and Erdélyi-Kober fractional integration operators are introduced by Saigo [9], [10] in terms of Gauss's hypergeometric function as given below.
Let α, β and η are complex numbers and let $y \in R_+ = (0,\infty)$. Following [9], [10] the fractional integral $(\mathrm{Re}\,(\alpha) > 0)$ and derivative $(\mathrm{Re}\,(\alpha) < 0)$ of the first kind of a function f(y) on R_+ are defined respectively in the following forms

$$I_{0,y}^{\alpha,\beta,\eta} f = \frac{y^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{y}\right) f(t)\,dt; \quad R(\alpha) > 0 \tag{1}$$

$$= \frac{d^n}{dy^n} I_{0,y}^{\alpha+n,\beta-n,\eta-n} f, \quad 0 < R(\alpha)+n \le 1, \quad (n = 1,2,...), \tag{2}$$

where ${}_2F_1(\alpha,\ \beta;\ \gamma;\ .)$ is Gauss's hypergeometric function. The fractional integral $(\mathrm{Re}\,(\alpha) > 0)$ and derivative $(\mathrm{Re}\,(\alpha) < 0)$ of the second kind are given by

$$J_{y,\infty}^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_y^\infty (t-y)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{y}{t}\right) f(t)\,dt, \quad R(\alpha)>0 \tag{3}$$

$$= (-1)^n \frac{d^n}{dy^n} I_{y,\infty}^{\alpha+n,\beta-n,\eta} f, \quad 0 < R\,(\alpha)+n \le 1 \quad (n = 1,2,...). \tag{4}$$

The Riemann-Liouville, Weyl and Erdélyi-Kober fractional calculus operators follow as special cases of the operators I and J as given below

$$R_{0,y}^{\alpha} f = I_{0,y}^{\alpha,-\alpha,\eta} f = \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} f(t)\,dt, \quad R\,(\alpha) > 0 \tag{5}$$

$$= \frac{d^n}{dy^n} R_{0,y}^{\alpha+n} f, \quad 0 < R\,(\alpha) + n \le 1, \quad (n = 1,2,...) \tag{6}$$

$$W_{y,\infty}^{\alpha} f = J_{y,\infty}^{\alpha,-\alpha,\eta} f = \frac{1}{\Gamma(\alpha)} \int_y^\infty (t-y)^{\alpha-1} f(t)\,dt, \quad R\,(\alpha) > 0 \tag{7}$$

$$= (-1)^n \frac{d^n}{dy^n} W_{y,\infty}^{\alpha+n} f, \quad 0 < R\,(\alpha) + n \le 1, \quad (n = 1,2,...) \tag{8}$$

$$E_{0,y}^{\alpha,\eta} f = I_{0,y}^{\alpha,0,\eta} f = \frac{y^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} t^{\eta} f(t)\,dt, \quad R(\alpha)>0, \tag{9}$$

$$K_{y,\infty}^{\alpha,\eta} f = J_{y,\infty}^{\alpha,0,\eta} f = \frac{y^{\eta}}{\Gamma(\alpha)} \int_y^{\infty} (t-y)^{\alpha-1} t^{-\alpha-\eta} f(t)\, dt, \quad R(\alpha) > 0. \tag{10}$$

Following Miller [8, p.82], we denote by u_1 the class of functions $f(x_1)$ on R_+ which are infinitely differentiable with partial derivatives of any order behaving as $0(|x_1|^{-\xi_1})$ when $x_1 \to \infty$ for all ξ_1. Similarly by u_2, we denote the class of functions $f(x_1, x_2)$ on $R_+ \times R_+$, which are infinitely differentiable with partial derivatives of any order behaving as $0(|x_1|^{-\xi_1} |x_2|^{-\xi_2})$ when $x_1 \to \infty, x_2 \to \infty$ for all $\xi_i (i = 1, 2)$.

On the same way, we denote the class of functions $f(x_1, x_2, \ldots, x_n)$ on $R_+ \times \ldots \times R_+$, which are infinitely differentiable with partial derivatives of any order behaving as $0(|x_1|^{-\xi_1} |x_2|^{-\xi_2} \ldots |x_n|^{\xi_n})$ when $x_i \to \infty, where\ i = 1, 2, \ldots, n$ for all $\xi_i (i = 1, 2, \ldots, n)$ by u_n.

The n-dimensional operator of Weyl type fractional integration of orders $\text{Re}(\alpha_i) > 0, where\ i = 1, 2, \ldots, n$ is defined in the class u_n by,

$$\prod_{i=1}^{n} \left[J_{p_i,\infty}^{\alpha_i,\beta_i,\gamma_i} \right] [f(p_1, p_2, \ldots, p_n)] = \prod_{i=1}^{n} \left[\frac{p_i^{\beta_i}}{\Gamma(\alpha_i)} \right]$$

$$. \int_{p_1}^{\infty} \int_{p_2}^{\infty} \ldots \int_{p_n}^{\infty} \prod_{i=1}^{n} \left\{ (t_i - p_i)^{\alpha_i - 1} t_i^{-\alpha_i - \beta_i} {}_2F_1\left(\alpha_i + \beta_i, -\gamma_i ; \alpha_i ; 1 - \frac{p_i}{t_i} \right) \right\} f(t_1, t_2, \ldots, t_n)\, dt_1 dt_2 \ldots dt_n, \tag{11}$$

where β_i *and* γ_i, $i = 1, 2, \ldots, n$ are real numbers.

More generally, the operator (11) of Weyl type fractional calculus in n-variables is defined by the differ-integral expression as,

$$\prod_{i=1}^{n} \left[J_{p_i,\infty}^{\alpha_i,\beta_i,\gamma_i} \right] [f(p_1, p_2, \ldots, p_n)] = \prod_{i=1}^{n} \left[\frac{p_i^{\beta_i}}{\Gamma(\alpha_i + r_i)} \right] (-1)^{\sum_{i=1}^{n} r_n} \frac{\partial^{r_1 + r_2 + \ldots + r_n}}{\partial p_1^{r_1} \partial p_2^{r_2} \ldots \partial p_n^{r_n}}$$

$$. \int_{p_1}^{\infty} \int_{p_2}^{\infty} \ldots \int_{p_n}^{\infty} \prod_{i=1}^{n} \left\{ (t_i - p_i)^{\alpha_i - 1} t_i^{-\alpha_i - \beta_i} {}_2F_1\left(\alpha_i + \beta_i, -\gamma_i ; \alpha_i ; 1 - \frac{p_i}{t_i} \right) \right\} f(t_1, t_2, \ldots, t_n)\, dt_1 dt_2 \ldots dt_n, \tag{12}$$

for arbitrary real (complex) α_i *and* $r_1, r_2 \ldots, r_n = 0, 1, 2, \ldots$.

In particular, if $R(\alpha_i)<0$ and r_i are positive integers such that $R(\alpha_i)+r_i>0, where\, i=1,2,...,n,$ then (12) yields the partial fractional derivative of $f(p_1,p_2,...,p_n)$.

On the other hand if we set $\beta_i=0$, (12) yields the Weyl type Erdélyi-Kober operators in n-dimensions

$$\prod_{i=1}^{n}\left[K_{p_i,\infty}^{\alpha_i,\gamma_i}\right][f(p_1,p_2,...,p_n)]=\prod_{i=1}^{n}\left[J_{p_i,\infty}^{\alpha_i,0,\gamma_i}\right][f(p_1,p_2,...,p_n)]$$

$$=\prod_{i=1}^{n}\left[\frac{p_i^{\beta_i}}{\Gamma(\alpha_i+r_i)}\right](-1)^{\sum_{i=1}^{n}r_i}\frac{\partial^{r_1+r_2+...+r_n}}{\partial p_1^{r_1}\partial p_2^{r_2}...\partial p_n^{r_n}}$$

$$.\left\{\int_{p_1}^{\infty}\int_{p_2}^{\infty}...\int_{p_n}^{\infty}\prod_{i=1}^{n}\left[(t_i-p_i)^{\alpha_i+r_i-1}t_i^{-\alpha_i-\gamma_i}\right]f(t_1,t_2,...,t_n)\,dt_1dt_2...dt_n\right\}. \tag{13}$$

3 n-Dimensional Laplace and $\overline{H}$-Transactions

The Laplace transform $\zeta(p_i)$ of a function $f(x_i)\in u_n$ is defined as

$$\zeta(p_1,p_2,...,p_n) = L[f(x_1,x_2,...,x_n);p_1,p_2,...,p_n]$$

$$= \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} e^{-\sum_{i=1}^{n}p_ix_i} f(x_1,x_2,...,x_n)\,dx_1dx_2...dx_n \tag{14}$$

where $R(p_i)>0, where\, i=1,2,...,n$ .Similarly, the Laplace transform of

$$f[u_1\sqrt{x_1^2-\lambda_1^2}\,H(x_1-\lambda_1),u_2\sqrt{x_2^2-\lambda_2^2}\,H(x_2-\lambda_2),...,u_n\sqrt{x_n^2-\lambda_n^2}\,H(x_n-\lambda_n)],$$

is defined by the Laplace transform of $F(x_1,x_2,...,x_n)$ where

$$F(x_1,x_2,...,x_n)=f[u_1\sqrt{x_1^2-\lambda_1^2}\,H(x_1-\lambda_1),u_2\sqrt{x_2^2-\lambda_2^2}\,H(x_2-\lambda_2),...,u_n\sqrt{x_n^2-\lambda_n^2}\,H(x_n-\lambda_n)],$$

$$x_i > \lambda_i>0,\ where\ i=1,2,...,n; \tag{15}$$

and H (t) denotes Heaviside’s unit step function.

Definition: *By n-dimensional* $\overline{H}$*-transform* $\varphi(p_1,p_{2,...,}p_n)$ *of a function* $F(x_1,x_2,...,x_n)$*, we mean the following repeated integral involving n-different* $\overline{H}$*-functions*

$$\varphi\left(p_1,p_{2,\ldots,}p_n\right)=\ \varphi^{M_1,N_1;M_2,N_2;\ldots;M_n,N_n}_{P_1,Q_1;P_2,Q_2;\ldots;P_n,Q_n}[F\left(x_1,x_2,\ldots,x_n\right)\ ;a_1,a_2,\ldots,a_n;p_1,p_{2,\ldots,}p_n]$$

$$=\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\ldots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}.\left[(p_ix_i)^{a_i-1}.\bar{H}^{M_i,N_i}_{P_i,Q_i}\left[(p_ix_i)^{k_i}\left|\begin{matrix}(a_{ij},\alpha_{ij};A_{ij})_{1,N_i},(a_{ij},\alpha_{ij})_{N_i+1,P_i}\\(b_{ij},\beta_{ij})_{1,M_i},(b_{ij},\beta_{ij};B_{ij})_{M_i+1,Q_i}\end{matrix}\right.\right]\right]$$

$$.F\ (x_1,x_2,\ldots,x_n)\ dx_1dx_2\ldots dx_n, \tag{16}$$

here we suppose that $\lambda_i>0, k_i>0$, *where* $i=1,2,\ldots,n;\varphi\left(p_1,p_{2,\ldots,}p_n\right)$, *exists and belongs to* u_n. *Further suppose that,*

$$|\arg p^{k_i}|<\frac{1}{2}\,T_i\pi, \tag{17}$$

where,

$$T_i=\sum_{j=1}^{M_i}|\beta_{ij}|+\sum_{j=1}^{N_i}A_{ij}\,a_{ij}-\sum_{j=M_i+1}^{Q_i}|B_{ij}\beta_{ij}|-\sum_{j=N_i+1}^{P_i}\alpha_{ij}>0, \tag{18}$$

where i=1,2,…,n.

The $\bar{H}$*-function appearing in (16), introduced by Inayat-Hussain ([1], see also [14]) in terms of Mellin-Barnes type contour integral, is defined by,*

$$\bar{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\psi\ (\xi)\ z^{\xi}\ d\xi\ , \tag{19}$$

where

$$\psi(\xi)\ =\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)^{B_j}\right\}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}, \tag{20}$$

which contains fractional powers of some of the Γ-functions. Here and throughout the paper $a_j\left(j=1,\ldots,P\right)$ and $b_j\left(j=1,\ldots,Q\right)$ *are complex parameters.* $\alpha_j\geq0\ \left(j=1,\ldots,P\right),\beta_j\geq0\ \left(j=1,\ldots,Q\right)$, *(not all zero simultaneously) and the exponents* $A_j\left(j=1,\ldots,N\right)$ and $B_j\left(j=M+1,\ldots,Q\right)$ *can take on non-integer values. The contour in (19) is imaginary axis* $R(\xi)=0$ *. It is suitably indented in order to avoid the singularities of the Γ-functions and to keep these singularities on appropriate sides. Again, for* $A_j\left(j=1,\ldots,N\right)$ *not an integer, the poles of the Γ-functions of the numerator in (16) are converted to branch points. However, as long as there is no coincidence of poles from any*

$\Gamma(b_j - \beta_j\xi)$, (j = 1,...,M) and $\Gamma(1 - a_j + \alpha_j\xi), (j = 1, \ldots, N)$ *pair the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.*

For the sake of brevity

$$T = \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} A_j\alpha_j - \sum_{j=M+1}^{Q} |B_j\,\beta_j| - \sum_{j=N+1}^{P} \alpha_j > 0. \tag{21}$$

4 Relationship between n-Dimensional $\bar{H}$-Transforms in Terms of n-Dimensional Saigo Operator of Weyl Type

To prove the theorem in this section, we need the following n-dimensional $\bar{H}$-transform $\phi_1\left(p_1, p_{2,\ldots,}p_n\right)$ of $F\left(x_1, x_{2,\ldots,}x_n\right)$ defined by,

$$\phi_1\left(p_1, p_{2,\ldots,}p_n\right) = \bar{H}^{M_1+1,N_1;M_2+1,N_2;\ldots;M_n+1,N_n}_{P_1+1,Q_1+1;P_2+1,Q_2+1;\ldots;P_n+1,Q_n+1}[F(x_1, x_{2,\ldots,}x_n); a_1, a_2, \ldots, a_n; p_1, p_{2,\ldots,}p_n]$$

$$= \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\ldots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\Big\{(p_i x_i)^{\alpha_i - 1}$$

$$.\bar{H}^{M_i+2,N_i}_{P_i+2,Q_i+2}\left[(p_i x_i)^{k_i}\left|\begin{matrix}(a_{ij},\delta_{ij};A_{ij})_{1,N_i},(a_{ij},\delta_{ij})_{N_i+1,P_i},(1-a_i,k_i),(\alpha_i+\beta_i+\gamma_i-a_i+1,k_i)\\(\beta_i-\alpha_i+1,k_i),(\gamma_i-\alpha_i+1,k_i),(b_{ij},\beta_{ij})_{1,M_i},(b_{ij},\beta_{ij},B_{ij})_{M_i+1,Q_i}\end{matrix}\right.\right]\Big\}$$

$$.F\,(x_1, x_2, \ldots, x_n)\; dx_1 dx_2 \ldots dx_n, \tag{22}$$

where it is assumed that $\phi_1\left(p_1, p_{2,\ldots,}p_n\right)$ exists and belongs to u_n as well as $k_i > 0$, where $i = 1, 2, \ldots, n$ and other conditions on the parameters, in which additional parameters $\alpha_i, \beta_i, \gamma_i$ *where* $i = 1, 2, \ldots, n$ included correspond to those in (11).

Theorem 1 *Let* $\phi_1\left(p_1, p_{2,\ldots,}p_n\right)$ *be given by definition (14) then for* $R\left(\alpha_i\right) > 0, \lambda_1 > 0, k_i > 0$, *where* $i = 1, 2, \ldots, n$ *there holds the formula*

$$\prod_{i=1}^{n}\left\{J^{\alpha_i,\beta_i,\gamma_i}_{p_i,\infty}\right\}[\phi(p_1, p_2, \ldots, p_n)] = \phi_1(p_1, p_2, \ldots, p_n) \tag{23}$$

provided that $\phi_1\,(p_1, p_2, \ldots, p_n)$ *exists and belong to* u_n.

Proof: Let $R(\alpha_i) > 0$, *where* $i = 1, 2, \ldots, n$ then in view of (11) and (18), we find that

$$\prod_{i=1}^{n}\left[J_{p_i,\infty}^{\alpha_i,\beta_i,\gamma_i}\right][\phi\,(p_1,p_2,\ldots,p_n)]=\prod_{i=1}^{n}\cdot\left[\frac{p_i^{\beta_i}}{\Gamma(\alpha_i)}\right]$$

$$\cdot\int_{p_1}^{\infty}\int_{p_2}^{\infty}\ldots\int_{p_n}^{\infty}\prod_{i=1}^{n}\left[(t_i-p_i)^{\alpha_i-1}t_i^{-\alpha_i-\beta_i}\,{}_2F_1\left(\alpha_i+\beta_i,-\gamma_i\,;\,\alpha_i\,;\,1-\frac{p_i}{t_i}\right)\right]\phi\,(t_1,t_2,\ldots,t_n)\;dt_1dt_2\ldots dt_n,$$

or

$$=\prod_{i=1}^{n}\cdot\left[\frac{p_i^{\beta_i}}{\Gamma(\alpha_i)}\right]\cdot\int_{p_1}^{\infty}\int_{p_2}^{\infty}\ldots\int_{p_n}^{\infty}\prod_{i=1}^{n}\left[(t_i-p_i)^{\alpha_i-1}t_i^{-\alpha_i-\beta_i}\,{}_2F_1\left(\alpha_i+\beta_i,-\gamma_i\,;\,\alpha_i\,;\,1-\frac{p_i}{t_i}\right)\right]$$

$$\cdot\left\{\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\ldots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\left[(x_it_i)^{a_i-1}\,.\bar{H}_{P_i,Q_i}^{M_i,N_i}\left[(t_ix_i)^{k_i}\left|\begin{matrix}(a_{ij},\delta_{ij};A_{ij})_{1,N_i},(a_{ij},\alpha_{ij})_{N_i+1,P_i}\\(b_{ij},\beta_{ij})_{1,M_i},(b_{ij},\beta_{ij};B_{ij})_{M_i+1,Q_i}\end{matrix}\right.\right]\right]\right.$$

$$\left..F\,(x_1,x_2,\ldots,x_n)\;dx_1dx_2\ldots dx_n\right\}dt_1dt_2\ldots dt_n. \qquad (24)$$

On interchanging the order of integration which is permissible and on evaluating the t_i *where* $i=1,2,\ldots,n$ integrals through the integral formula

$$\int_{x}^{\infty}u^{-\mu-\nu}(u-x)^{\nu-1}\,{}_2F_1\left(\tau,\omega,\nu;1-\frac{x}{u}\right).\bar{H}_{P,Q}^{M,N}\left[(au)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right]du$$

$$=\frac{\Gamma(\nu)}{x^{\mu}}\bar{H}_{P+2,Q+2}^{M+2,N}\left[(ax)^k\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},(\mu+\nu-\tau,k),\,(\mu+\nu-\omega,k)\\(\mu,k),\,(\mu+\nu-\tau-\omega,\,k),(b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right], \qquad (25)$$

where, $R\,(\nu)>0,\quad R\left(\mu+\nu+\frac{k\,(1-a_j)}{\alpha_j}\right)>0$

$$R\left(\mu+\nu-\tau-\omega+\frac{k\,(1-a_j)}{\alpha_j}\right)>0,|\arg z|\;<\;\frac{1}{2}T\,\pi\quad \text{(T is given in (21))}$$

(23) can be established by means of the following formula [2, p.399].

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1}\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx\;=\frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)} \qquad (26)$$

for $R(\gamma) > 0$, $R(\rho) > 0$, $R(\gamma+\rho-\alpha-\beta) > 0$.

by using the formula, left hand side of (24) becomes

$$= \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\ldots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\Bigg[(t_i x_i)^{a_i-1}$$

$$.\bar{H}_{P_i+2,Q_i+2}^{M_i+2,N_i}\left[(t_i x_i)^{k_i}\left|\begin{matrix}(a_{ij},\delta_{ij};A_{ij})_{1,N_i},(a_j,\delta_j)_{N_i+1,P_i},(1-a_i,k_i),(\alpha_i+\beta_i+\gamma_i-a_i+1,k_i)\\(\beta_i-a_i+1,k_i),(\gamma_i-a_i+1,k_i),(b_{ij},\beta_{ij})_{1,M_i},(b_{ij},\beta_{ij})_{M_i+1,Q_i}\end{matrix}\right.\right]\Bigg]$$

$$.F(x_1,x_2,\ldots,x_n)\,dx_1dx_2\ldots dx_n\,.$$

$$=\bar{H}_{P_1+2,Q_1+2;P_2+2,Q_2+2;\ldots;P_n+2,Q_n+2}^{M_1+2,N_1\,;M_2+2,N_2;\ldots;M_n+2,N_n}[F(x_1,x_2,\ldots,x_n)\,;\,\alpha_1,\alpha_2,\ldots,\alpha_n\,;p_1,p_2,\ldots,p_n]$$

$$=\varphi_1(p_1,p_2,\ldots,p_n).$$

As far as the n- dimensional Weyl type operators $\prod_{i=1}^{n}\left[J_{p_i,\infty}^{\alpha_i,\beta_i,\gamma_i}\right]$ preserves the class u_n, it follows that $\varphi_1(p_1,p_2,\ldots,p_n)$ also belongs to u_n.

It is interesting to note that the statement of Theorem 1 can easily be extended for arbitrary real α_i *where* $i=1,2,\ldots,n$ by using the definition (12) for the generalized Weyl type fractional calculus operators and differentiating under the signs of the integrals.

5 Interesting Special Cases

Putting $\gamma_i = 0$ *where* $i=1,2,\ldots,n$ *in theorem 1, we can easily prove Theorem 1(a).*

Theorem 1(a). *For* $R(\alpha_i)>0$, $\beta_i > 0$, $r_i > 0$; *where* $i=1,2,\ldots,n$ *and also let* $\varphi(p_1,p_2,\ldots,p_n)$ *be given by (14) then there holds the following formula,*

$$\prod_{i=1}^{n}.\left[J_{p_i,\infty}^{\alpha_i,\beta_i,0}\right][\varphi(p_1,p_2,\ldots,p_n)] = \varphi_2(p_1,p_2,\ldots,p_n) \tag{27}$$

provided that $\varphi_2(p_1,p_2,\ldots,p_n)$ *exists and belongs to* u_n *where* φ_2 *is represented by the repeated integral,*

$$\varphi_2(p_1,p_2,\ldots,p_n)=\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}\ldots\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\Bigg[(p_i x_i)^{a_i-1}$$

$$.\bar{H}^{M_i+1,N_i}_{P_i+1,Q_i+1}\left[(p_i x_i)^{k_i}\left|\begin{matrix}(a_{ij},\delta_{ij};A_{ij})_{1,N_i},(a_{ij},\delta_{ij})_{N_i+1,P_i},(\alpha_i+\beta_i-a_i+1,k_i)\\(\beta_i-a_i+1,k_i),(b_{ij},\beta_{ij})_{1,M_i},(b_{ij},\beta_{ij},B_{ij})_{M_i+1,Q_i}\end{matrix}\right.\right]F(x_1,x_2,...,x_n)dx_1dx_2...dx_n. \tag{28}$$

For $A_{ij}=1$, the $\bar{H}$-function reduces to Fox's H-function [5], [6] and then Theorem 1 (a) reduces to,

$$\prod_{i=1}^{n}\left[J^{\alpha_i,\beta_i,0}_{p_i,\infty}\right][\varphi(p_1,p_2,...,p_n)] = \varphi_3(p_1,p_2,...,p_n), \tag{29}$$

provided that $\varphi_3(p_1,p_2,...,p_n)$ exists and belongs to u_n, where φ_3 is represented by the repeated integral,

$$\varphi_3(p_1,p_2,...,p_n)=\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}...\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\left\{(p_i x_i)^{a_i-1}.H^{M_i+1,N_i}_{P_i+1,Q_i+1}\left[(p_i x_i)^{k_i}\left|\begin{matrix}(a_{iP_i},\delta_{iP_i}),(\alpha_i+\beta_i-a_i+1,k_i)\\(\beta_i-a_i+1,k_i),(b_{iQ_i},\beta_{iQ_i})\end{matrix}\right.\right]\right\}$$

$$.F(x_1,x_2,...,x_n)\,dx_1dx_2...dx_n. \tag{30}$$

On employing the identity

$$H^{M,N}_{P,Q}\left[x\left|\begin{matrix}(a_P,1)\\(b_Q,1)\end{matrix}\right.\right]=G^{M,N}_{P,Q}\left[x\left|\begin{matrix}a_1,...,a_P\\b_1,...,b_Q\end{matrix}\right.\right], \tag{31}$$

we see that the n- dimensional H-transform reduces to the corresponding n- dimensional G-transform $\theta(p_1,p_2,...,p_n)$ defined by

$$\theta(p_1,p_2,...,p_n)=G^{M_1,N_1;M_2,N_2;...;M_n,N_n}_{P_1,Q_1;P_2,Q_2;...;P_n,Q_n}[F(x_1,x_2,...,x_n);\alpha_1,\alpha_2,...,\alpha_n;p_1,p_2,...,p_n]$$

$$=\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}...\int_{\lambda_n}^{\infty}(p_i x_i)^{\alpha_i-1}G^{M_i,N_i}_{P_i,Q_i}\left[(p_i x_i)^{k_i}\left|\begin{matrix}a_{1i},...,a_{P_i}\\b_{1i},...,b_{Q_i}\end{matrix}\right.\right].F(x_1,x_2,...,x_n)\,dx_1dx_2...dx_n, \tag{32}$$

provided that $\theta(p_1,p_2,...,p_n)$ exists and belongs to class u_n, where k_i, are positive integers, $\lambda_i>0, P_i\le Q_i$,

$$|\arg p^{k_i}| < \frac{1}{2} T_i^* \pi, \tag{33}$$

with

$$T_i^* = 2N_i + 2M_i - P_i - Q_i, \tag{34}$$

where $i=1,2,\ldots,n$, $G^{M,N}_{P,Q}[.]$, *appealing in (31) and (32) represents Meijer's G-function whose detailed account is available from the monograph of Mathai and Saxena [4].*

Thus, we obtain the following Theorem 1 (b).

Theorem 1(b). *For* $R(\alpha_i) > 0,\ \beta_i > 0,\ k_i > 0;$ *where* $i = 1, 2, \ldots, n$ *being positive integers and also let* $\theta(p_1, p_2, \ldots, p_n)$ *be given by (31) then the following formula*

$$\prod_{i=1}^{n} \cdot \left[J^{\alpha_i, \beta_i, 0}_{p_i, \infty} \right] [\theta(p_1, p_2, \ldots, p_n)] = \theta_1(p_1, p_2, \ldots, p_n) \tag{35}$$

holds, provided that $\theta_1(p_1, p_2, \ldots, p_n)$ *exists and belongs to class* u_n *for other conditions on the parameters, in which additional parameters* α_i, β_i *and* γ_i *where* $i = 1, 2, \ldots, n$ *included correspond to those in (32). Here*

$$\theta_1(p_1, p_2, \ldots, p_n) = \prod_{i=1}^{n} \left[k_i^{-\alpha_i} \right] \int_{\lambda_1}^{\infty} \int_{\lambda_2}^{\infty} \ldots \int_{\lambda_n}^{\infty} \prod_{i=1}^{n} \left\{ (p_i x_i)^{a_i - 1} \right.$$

$$\left. . G^{M_i+2, N_i}_{P_i+2, N_i} \left[(px)^{k_i} \left| \begin{matrix} a_{1_i}, \ldots, a_{P_i}, \Delta(k_i, 1-a_i), \Delta(k_i, \alpha_i + \beta_i + \gamma_i - a_i + 1) \\ \Delta(k_i, \beta_i - a_i + 1), \Delta(k_i, \gamma_i - a_i + 1), b_{Q_i}, \ldots, b_{Q_i} \end{matrix} \right. \right] \right\}$$

$$. F(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n, \tag{36}$$

and the symbol $\Delta(n, \alpha)$ *represents the sequence of parameters*

$$\frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n}$$

On taking $\gamma_i = 0,$ *where* $i = 1, 2, \ldots, n,$ *(36) becomes*

$$\prod_{i=1}^{n}\cdot\left[J_{p_i,\infty}^{\alpha_i,\beta_i,0}\right][\theta(p_1,p_2,...,p_n)]=\theta_2(p_1,p_2,...,p_n) \qquad (37)$$

provided $\theta_2(p_1,p_2,...,p_n)$ *exists and belongs to class* u_n*, where* θ_2 *is represented by the integral*

$$\theta_2(p_1,p_2,...,p_n) = \prod_{i=1}^{n}\left[k_i^{-\alpha_i}\right]\int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty}...\int_{\lambda_n}^{\infty}\prod_{i=1}^{n}\left\{(p_i x_i)^{a_i-1}\right.$$

$$.G_{P_i+1,N_i}^{M_i+1,N_i}\left[(px)^{k_i}\left|\begin{matrix}a_{1_i},...,a_{P_i},\Delta(k_i,\alpha_i+\beta_i-a_i+1)\\ \Delta(k_i,\beta_i-a_i+1),b_{Q_i},...,b_{Q_i}\end{matrix}\right.\right]\Bigg\}$$

$$.F(x_1,x_2,...,x_n)\,dx_1dx_2...dx_n. \qquad (38)$$

6 Special Case

(i) Converting our Theorem 1, 1(a) and 1(b) for i=1,2,3; we find the known result defined by Chaurasia and jain [19]

(ii) Converting our Theorem 1, 1(a) and 1(b) for i=1,2; we find the known result defined by Chaurasia and Shrivastava [18],if we tack $N = N' = 0$.

(iii) Taking $A_j = B_j = 1$, then Theorem 1, 1(a) and 1(b) for i=1,2; we find the known result defined by Saigo, Saxena and Ram [13].

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